Resonances and their $N_C$ fates in $U(3)$ chiral perturbation theory

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Hadron 2011, 13 June -17 June 2011, Munich, Germany

in collaboration with Jose Oller, based on arXiv:1104.2849[hep-ph]

A work dedicated to Joaquim Prades
Outline

1. Preface
2. Analytical calculation
   - Chiral Lagrangian & perturbative amplitudes
   - Resummation of s-channel loops: a variant N/D method
3. Phenomenological discussion
   - Fit quality
   - Poles in the complex energy plane & their residues
   - $N_C$ trajectories
4. Conclusions
In the chiral limit $m_u = m_d = m_s = 0$ the QCD Lagrangian is invariant under $U_L(3) \otimes U_R(3)$ symmetry at the classical level.

$U_A(1) \equiv U_{L-R}$: violated at the quantum level, i.e. $U_A(1)$ anomaly, which is also responsible for the massive $\eta_1$.

$U_V(1) \equiv U_{L+R}$: conserved baryon number.

$SU_L(3) \otimes SU_R(3) \rightarrow SU_V(3)$ is spontaneously broken. Goldstone bosons appear $\pi, K, \eta_8$: $SU(3) \chi$PT [Gasser, Leutwyler, NPB’85].

In large $N_C$ limit, $U_A(1)$ anomaly disappears and the $\eta_1$ mass vanishes: $M_{\eta_1}^2 \sim \mathcal{O}(1/N_C)$. So $\eta_1$ together with $\pi, K, \eta_8$ constitute the nonet of pesudo Goldstone bosons.

[t’Hooft, NPB’74] [Witten, NPB’79] [Coleman & Witten, PRL’80]
$U(3)\chi$PT takes $\pi$, $K$, $\eta_8$ and $\eta_1$ as its dynamical degrees of freedom and employs the triple expansion scheme: momentum, quark masses and $1/N_C$, i.e. $\delta \sim p^2 \sim m_q \sim 1/N_C$.

- Set up in: [Witten, PRL’80] [Di Vecchia & Veneziano,’80 ] [Rosenzweig, Schechter & Trahern, ’80 ]

- Chiral Lagrangian to $O(p^4)$ completed in:
  [Herrera-Siklody, Latorre, Pascual, Taron, NPB’97 ] . See also [Kaiser, Leutwyler, EPJC’00 ] .

- Applications
  Light quark masses:  [Leutwyler, PLB’96 ]
  $\eta - \eta'$ mixing:  [Herrera-Siklody, Latorre, Pascual, Taron, PLB’98]
  [Leutwyler, NPB(Proc.Suppl)’98 ]
  $\eta' \rightarrow \eta \pi \pi$ decay:  [Escribano,Masjuan, Sanz-Cillero, JHEP’11]
Our current work offers the complete one-loop amplitudes of the meson-meson scattering within $U(3)\chi$PT. And then we study the properties of various resonances, such as their pole positions, residues and $N_C$ behaviour, by unitarizing the $U(3)\chi$PT amplitudes.
There are variant methods to treat $\eta'$ in the market

- Matter filed: $M_{\eta'}^2 \sim \mathcal{O}(1)$ and Infrared Regularization method used to handle the loops. [Beisert, Borasoy, NPA’02, PRD’03]
- Non-relativistic field
  [Kubis, Schneider, EPJC’09]
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Relevant Chiral Lagrangian
\[ \mathcal{L}^{(\delta^0)} = \frac{F^2}{4} \langle u_{\mu} u^{\mu} \rangle + \frac{F^2}{4} \langle \chi^+ \rangle + \frac{F^2}{3} M_0^2 \ln^2 \det u, \] (1)

where

\[ u = e^{i \frac{\Phi}{\sqrt{2} F}}, \quad U = u^2, \]

\[ u_{\mu} = i u^\dagger D_{\mu} U u^\dagger = u^\dagger_{\mu}, \quad \chi_{\pm} = u^\dagger \chi u^\dagger \pm u \chi^\dagger u, \]

\[ \Phi = \begin{pmatrix} \sqrt{3} \pi^0 + \eta_8 + \sqrt{2} \eta_1 & \pi^+ & K^+ \\ \sqrt{6} & -\sqrt{3} \pi^0 + \eta_8 + \sqrt{2} \eta_1 & K^0 \\ \pi^- & \sqrt{6} & -2 \eta_8 + \sqrt{2} \eta_1 \\ K^- & \bar{K}^0 & \sqrt{6} \end{pmatrix}. \] (2)
\( L_i \)s correspond to the higher order local operators.
At \( \mathcal{O}(\delta) \) one has \( \mathcal{O}(N_C p^4) \) and \( \mathcal{O}(N_C^0 p^2) \) operators:

\[
\mathcal{L}^{(\delta)} = L_2 \langle u_\mu u_\nu u_\mu u_\nu \rangle + (2L_2 + L_3) \langle u_\mu u_\mu u_\nu u_\nu \rangle \\
+ L_5 \langle u_\mu u_\mu \chi_+ \rangle + L_8/2 \langle \chi_+ \chi_+ + \chi_- \chi_- \rangle + \ldots \\
+ F^2 \Lambda_1/12 D_\mu \psi D^\mu \psi - i F^2 \Lambda_2/12 \psi \langle U^\dagger \chi - \chi^\dagger U \rangle + \ldots
\]

At \( \mathcal{O}(\delta^2) \) (same order as the one-loop contribution), one then has \( \mathcal{O}(\tilde{N}_C^{-2} p^0) \), \( \mathcal{O}(\tilde{N}_C^{-1} p^2) \), \( \mathcal{O}(N_C^0 p^4) \) and \( \mathcal{O}(N_C p^6) \) operators:

\[
\mathcal{L}^{(\delta^2)} = \tilde{\psi}_0^{(4)} X^4 + \tilde{\psi}_1^{(2)} X^2 \langle u_\mu u^\mu \rangle + L_4 \langle u_\mu u^\mu \rangle \langle \chi_+ \rangle \\
+ C_1 \langle u_\rho u^\rho h_{\mu\nu} h^{\mu\nu} \rangle + \ldots
\]

with \( \psi = -i \ln \det U \), \( X = \log \det(U) \) and \( h_{\mu\nu} = \nabla_\mu u_\nu + \nabla_\nu u_\mu \).

[Herrera-Siklody, Latorre, Pascual, Taron, NPB’97]
[Bijnens, Colangelo, Ecker, JHEP’99]
Alternatively, one could use resonances to estimate the higher order low energy constants:

\[ \mathcal{L}_S = c_d \langle S_8 u_\mu u^\mu \rangle + c_m \langle S_8 \chi_+ \rangle + \tilde{c}_d S_1 \langle u_\mu u^\mu \rangle + \tilde{c}_m S_1 \langle \chi_+ \rangle + \ldots \]  

(3)

\[ \mathcal{L}_V = \frac{iG_V}{2\sqrt{2}} \langle V_{\mu\nu} [u^\mu, u^\nu] \rangle + \ldots, \]  

(4)

[Ecker, Gasser, Pich, de Rafael, NPB’89]

In the current discussion, we assume the resonance saturation and exploit the above resonance operators to calculate the meson-meson scattering.

The monomials proportional to \( \Lambda_1 \) and \( \Lambda_2 \) are not generated through resonance exchange. No double counting.
Perturbative calculation of the scattering amplitudes
Figure: Relevant Feynman diagrams for mass, wave function renormalization and $\eta - \eta'$ mixing

The leading order $\eta$-$\eta'$ mixing has to be solved exactly

Figure: The dot denotes the mixing of $\eta_8$ and $\eta_1$ at leading order, which is proportional to $m_K^2 - m_\pi^2$. 
Scattering amplitudes consist of

(a) + (b) + (c) + crossed

(d) + (e) + crossed
Figure: Relevant Feynman diagrams for the pseudo Goldstone decay constant. The wiggly line corresponds to the axial-vector external source.

We expressed all the amplitudes in terms of physical masses and $F_\pi$, i.e. reshuffling the leading order contributions.
Partial wave amplitude and its unitarization
Partial wave projection:

\[ T_J^I(s) = \frac{1}{2(\sqrt{2})^N} \int_{-1}^{1} dx \, P_J(x) \, T^I[s, t(x), u(x)], \tag{5} \]

where \( P_J(x) \) denote the Legendre polynomials and \( (\sqrt{2})^N \) is a symmetry factor to account for the identical particles, such as \( \pi\pi, \eta\eta, \eta'\eta' \).
The essential of the $N/D$ method is to construct the unitarized $T_J$: [Chew, Mandelstam, PR’60]

$$T_J = \frac{N}{D},$$ (6)

where

$$\text{Im} D = N \text{Im} T_J = -\rho N, \quad \text{for } s > 4m^2,$$

$$\text{Im} N = 0, \quad \text{for } s < 4m^2,$$

$$\text{Im} N = D \text{Im} T_J, \quad \text{for } s < 0,$$

$$\text{Im} N = 0, \quad \text{for } s > 0,$$ (7)

due to the fact that the unitarity condition for the elastic channel is

$$\text{Im} T_J^{-1} = -\rho, \quad s > 4m^2$$ (8)

where $\rho = \sqrt{1 - 4m^2/s/16\pi}$. 
One can now write the dispersion relations for $N$ and $D$:

$$D(s) = \tilde{a}^{SL}(s_0) - \frac{s - s_0}{\pi} \int_{4m^2}^{\infty} \frac{N(s') \rho(s')}{(s' - s)(s' - s_0)} ds' + ...,$$  \hspace{1cm} (9)

$$N(s) = \int_{-\infty}^{0} \frac{D(s') \text{Im} T_J(s')}{s' - s} ds'.$$  \hspace{1cm} (10)

It can be greatly simplified if one imposes the perturbative solution for $N(s)$ instead of the left hand discontinuity [Oller, Oset, PRD’99],

$$T_J(s) = \frac{N(s)}{1 + g(s) N(s)},$$ \hspace{1cm} (11)

where

$$g(s) = \frac{a^{SL}(s_0)}{16\pi^2} - \frac{s - s_0}{\pi} \int_{4m^2}^{\infty} \frac{\rho(s')}{(s' - s)(s' - s_0)} ds'. \hspace{1cm} (12)$$
Matching the $T_J(s) = N(s)/[1 + g(s) N(s)]$ with
$T_J(s)|_{\chi PT} = T_2 + T_{\text{Resonance}} + T_{\text{Loop}}$ up to one-loop:

$$N(s) = T_2 + T_{\text{Resonance}} + T_{\text{Loop}} + T_2 g(s) T_2. \quad (13)$$

The generalization to the inelastic case is straightforward:

$$T_J(s) = N(s) \cdot [1 + g(s) \cdot N(s)]^{-1}. \quad (14)$$

This formalism has been explored in many areas. See in this conference the talks already done by Alarcon, F.K.Guo, Magalaes, Molina, Oset.
For $IJ = 00$ case, we have 5 channels: $\pi\pi$, $K\bar{K}$, $\eta\eta$, $\eta\eta'$ and $\eta'\eta'$

$N_0^0(s) = \begin{pmatrix} N_{\pi\pi \rightarrow \pi\pi} & N_{\pi\pi \rightarrow K\bar{K}} & N_{\pi\pi \rightarrow \eta\eta} & N_{\pi\pi \rightarrow \eta'\eta'} & N_{\pi\pi \rightarrow \eta'\eta'} \\ N_{\pi\pi \rightarrow K\bar{K}} & N_{K\bar{K} \rightarrow K\bar{K}} & N_{K\bar{K} \rightarrow \eta\eta} & N_{K\bar{K} \rightarrow \eta'\eta'} & N_{K\bar{K} \rightarrow \eta'\eta'} \\ N_{\pi\pi \rightarrow \eta\eta} & N_{K\bar{K} \rightarrow \eta\eta} & N_{\eta\eta \rightarrow \eta\eta} & N_{\eta\eta \rightarrow \eta'\eta'} & N_{\eta\eta \rightarrow \eta'\eta'} \\ N_{\pi\pi \rightarrow \eta'\eta'} & N_{K\bar{K} \eta'\eta'} & N_{\eta\eta \rightarrow \eta'\eta'} & N_{\eta'\eta' \rightarrow \eta'\eta'} & N_{\eta'\eta' \rightarrow \eta'\eta'} \\ N_{\pi\pi \rightarrow \eta'\eta'} & N_{K\bar{K} \eta'\eta'} & N_{\eta\eta \rightarrow \eta'\eta'} & N_{\eta'\eta' \rightarrow \eta'\eta'} & N_{\eta'\eta' \rightarrow \eta'\eta'} \end{pmatrix}$

$g_0^0(s) = \begin{pmatrix} g_{\pi\pi} & 0 & 0 & 0 & 0 \\ 0 & g_{K\bar{K}} & 0 & 0 & 0 \\ 0 & 0 & g_{\eta\eta} & 0 & 0 \\ 0 & 0 & 0 & g_{\eta'\eta'} & 0 \\ 0 & 0 & 0 & 0 & g_{\eta'\eta'} \end{pmatrix}$. 
For $IJ = 10$, we have 3 channels: $\pi \eta$, $K \bar{K}$ and $\pi \eta'$

\[
N(s)_{0}^{1} = \begin{pmatrix}
N_{\pi \eta \rightarrow \pi \eta} & N_{\pi \eta \rightarrow K \bar{K}} & N_{\pi \eta \rightarrow \pi \eta'} \\
N_{\pi \eta \rightarrow K \bar{K}} & N_{K \bar{K} \rightarrow K \bar{K}} & N_{K \bar{K} \rightarrow \pi \eta'} \\
N_{\pi \eta \rightarrow \pi \eta'} & N_{K \bar{K} \rightarrow \pi \eta'} & N_{\pi \eta' \rightarrow \pi \eta'}
\end{pmatrix},
\]

\[
g(s)_{0}^{1} = \begin{pmatrix}
g_{\pi \eta} & 0 & 0 \\
0 & g_{K \bar{K}} & 0 \\
0 & 0 & g_{\pi \eta'}
\end{pmatrix}.
\]
For $IJ = 1/2 \ 0$, there are three channels: $K\pi$, $K\eta$ and $K\eta'$

$$N(s)^{1/2} = \begin{pmatrix} N_{K\pi \rightarrow K\pi} & N_{K\pi \rightarrow K\eta} & N_{K\pi \rightarrow K\eta'} \\ N_{K\pi \rightarrow K\eta} & N_{K\eta \rightarrow K\eta} & N_{K\eta \rightarrow K\eta'} \\ N_{K\pi \rightarrow K\eta'} & N_{K\eta \rightarrow K\eta'} & N_{K\eta' \rightarrow K\eta'} \end{pmatrix},$$

$$g(s)^{1/2} = \begin{pmatrix} g_{K\pi} & 0 & 0 \\ 0 & g_{K\eta} & 0 \\ 0 & 0 & g_{K\eta'} \end{pmatrix}.$$

The same expressions hold for $IJ = 1/2 \ 1$. 
For $IJ = 1\ 1$ there are 2 channels

$$N_1^1(s) = \begin{pmatrix} N_{\pi\pi\to\pi\pi} & N_{\pi\pi\to K\bar{K}} \\ N_{\pi\pi\to K\bar{K}} & N_{K\bar{K}\to K\bar{K}} \end{pmatrix},$$

$$g_1^1(s) = \begin{pmatrix} g_{\pi\pi} & 0 \\ 0 & g_{K\bar{K}} \end{pmatrix}.$$

For $IJ = 3/2\ 0$, it is an elastic channel

$$N(s)^{3/2}_0 = N_{K\pi\to K\pi},$$

$$g(s)^{3/2}_0 = g_{K\pi}.$$

For $IJ = 2\ 0$, it is

$$N(s)^2_0 = N_{\pi\pi\to\pi\pi},$$

$$g(s)^2_0 = g_{\pi\pi}.$$
Phenomenological discussion
Resonances and their $N_C$ fates in $U(3)$ chiral perturbation theory

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Resonances and their $N_C$ fates in $U(3)$ chiral perturbation theory
Resonances and their $N_C$ fates in $U(3)$ chiral perturbation theory
We have 16 free parameters with 348 data and the fitted results are

\[
\begin{align*}
  c_d &= (15.6^{+4.2}_{-3.4}) \text{ MeV}, \\
  \bar{c}_d &= (8.7^{+2.5}_{-1.7}) \text{ MeV}, \\
  M_{S_8} &= (1370^{+132}_{-57}) \text{ MeV}, \\
  M_\rho &= (801.0^{+7.0}_{-7.5}) \text{ MeV}, \\
  G_V &= (61.9^{+1.9}_{-1.9}) \text{ MeV}, \\
  a_{SL}^{00} &= (-1.15^{+0.07}_{-0.09}), \\
  a_{SL}^{10,\pi\eta} &= 2.0^{+3.1}_{-3.4}, \\
  a_{SL}^{1/2,0} &= (-0.96^{+0.10}_{-0.09}), \\
  N &= (0.6^{+0.3}_{-0.3}) \text{ MeV}^{-2}, \\
  M_0 &= (954^{+102}_{-95}) \text{ MeV}, \\
  M_{0'} &= (954^{+102}_{-95}) \text{ MeV}, \\
  c_{\pi} &= (1.0^{+0.6}_{-0.4}), \\
  \Lambda_2 &= (-0.6^{+0.5}_{-0.4}),
\end{align*}
\]

with \( \chi^2/\text{d.o.f} = 714/(348 - 16) \simeq 2.15 \).

\[ n_\sigma = \Delta \chi^2 / \sqrt{2 \chi^2} \leq 2 \] to get the errors, \( n_\sigma = 2 \) Etkin et al. PRD’82
Poles from the unitarized amplitudes

$\sigma$ or $f_0(600)$, $IJ = 0\ 0$

\[
M_\sigma = 440^{+3}_{-3} \text{ MeV}, \quad \Gamma_\sigma / 2 = 258^{+5}_{-7} \text{ MeV},
\]
\[
|g_{\sigma \pi \pi}| = 3.02^{+0.03}_{-0.03} \text{ GeV},
\]
\[
|g_{\sigma K \bar{K}}| / |g_{\sigma \pi \pi}| = 0.51^{+0.03}_{-0.02}, \quad |g_{\sigma \eta \eta}| / |g_{\sigma \pi \pi}| = 0.06^{+0.03}_{-0.01}
\]
\[
|g_{\sigma \eta \eta'}| / |g_{\sigma \pi \pi}| = 0.16^{+0.03}_{-0.02}, \quad |g_{\sigma \eta' \eta'}| / |g_{\sigma \pi \pi}| = 0.05^{+0.03}_{-0.03}
\]

Other approaches:

$M_\sigma = 470 \pm 50, \quad \Gamma_\sigma / 2 = 285 \pm 25$ Zhou, et al. JHEP’05

$M_\sigma = 441^{+16}_{-8}, \quad \Gamma_\sigma / 2 = 272^{+9}_{-13}$ Caprini et al. PRL’06

$M_\sigma = 484 \pm 17, \quad \Gamma_\sigma / 2 = 255 \pm 10$ García-Martín et al. PRD’07

$M_\sigma = 456 \pm 6, \quad \Gamma_\sigma / 2 = 241 \pm 17$ Albaladejo, Oller PRL’08
\[ f_0(980), IJ = 00 \]

\[ M_{f_0} = 981^{+9}_{-7} \text{ MeV}, \quad \Gamma_{f_0}/2 = 22^{+5}_{-7} \text{ MeV}, \]
\[ |g_{f_0\pi\pi}| = 1.7^{+0.3}_{-0.3} \text{ GeV} \]
\[ |g_{f_0K\bar{K}}|/|g_{f_0\pi\pi}| = 2.3^{+0.3}_{-0.2}, \quad |g_{f_0\eta\eta}|/|g_{f_0\pi\pi}| = 1.6^{+0.3}_{-0.3} \]
\[ |g_{f_0\eta\eta'}|/|g_{f_0\pi\pi}| = 1.2^{+0.1}_{-0.2}, \quad |g_{f_0\eta'\eta'}|/|g_{f_0\pi\pi}| = 0.7^{+0.4}_{-0.5} \]

\[ f_0(1370), IJ = 00 \]

\[ M_{f_0} = 1401^{+58}_{-37} \text{ MeV}, \quad \Gamma_{f_0}/2 = 106^{+36}_{-23} \text{ MeV}, \]
\[ |g_{f_0\pi\pi}| = 2.4^{+0.2}_{-0.1} \text{ GeV} \]
\[ |g_{f_0K\bar{K}}|/|g_{f_0\pi\pi}| = 0.62^{+0.04}_{-0.05}, \quad |g_{f_0\eta\eta}|/|g_{f_0\pi\pi}| = 0.9^{+0.1}_{-0.1} \]
\[ |g_{f_0\eta\eta'}|/|g_{f_0\pi\pi}| = 1.7^{+0.4}_{-0.6}, \quad |g_{f_0\eta'\eta'}|/|g_{f_0\pi\pi}| = 1.1^{+0.4}_{-0.5} \]

Both resonances have strong couplings to states with \( \eta, \eta' \)
\( \kappa \) or \( K_0^*(800) \), \( IJ = 1/2 \)

\[
M_\kappa = 665^{+9}_{-9} \text{ MeV}, \quad \Gamma_\kappa/2 = 268^{+21}_{-6} \text{ MeV},
\]

\[
|g_\kappa K\pi| = 4.2^{+0.2}_{-0.2} \text{ GeV}
\]

\[
|g_\kappa K\eta|/|g_\kappa K\pi| = 0.7^{+0.1}_{-0.1} , \quad |g_\kappa K\eta'|/|g_\kappa K\pi| = 0.50^{+0.1}_{-0.1}
\]

Other approaches:
\[
\sqrt{s} = (594 \pm 79 - i \ 362 \pm 166) \text{ MeV} \quad \text{Zheng, et al. NPA'04}
\]
\[
\sqrt{s} = (658 \pm 13 - i \ 278 \pm 12) \text{ MeV} \quad \text{Descotes, Moussallam EPJC'06}
\]

\( K_0^*(1430) \), \( IJ = 1/2 \)

\[
M_{K_0^*} = 1428^{+56}_{-23} \text{ MeV}, \quad \Gamma_{K_0^*}/2 = 87^{+53}_{-28} \text{ MeV},
\]

\[
|g_{K_0^* K\pi}| = 3.3^{+0.5}_{-0.4} \text{ GeV}
\]

\[
|g_{K_0^* K\eta}|/|g_{K_0^* K\pi}| = 0.54^{+0.07}_{-0.02} , \quad |g_{K_0^* K\eta'}|/|g_{K_0^* K\pi}| = 1.2^{+0.2}_{-0.3}
\]
- $a_0(980)$, $IJ = 10$

\[ M_{a_0} = 1012^{+25}_{-7} \text{ MeV}, \quad \Gamma_{a_0}/2 = 16^{+50}_{-13} \text{ MeV}, \]
\[ |g_{a_0 \pi \eta}| = 2.5^{+1.3}_{-0.8} \text{ GeV} \]
\[ |g_{a_0 K \bar{K}}|/|g_{a_0 \pi \eta}| = 1.9^{+0.2}_{-0.3}, \quad |g_{a_0 \pi \eta'}|/|g_{a_0 \pi \eta}| = 0.01^{+0.03}_{-0.01} \]

- $a_0(1450)$, $IJ = 10$

\[ M_{a_0} = 1368^{+68}_{-68} \text{ MeV}, \quad \Gamma_{a_0}/2 = 71^{+48}_{-23} \text{ MeV}, \]
\[ |g_{a_0 \pi \eta}| = 2.3^{+0.4}_{-0.5} \text{ GeV} \]
\[ |g_{a_0 K \bar{K}}|/|g_{a_0 \pi \eta}| = 0.6^{+0.7}_{-0.2}, \quad |g_{a_0 \pi \eta'}|/|g_{a_0 \pi \eta}| = 0.6^{+0.2}_{-0.1} \]
\[ \rho(770) \quad IJ = 11 \]

\[ M_\rho = 762^{+4}_{-4} \text{MeV} \quad \Gamma_\rho/2 = 72^{+2}_{-2} \text{MeV} \]

\[ |g_\rho \pi \pi| = 2.48^{+0.03}_{-0.05} \text{GeV} \quad |g_\rho K \bar{K}|/|g_\rho \pi \pi| = 0.64^{+0.01}_{-0.01} \]

\[ K^*(892) \quad IJ = 1/2 1 \]

\[ M_{K^*} = 891^{+3}_{-4} \text{MeV} \quad \Gamma_{K^*}/2 = 25^{+2}_{-1} \text{MeV} \]

\[ |g_{K^* \pi K}| = 1.86^{+0.05}_{-0.05} \text{GeV} \]

\[ |g_{K^* K \eta}|/|g_{K^* K \pi}| = 0.91^{+0.03}_{-0.02} \quad |g_{K^* K \eta'}|/|g_{K^* K \pi}| = 0.45^{+0.08}_{-0.08} \]

\[ \phi(1020) \quad IJ = 0 1 \]

\[ M_\phi = 1019.5^{+0.3}_{-0.3} \text{MeV} \quad \Gamma_\phi/2 = 2.00^{+0.04}_{-0.08} \text{MeV} \]

\[ |g_\phi K \bar{K}| = 0.85^{+0.01}_{-0.02} \text{GeV} \]
Running of pole positions with $N_C$

For the first time the $N_C$ dependence of the pseudo-Goldstone masses and mixing angle are taken into account for determining resonance properties with increasing $N_C$.

In $SU(3)\chi$PT, there is one mixing ingredient for the large $N_C$ limit: the singlet $\eta_1$.

The leading order behaviours of the parameters at large $N_C$ are

\[
M_0^2 \sim \Lambda_2 \sim 1/N_c \\
c_d \sim c_m \sim \tilde{c}_d \sim \tilde{c}_m \sim G_V \sim F \sim \sqrt{N_c} \\
M_V^2 \sim M_S^2 \sim M_{S_1}^2 \sim B \sim a_{SL} \sim O(N_c^0)
\]

with $\bar{m}_\pi^2 = 2Bm_u, \bar{m}_K^2 = B(m_u + m_s)$.

[Ecker, et al., NPB’89] [Kaiser,Leutwyler, EPJC’00]
The next-to-leading order of $1/N_C$ running can be read out from our prediction for $F_\pi$

$$F_\pi = F \left\{ 1 + \frac{1}{16\pi^2 F_\pi^2} \left[ A_0(m_\pi^2) + \frac{1}{2} A_0(m_K^2) \right] \right.$$ 

$$+ \left[ \frac{4\tilde{c}_d \tilde{c}_m(m_\pi^2 + 2m_K^2)}{F_\pi^2 M_{S_1}^2} - \frac{8c_d c_m (m_K^2 - m_\pi^2)}{3F_\pi^2 M_{S_8}^2} \right] \right\}.$$

In addition we also take the following assumptions for the next-to-leading order of $1/N_C$ pieces for the other resonance couplings

$$c_d(N_C) = c_d(N_C = 3) \frac{F_\pi(N_C)}{F_\pi(N_C = 3)},$$

similar expressions also apply for $c_m, \tilde{c}_d, \tilde{c}_m, G_V$ due to the high energy constraint from QCD

$$c_d = c_m = \sqrt{3}\tilde{c}_d = \sqrt{3}\tilde{c}_m = \frac{F_\pi}{2}, \quad G_V = \frac{F_\pi}{\sqrt{2}} \text{ or } \frac{F_\pi}{\sqrt{3}}.$$

[Ecker, et al., PLB’89] [Jamin, et al., NPB’00] [Guo, et al., JHEP’07]
Pseudoscalar masses with varying $N_C$

Leading order $1/N_c \to \infty$ prediction ($M_0 \to 0$):

$$m_\eta^2 = \overline{m}_\pi^2 = (139.5^{+4.4}_{-4.6})^2 \text{ MeV}^2, \quad m_{\eta'}^2 = 2\overline{m}_K^2 - \overline{m}_\pi^2 = (721.5^{+17.4}_{-11.1})^2 \text{ MeV}^2.$$
Ideal Mixing (OZI rule is exact): leading order mixing angle $\theta = -54.7^\circ$
Two approximations of our full results are studied for the resonance poles

- **vector reduced**:

  \[
  \frac{1}{M_V^2 - t} \rightarrow \frac{1}{M_V^2},
  \]

  We only includes the NLO local terms in \( \chi \)PT in this scheme.

- **Mimic SU(3)**: Mixing is set to zero and \( \eta_1 \) is kept in the loops. \( \pi, K, \eta_8, \eta_1 \) masses are frozen. Differences highlight the role of \( \eta \) and \( \eta' \).
The results from one-loop inverse amplitude (IAM) are quite similar with the vector reduced case. [Pelaez, ’04][Sun, et al. ’07][Ruiz-Arriolla, Nieves, ’09]

Two-loop ($SU(2)$) IAM shows a quite different picture: $\sigma$ moves to a pole with zero width at 1 GeV. [Pelaez, Rios, ’06][Sun, et al. ’07] We also obtain such a pole but it comes from the bare scalar singlet $M_{S_1} \simeq 1$ GeV (At $N_C = 3$ it contributes to the $f_0(980).$)
A short summary of our finding for $\sigma$:

- The one-loop IAM study reflects a specific approximation of our full result: *vector reduced*. Whereas the *scalar reduced* approximation perfectly agrees with the full result.

- The *mimic SU(3)* approximation turns out to be quite similar to the full result of the $\sigma$ trajectory, indicating $\sigma$ is insensitive to $\eta$ and $\eta'$ even for large $N_C$.

- The possible source of the disagreement of our result and the two-loop IAM is the higher order local terms, because much more resonance operators will be involved to produce the $\mathcal{O}(p^6)$ LECs.

[Cirigliano, et al., NPB'06]
Figure: $N_C$ trajectory for $a_0(980)$
Figure: $N_C$ trajectory for $\kappa$
Resonances and their $N_C$ fates in $U(3)$ chiral perturbation theory
Figure: $N_C$ running of the residues for $K^*$
Conclusions

- A complete one-loop calculation of all meson-meson scattering amplitudes within $U(3)$ $\chi$PT has been worked out for the first time in literature.
- A variant N/D method has been employed to resum the $s$-channel loops. Various resonance poles in the complex plane and their residues have been calculated.
- $N_C$ dependence of the resonance pole positions and the residuals, are studied, also for the first time in literature, by taking into account the $N_C$ running of the pseudo-Goldstone masses and the $\eta - \eta'$ mixing angle.

Danke !
\[
\bar{\eta} = \cos \theta \eta_8 - \sin \theta \eta_1,
\]
\[
\bar{\eta}' = \sin \theta \eta_8 + \cos \theta \eta_1,
\]
\[
m_{\bar{\eta}}^2 = \frac{M_0^2}{2} + m_K^2 - \frac{\sqrt{M_0^4 - \frac{4M_0^2\Delta^2}{3} + 4\Delta^2}}{2},
\]
\[
m_{\bar{\eta}'}^2 = \frac{M_0^2}{2} + m_K^2 + \frac{\sqrt{M_0^4 - \frac{4M_0^2\Delta^2}{3} + 4\Delta^2}}{2},
\]
\[
\sin \theta = -1/\sqrt{1 + (3M_0^2 - 2\Delta^2 + \sqrt{9M_0^4 - 12M_0^2\Delta^2 + 36\Delta^4})^2/32\Delta^4}
\]
\[
\Delta^2 = m_K^2 - m_\pi^2, \quad \sin \theta \rightarrow 0 \text{ for } \Delta^2 \rightarrow 0, \text{ i.e. in } SU(3) \text{ limit.}
\]
The NLO $\bar{\eta}$-$\bar{\eta}'$ mixing can be treated perturbatively

$$
\mathcal{L} = \frac{1 + \delta_{\bar{\eta}}}{2} \partial_\mu \bar{\eta} \partial^\mu \bar{\eta} + \frac{1 + \delta_{\bar{\eta}'}}{2} \partial_\mu \bar{\eta}' \partial^\mu \bar{\eta}' + \delta_k \partial_\mu \bar{\eta} \partial^\mu \bar{\eta}'
- \frac{m^2_{\bar{\eta}} + \delta m^2_{\bar{\eta}}}{2} \bar{\eta} \bar{\eta} - \frac{m^2_{\bar{\eta}'} + \delta m^2_{\bar{\eta}'} }{2} \bar{\eta}' \bar{\eta}' - \delta m^2 \bar{\eta} \bar{\eta}'.
$$

$$
\begin{pmatrix}
\eta \\
\eta'
\end{pmatrix} =
\begin{pmatrix}
\cos \theta_\delta & -\sin \theta_\delta \\
\sin \theta_\delta & \cos \theta_\delta
\end{pmatrix}
\begin{pmatrix}
1 + \frac{\delta_{\bar{\eta}}}{2} & \frac{\delta_k}{2} \\
\frac{\delta_k}{2} & 1 + \frac{\delta_{\bar{\eta}'} }{2}
\end{pmatrix}
\begin{pmatrix}
\bar{\eta} \\
\bar{\eta}'
\end{pmatrix}.
$$
Observables fitted:

- $I = J = 0$: $\delta^{00}_{\pi\pi\rightarrow\pi\pi}$, $|S^{00}_{\pi\pi\rightarrow\pi\pi}|$, $\frac{1}{2} |S^{00}_{\pi\pi\rightarrow K\bar{K}}|$, $\delta^{00}_{\pi\pi\rightarrow K\bar{K}}$

- $I = J = 1$: $\delta^{11}_{\pi\pi\rightarrow\pi\pi}$

- $I = 1/2 J = 0, 1$: $\delta^{1/2 0}_{\pi K\rightarrow\pi K}$, $\delta^{1/2 1}_{\pi K\rightarrow\pi K}$

- $I = 2 J = 0$: $\delta^{20}_{\pi\pi\rightarrow\pi\pi}$

- $I = 3/2 J = 0$: $\delta^{3/2 0}_{\pi\pi\rightarrow\pi\pi}$

- $I = 1 J = 0$: $\pi\eta$ event distribution around $a_0(980)$

\[
\frac{dN_{\pi\eta}}{dE_{\pi\eta}} = q_{\pi\eta} N \left| T_{K\bar{K}\rightarrow\pi\eta}(s) + c T_{\pi\eta\rightarrow\pi\eta}(s) \right|^2.
\]

- $m_\eta$, $m_\eta'$
Subtraction Constants: The number of free ones can be reduced enormously by applying Isospin and U(3) symmetry. Jido, Oller, Oset, Ramos, Meißner, NPA’03

- Isospin Symmetry requires that all the $a_{SL}^{IJ}$ are the same separately for $\pi\pi$, $K\bar{K}$ and $K\pi$

- U(3) Symmetry requires that all $a_{SL}^{IJ}$ are the same for a given $J$

\[
\begin{align*}
a_{SL}^{00} &= a_{SL}^{00, \pi\pi} = a_{SL}^{00, K\bar{K}} = a_{SL}^{00, \eta\eta} = a_{SL}^{00, \eta'\eta'} = a_{SL}^{20, \pi\pi} \\
&= a_{SL}^{10, \pi\eta'} = a_{SL}^{10, K\bar{K}}, \\
&= a_{SL}^{1\frac{1}{2}0, K\pi} = a_{SL}^{1\frac{1}{2}0, K\eta} = a_{SL}^{1\frac{1}{2}0, K\eta'} = a_{SL}^{3\frac{3}{2}0, K\pi} \\
&= a_{SL}^{10, \pi\eta}
\end{align*}
\]

All the subtraction constants in the vector channels are set equal to $a_{SL}^{00}$ (play a little role).
We have 16 free parameters with 348 data and the fitted results are

\[ c_d = (15.6^{+4.2}_{-3.4}) \text{ MeV}, \]
\[ \tilde{c}_d = (8.7^{+2.5}_{-1.7}) \text{ MeV}, \]
\[ M_{S_8} = (1370^{+132}_{-57}) \text{ MeV}, \]
\[ M_\rho = (801.0^{+7.0}_{-7.5}) \text{ MeV}, \]
\[ G_V = (61.9^{+1.9}_{-1.9}) \text{ MeV}, \]
\[ a_{SL}^{00} = (-1.15^{+0.07}_{-0.09}), \]
\[ a_{SL}^{10} = (0.6^{+0.3}_{-0.3}) \text{ MeV}^{-2}, \]
\[ M_0 = (954^{+102}_{-95}) \text{ MeV}, \]
\[ c_m = (31.5^{+19.5}_{-22.5}) \text{ MeV}, \]
\[ \tilde{c}_m = (15.8^{+3.3}_{-3.0}) \text{ MeV}, \]
\[ M_{S_1} = (1063^{+53}_{-31}) \text{ MeV}, \]
\[ M_{K^*} = (909.0^{+7.5}_{-6.9}) \text{ MeV}, \]
\[ a_{SL}^{10,\pi\eta} = 2.0^{+3.1}_{-3.4}, \]
\[ a_{SL}^{1/2,0} = (-0.96^{+0.10}_{-0.16}), \]
\[ c = (1.0^{+0.6}_{-0.4}), \]
\[ \Lambda_2 = (-0.6^{+0.5}_{-0.4}), \]

with \( \chi^2/\text{d.o.f} = 714/(348 - 16) \approx 2.15. \)
\[ n_\sigma = \Delta \chi^2 / \sqrt{2\chi^2} \leq 2 \] to get the errors, \( n_\sigma = 2 \) Etkin \textit{et al.} PRD'82
Another strategy to perform the fit

The number of parameters can be reduced by imposing the following constraints [Ecker, Gasser, Pich, de Rafael, NPB’88]

\[ \tilde{c}_d = \frac{c_d}{\sqrt{3}}, \quad \tilde{c}_m = \frac{c_m}{\sqrt{3}}, \]  

and some of the parameters can be taken from other works:
- \( M_{S_1} = 1020 \text{ MeV}, \ M_{S_8} = 1390 \text{ MeV} \) [Oller, Oset, PRD’99];
- \( M_0 = 850 \text{ MeV} \) from [Feldmann, IJMPLA’00];
- \( G_V = 60.0 \text{ MeV} \), average value from [Ecker, Gasser, Pich, de Rafael, PLB’89]
  [Guo, Sanz-Cillero, Zheng, JHEP’07]
  [Guo, Sanz-Cillero, PRD’09].
We have 10 free parameters with 348 data now and the fitted results are

\[
\begin{align*}
    c_d &= 17.4 \text{ MeV}, \\
    M_\rho &= 800.4 \text{ MeV}, \\
    a_{SL}^{00} &= -1.14, \\
    \Lambda_2 &= -0.22, \\
    N &= 0.55 \text{ MeV}^{-2}, \\
    c_m &= 28.1 \text{ MeV}, \\
    M_{K^*} &= 910.0 \text{ MeV}, \\
    a_{SL}^{10} &= -0.89, \\
    a_{SL}^{10,\pi\eta} &= 2.0, \\
    c &= 0.84,
\end{align*}
\]

with \( \chi^2 / \text{d.o.f} = 842 / (348 - 10) \approx 2.5 \).