Roy–Steiner equations for $\gamma\gamma \to \pi\pi$

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Outline

1. Roy equations for $\pi\pi$ scattering
2. Roy–Steiner equations for $\gamma\gamma \to \pi\pi$
3. Muskhelishvili–Omnès solution for $\gamma\gamma \to \pi\pi$
4. Results

Motivation

Roy equations $=$ coupled system of partial wave dispersion relations

$+$ **crossing symmetry** $+$ **unitarity**

- Roy equations respect **analyticity**, **unitarity**, and **crossing symmetry**
- Partial wave dispersion relations in combination with unitarity (and chiral symmetry) allow for **high-precision** studies of low-energy processes
  - $\pi\pi$ scattering: Roy (1971), Ananthanarayan et al. (2001), García-Martín et al. (2011)
  - πK scattering: Büttiker et al. (2004)

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  - $\pi\pi$ scattering: Roy (1971), Ananthanarayan et al. (2001), García-Martín et al. (2011)
  - πK scattering: Büttiker et al. (2004)
- Application: determination of the pole position of the **σ-meson**
- $\pi\pi$ Roy equations + Chiral Perturbation Theory (ChPT) Caprini et al. (2006)

$$M_\sigma = 441^{+16}_{-8}\,\text{MeV} \qquad \Gamma_\sigma = 544^{+18}_{-25}\,\text{MeV}$$

- $\gamma\gamma \to \pi\pi$ provides alternative access to the σ $\Rightarrow$ **two-photon width $\Gamma_{\sigma\gamma\gamma}$**
- Aim: constrain $\Gamma_{\sigma\gamma\gamma}$ at a similar level of rigor as M_σ and Γ_σ

Roy equations for $\pi\pi$ scattering

- Start from twice-subtracted dispersion relation at fixed Mandelstam t

$$T(s,t) = c(t) + \frac{1}{\pi}\int\limits_{4M_\pi^2}^{\infty} ds' \left\{ \frac{s^2}{s'^2(s'-s)} + \frac{u^2}{s'^2(s'-u)} \right\} \operatorname{Im} T(s',t)$$

- Determine subtraction functions $c(t)$ from **crossing symmetry**
- Partial wave projection (angular momentum J and isospin I)

$\Rightarrow$ **coupled system of integral equations** for partial waves $t_J^I(s)$

$$t_J^I(s) = k_J^I(s) + \sum_{I'=0}^{2}\sum_{J'=0}^{\infty}\int\limits_{4M_\pi^2}^{\infty} ds' K_{JJ'}^{II'}(s,s') \operatorname{Im} t_{J'}^{I'}(s')$$

- Kernel functions $K_{JJ'}^{II'}$ known analytically

$$K_{JJ'}^{II'}(s,s') = \frac{\delta_{JJ'}\delta_{II'}}{s'-s-i\varepsilon} + \bar{K}_{JJ'}^{II'}(s,s')$$

Roy equations for $\pi\pi$ scattering

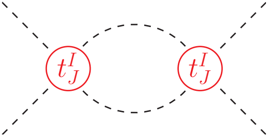
$$t_J^I(s) = k_J^I(s) + \sum_{I'=0}^{2} \sum_{J'=0}^{\infty} \int_{4M_\pi^2}^{\infty} \mathrm{d}s' K_{JJ'}^{II'}(s,s') \,\mathrm{Im}\, t_{J'}^{I'}(s')$$

- Free parameters: $\pi\pi$ scattering lengths in $k_J^I(s)$ ("subtraction constants")

  $\Rightarrow$ Matching to ChPT Colangelo et al. (2001)

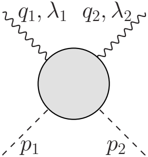
- Use **elastic unitarity** to obtain a coupled integral equation for the phase shifts

$$\mathrm{Im}\, t_J^I(s) = \sigma(s)\left|t_J^I(s)\right|^2$$



$$t_J^I(s) = \frac{e^{2i\delta_J^I(s)} - 1}{2i\sigma(s)} \qquad \sigma(s) = \sqrt{1 - \frac{4M_\pi^2}{s}}$$

Roy–Steiner equations for $\gamma\gamma \to \pi\pi$



- Kinematics: $s = (p_1 + q_1)^2$, $t = (q_1 - q_2)^2$, $u = (q_1 - p_2)^2$
- Amplitude for $\gamma\pi \to \gamma\pi$:

$$F_{\lambda_1\lambda_2}(s,t) = \varepsilon_\mu(q_1,\lambda_1)\varepsilon^*_\nu(q_2,\lambda_2)W^{\mu\nu}(s,t) \qquad \Delta_\mu = p_{1\mu} + p_{2\mu}$$

$$W_{\mu\nu}(s,t) = A(s,t)\left(\frac{t}{2}g_{\mu\nu} + q_{2\mu}q_{1\nu}\right) + B(s,t)\left(2t\Delta_\mu\Delta_\nu - (s-u)^2 g_{\mu\nu} + 2(s-u)(\Delta_\mu q_{1\nu} + \Delta_\nu q_{2\mu})\right)$$

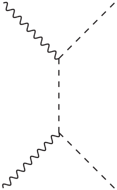
- Use dispersion relations for $A(s,t)$ and $B(s,t)$

  $\Rightarrow$ constraints from **gauge invariance** automatically fulfilled

- Crossing symmetry couples $\gamma\gamma \to \pi\pi$ and $\gamma\pi \to \gamma\pi$ $\qquad (s-a)(u-a) = (s'-a)(u'-a)$

  $\Rightarrow$ use **hyperbolic dispersion relations** Hite, Steiner (1973)

$$A(s,t) = \frac{1}{M_\pi^2 - s} + \frac{1}{M_\pi^2 - u} - \frac{1}{M_\pi^2 - a} + \frac{1}{\pi}\int\limits_{4M_\pi^2}^{\infty} \mathrm{d}t' \frac{\mathrm{Im}A(t',z_t')}{t'-t}$$
$$+ \frac{1}{\pi}\int\limits_{M_\pi^2}^{\infty} \mathrm{d}s' \,\mathrm{Im}A(s',t')\left(\frac{1}{s'-s} + \frac{1}{s'-u} - \frac{1}{s'-a}\right)$$



- Coupled system for $\gamma\gamma \to \pi\pi$ partial waves $h^I_{J,\pm}(t)$ and $\gamma\pi \to \gamma\pi$ partial waves $f^I_{J,\pm}(s)$ (photon helicities $\pm$), e.g.

$$h^I_{J,-}(t) = \tilde{N}^-_J(t) + \frac{1}{\pi}\int\limits_{M_\pi^2}^{\infty} ds' \sum_{J'=1}^{\infty} \tilde{G}^{-+}_{JJ'}(t,s')\,\text{Im}\, f^I_{J',+}(s') + \frac{1}{\pi}\int\limits_{4M_\pi^2}^{\infty} dt' \sum_{J'} \tilde{K}^{--}_{JJ'}(t,t')\,\text{Im}\, h^I_{J',-}(t')$$

- Subtraction constants $\Leftrightarrow$ **pion polarizabilities**

$$\pm\frac{2\alpha}{M_\pi t}\hat{F}_{+\pm}(s=M_\pi^2,t) = \alpha_1 \pm \beta_1 + \frac{t}{12}(\alpha_2 \pm \beta_2) + \mathcal{O}(t^2)$$

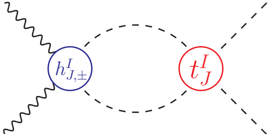
- Transition between isospin and particle basis

$$\begin{pmatrix} h^{\pi^\pm}_{J,\pm} \\ h^{\pi^0}_{J,\pm} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \end{pmatrix} \begin{pmatrix} h^0_{J,\pm} \\ h^2_{J,\pm} \end{pmatrix} \qquad \text{etc.}$$

$$h^I_{J,-}(t)=\tilde{N}^-_J(t)+\frac{1}{\pi}\int_{M_\pi^2}^{\infty}ds'\sum_{J'=1}^{\infty}\tilde{G}^{-+}_{JJ'}(t,s')\,\mathrm{Im}\,f^I_{J',+}(s')+\frac{1}{\pi}\int_{4M_\pi^2}^{\infty}dt'\sum_{J'}\tilde{K}^{--}_{JJ'}(t,t')\,\mathrm{Im}\,h^I_{J',-}(t')$$

- Unitarity relation is linear in $h^I_{J,\pm}(t)$

$$\mathrm{Im}\,h^I_{J,\pm}(t)=\sigma(t)h^I_{J,\pm}(t)t^I_J(t)^*$$



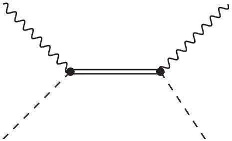
$\Rightarrow$ less restrictive than for $\pi\pi$ scattering

- **"Watson's theorem"**: phase of $h^I_{J,\pm}(t)$ equals $\delta^I_J(t)$ Watson (1954)

  $\Rightarrow$ Muskhelishvili–Omnès problem for $h^I_{J,\pm}(t)$ Muskhelishvili (1953), Omnès (1958)

- Equations are valid up to $t_{\max}=(1\,\mathrm{GeV})^2$ (assuming Mandelstam analyticity)

Muskhelishvili–Omnès solution for $\gamma\gamma \to \pi\pi$



- Truncate the system at $J = 2$
- Input for $\text{Im} f^I_{J,\pm}(s)$: approximate multi-pion states by sum of resonances García-Martín, Moussallam (2010)
- Assume $h^I_{J,\pm}(t)$ to be known above $t_\text{m} = (0.98\,\text{GeV})^2$

  $\Rightarrow$ **Muskhelishvili–Omnès problem with finite matching point** Büttiker et al. (2004)
- Solution in terms of Omnès functions, e.g. for $h^I_{0,+}(t)$ (one subtraction)

$$h^I_{0,+}(t) = \Delta^I_{0,+}(t) + \frac{M_\pi}{2\alpha}(\alpha_1 - \beta_1)^I t\Omega^I_0(t) + \frac{t^2\Omega^I_0(t)}{\pi}\left\{\int\limits_{4M_\pi^2}^{t_\text{m}} \text{d}t' \frac{\sin\delta^I_0(t')\Delta^I_{0,+}(t')}{t'^2(t'-t)|\Omega^I_0(t')|} + \int\limits_{t_\text{m}}^{\infty} \text{d}t' \frac{\text{Im}\, h^I_{0,+}(t')}{t'^2(t'-t)|\Omega^I_0(t')|}\right\}$$

with the Omnès function

$$\Omega^I_J(t) = \exp\left\{\frac{t}{\pi}\int\limits_{4M_\pi^2}^{t_\text{m}} \text{d}t' \frac{\delta^I_J(t')}{t'(t'-t)}\right\}$$

Muskhelishvili–Omnès solution for $\gamma\gamma \to \pi\pi$

$$h^I_{0,+}(t) = \Delta^I_{0,+}(t) + \frac{M_\pi}{2\alpha}(\alpha_1 - \beta_1)^I t\Omega^I_0(t) + \frac{t^2\Omega^I_0(t)}{\pi}\left\{\int_{4M_\pi^2}^{t_\mathrm{m}} \mathrm{d}t' \frac{\sin\delta^I_0(t')\Delta^I_{0,+}(t')}{t'^2(t'-t)|\Omega^I_0(t')|} + \int_{t_\mathrm{m}}^{\infty}\mathrm{d}t' \frac{\mathrm{Im}\, h^I_{0,+}(t')}{t'^2(t'-t)|\Omega^I_0(t')|}\right\}$$

- $\Delta^I_{0,+}(t)$ describes left-hand cut

$$\Delta^I_{0,+}(t) = N^I_{0,+}(t) + \frac{1}{\pi}\int_{4M_\pi^2}^{\infty}\mathrm{d}t'\Big(\tilde{K}^{++}_{02}(t,t')\mathrm{Im}\, h^I_{2,+}(t') + \tilde{K}^{+-}_{02}(t,t')\mathrm{Im}\, h^I_{2,-}(t')\Big)$$
$$+ \frac{1}{\pi}\int_{M_\pi^2}^{\infty}\mathrm{d}s' \sum_{J'=1,2}\Big(\tilde{G}^{++}_{0J'}(t,s')\mathrm{Im} f^I_{J',+}(s') + \tilde{G}^{+-}_{0J'}(t,s')\mathrm{Im} f^I_{J',-}(s')\Big)$$

Input

- Above t_m use Breit–Wigner description of $f_2(1270)$
- $\pi\pi$ phases: Caprini et al. (in preparation), García-Martín et al. (2011)

Muskhelishvili–Omnès solution for $\gamma\gamma \to \pi\pi$: sum rule for $I = 2$

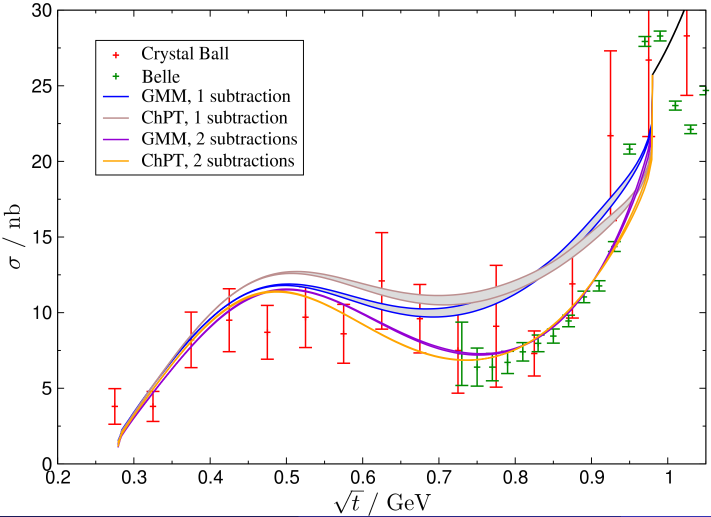
- If $\delta_J^I(t_\mathrm{m}) < 0$, can derive sum rules for pion polarizabilities, e.g.

$$0 = \frac{M_\pi}{2\alpha}(\alpha_1 - \beta_1)^{I=2} t_\mathrm{m}\big(1 - t_\mathrm{m}\,\dot{\Omega}_0^2(0)\big) + \frac{M_\pi}{24\alpha}(\alpha_2 - \beta_2)^{I=2} t_\mathrm{m}^2$$
$$+ \frac{t_\mathrm{m}^3}{\pi}\left\{\int\limits_{4M_\pi^2}^{t_\mathrm{m}} \mathrm{d}t' \frac{\sin\delta_0^2(t')\Delta_{0,+}^2(t')}{t'^3(t'-t_\mathrm{m})|\Omega_0^2(t')|} + \int\limits_{t_\mathrm{m}}^{\infty} \mathrm{d}t' \frac{\operatorname{Im} h_{0,+}^2(t')}{t'^3(t'-t_\mathrm{m})|\Omega_0^2(t')|}\right\}$$

- Gasser et al. (2006): $(\alpha_2 - \beta_2)^{\pi^\pm}$ strongly dependent on poorly known low-energy constants $\Rightarrow$ $(\alpha_2 - \beta_2)^{\pi^\pm} = 16.2[21.6] \cdot 10^{-4}\mathrm{fm}^5$ for two sets of LECs
- Sum rule + ChPT prediction for $(\alpha_1 - \beta_1)^{\pi^\pm,\pi^0}$ and $(\alpha_2 - \beta_2)^{\pi^0}$ Gasser et al. (2005, 2006) yields

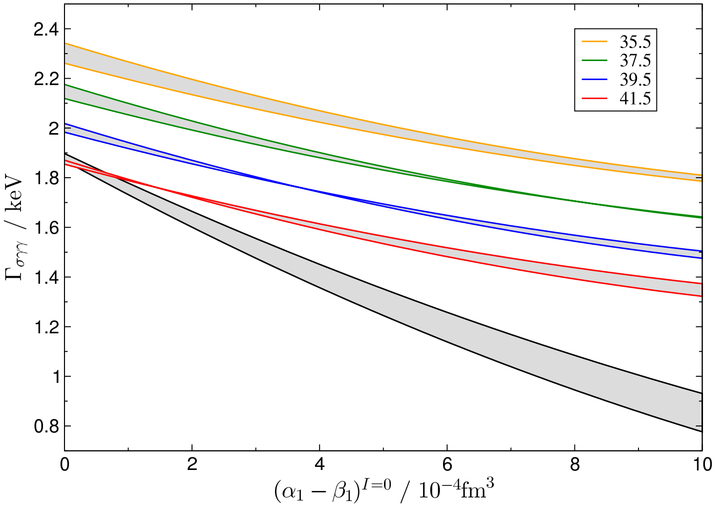
$$(\alpha_2 - \beta_2)^{\pi^\pm} = (15.3 \pm 3.7) \cdot 10^{-4}\mathrm{fm}^5$$

- Pion polarizabilities
  - ChPT: Gasser et al. (2005, 2006) + sum rule
  - GMM: two-channel Omnès fit to $\gamma\gamma \to \pi\pi$ data García-Martín, Moussallam (2010)



Results for $\Gamma_{\sigma\gamma\gamma}$: correlation plot

- Obtain $\Gamma_{\sigma\gamma\gamma}$ by analytic continuation to the σ pole

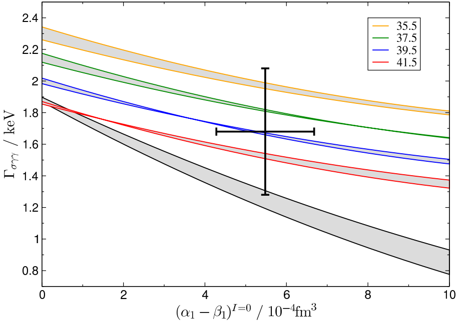


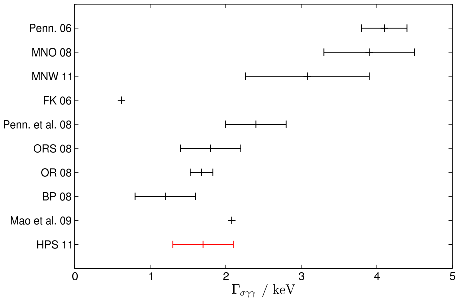
black: 1 subtraction

colored: 2 subtractions with $(\alpha_2-\beta_2)^{I=0}$ as indicated

$\Rightarrow$ **Correlation between $\Gamma_{\sigma\gamma\gamma}$ and pion polarizabilities**

- Combine correlation plot with ChPT predictions for pion polarizabilities





Roy–Steiner equations + ChPT

$$\Gamma_{\sigma\gamma\gamma} = (1.7 \pm 0.4)\,\text{keV}$$

Conclusions

- Construction of Roy–Steiner equations for $\gamma\gamma \to \pi\pi$
- Coupling between S- and D-waves
- Solution of Muskhelishvili–Omnès problem
- Sum rule to provide error estimate for chiral prediction of $(\alpha_2 - \beta_2)^{\pi^\pm}$
- Correlation between $\Gamma_{\sigma\gamma\gamma}$ and pion polarizabilities $\Rightarrow$ COMPASS

- Omnès function behaves as $|\Omega^I_J(t)| \sim |t_\mathrm{m}-t|^{\frac{\delta^I_J(t_\mathrm{m})}{\pi}} \Rightarrow$ If $\delta^I_J(t_\mathrm{m})<0$, $\Omega^I_J(t_\mathrm{m})^{-1}=0$
- Multiply

$$h^2_{0,+}(t)=\Delta^2_{0,+}(t)+\frac{M_\pi}{2\alpha}(\alpha_1-\beta_1)^{I=2}t\Omega^2_0(t)+\frac{t^2\Omega^2_0(t)}{\pi}\left\{\int\limits_{4M_\pi^2}^{t_\mathrm{m}}\mathrm{d}t'\frac{\sin\delta^2_0(t')\Delta^2_{0,+}(t')}{t'^2(t'-t)|\Omega^2_0(t')|}+\int\limits_{t_\mathrm{m}}^{\infty}\mathrm{d}t'\frac{\mathrm{Im}\,h^2_{0,+}(t')}{t'^2(t'-t)|\Omega^2_0(t')|}\right\}$$

with $\Omega^2_0(t)^{-1}$ and then put $t=t_\mathrm{m}$

$$0=\frac{M_\pi}{2\alpha}(\alpha_1-\beta_1)^{I=2}t_\mathrm{m}+\frac{t_\mathrm{m}^2}{\pi}\left\{\int\limits_{4M_\pi^2}^{t_\mathrm{m}}\mathrm{d}t'\frac{\sin\delta^2_0(t')\Delta^2_{0,+}(t')}{t'^2(t'-t_\mathrm{m})|\Omega^2_0(t')|}+\int\limits_{t_\mathrm{m}}^{\infty}\mathrm{d}t'\frac{\mathrm{Im}\,h^2_{0,+}(t')}{t'^2(t'-t_\mathrm{m})|\Omega^2_0(t')|}\right\}$$

- Integrals and individual contributions

	full	$a\to\infty$	no resonances
$I^{(2)}$, CCL	3.45	3.58	2.08
$I^{(2)}$, GKPRY	3.40	3.53	2.03

	$(\alpha_1-\beta_1)^{I=2}$	$(\alpha_2-\beta_2)^{I=2}$	total
ChPT	1.03 ± 0.14	-4.29 ± 0.78	0.18 ± 0.85
GMM	0.80 ± 0.14	-3.49 ± 0.60	0.76 ± 0.68

Input above t_{m}

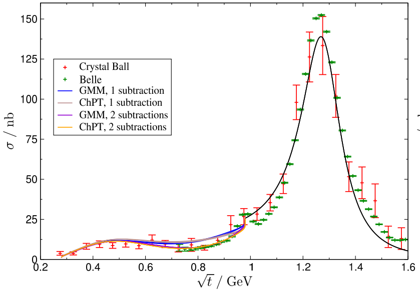
- Cross section above t_{m} dominated by $f_2(1270) \Rightarrow$ Breit–Wigner description

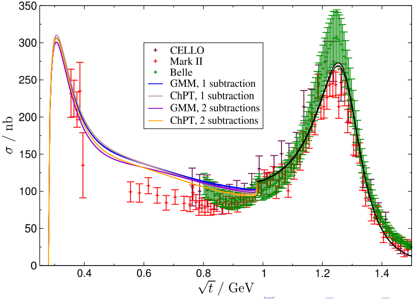
$$\mathscr{L}_{f_2\pi\pi} = C^{\pi}_{f_2} f_2^{\mu\nu} \partial_\mu \pi \partial_\nu \pi \qquad \mathscr{L}_{f_2\gamma\gamma} = e^2 C^{\gamma}_{f_2} f_2^{\mu\nu} F_{\mu\alpha} F_\nu{}^{\alpha}$$

- Amounts to putting all partial waves to zero except for $h^0_{2,-}(t)$

$$h^0_{2,-}(t) = \frac{C^{\pi}_{f_2} C^{\gamma}_{f_2}}{5\sqrt{6}} \frac{t^2 \sigma(t)}{t - m^2_{f_2} + i m_{f_2} \Gamma_{f_2}} = \frac{C^{\pi}_{f_2} C^{\gamma}_{f_2}}{5\sqrt{6}} \frac{m^4_{f_2} \sigma(m^2_{f_2})}{t - m^2_{f_2} + i m_{f_2} \Gamma_{f_2}} + \text{background}$$

- Need background for charged channel $\Rightarrow$ taking Born terms + background from f_2 works satisfactorily Drechsel et al. (1999)





Analytic continuation

- On the second Riemann sheet near the σ-pole t_σ we may write

$$h^0_{0,+,\text{II}}(t)=\frac{g_{\sigma\pi\pi}g_{\sigma\gamma\gamma}}{t_\sigma-t}\qquad 32\pi t^0_{0,\text{II}}(t)=\frac{g^2_{\sigma\pi\pi}}{t_\sigma-t}\qquad t_\sigma=\left(M_\sigma-i\frac{\Gamma_\sigma}{2}\right)^2$$

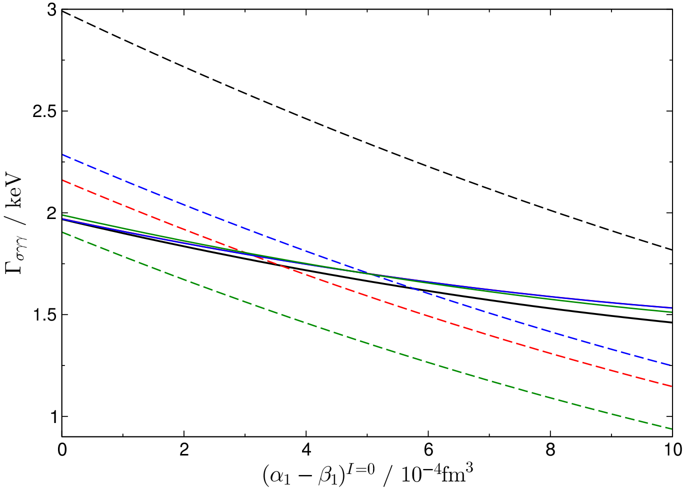
- Continuity at the cut relates amplitudes on the first and second Riemann sheet

$$h^0_{0,+,\text{II}}(t)=\left(1-2i\sigma(t)t^0_{0,\text{II}}(t)\right)h^0_{0,+,\text{I}}(t)$$

- Two-photon width $\Gamma_{\sigma\gamma\gamma}$ thus follows from

$$\frac{g^2_{\sigma\gamma\gamma}}{g^2_{\sigma\pi\pi}}=-\left(\frac{\sigma(t_\sigma)}{16\pi}\right)^2\left(h^0_{0,+,\text{I}}(t_\sigma)\right)^2\qquad \Gamma_{\sigma\gamma\gamma}=\frac{\pi\alpha^2|g_{\sigma\gamma\gamma}|^2}{M_\sigma}$$

S- and D-wave coupling



dashed: 1 subtraction

solid: 2 subtractions

black: Born terms only

red: + resonances for $a \to \infty$

blue: + terms for finite a

green: + D-waves