Precision calculation of the pion electromagnetic form factor from lattice QCD

In collaboration with Andreas Juettner (CERN) and Hartmut Wittig (Mainz)

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1. Introduction:

The electromagnetic vector form factor of the pion as a high precision lattice observable
Lattice QCD and experiment

- Recently lattice QCD has evolved in producing reliable results for a lot of physical quantities. Some of them collected in the reviews:


(See plenary talk by Andreas Jüttner tomorrow)

- Nevertheless: Some quantities fail to reproduce physical results! (e.g. baryonic form factors, etc.)

- In general this might be due to the fact that simulations still lack full control of systematic effects.

- The charge radius of the pion, connected with the form factor, is the easiest case of an observable where these systematic effects enters and is thus well suited for a high precision benchmark of new techniques.
Vector form factor of the pion in the space-like regime

- Known from experiment with high precision.
- Relatively easy to extract from lattice simulations.
- Connected to the pion charge radius:

\[
\langle r_{\pi}^2 \rangle = 6 \left. \frac{d f_\pi(q^2)}{dq^2} \right|_{q^2=0}
\]
Vector form factor of the pion in the space-like regime: Experiment

Space-like regime with low $q^2$: [NA7 collaboration, Nucl. Phys. B277 (1986)]

Charge radius: $\langle r_{\pi}^2 \rangle = 0.431(10)$ fm$^2$
Vector form factor of the pion in the space-like regime:

Lattice

Related to the matrix element of the vector current by:

$$\langle \pi^+(p_f) | V_\mu(q^2) | \pi^+(p_i) \rangle = (p_f + p_i)_\mu f_\pi(q^2)$$

- no quark-disconnected diagrams contribute
- noise reduction techniques can be applied efficiently
- in principle only space-like momenta are available due to euclidean signature of spacetime
  (recently extended also to time-like momenta)

$\Rightarrow$ Allows for a high precision simulation!
Vector form factor of the pion in the space-like regime:

Lattice – results

\[ f_{\pi \pi} - (q^*r_0)^2 \]

- PACS-CS \((N_f=2+1)\) \(m_\pi=411\) MeV
- PACS-CS \((N_f=2+1)\) \(m_\pi=296\) MeV
- ETMC \((N_f=2)\) physical point
- UKQCD \((N_f=2+1)\) \(m_\pi=330\) MeV

\(r_0\): Sommer scale \((\approx 0.5\) fm\) [Sommer (1994)]
2. Lattice regularisation:

Form factors from the lattice
The lattice

- Euclidean spacetime discretized on a 4d hypercubic lattice.
- Expectation values of observable (fermions integrated out):
  \[ \langle O \rangle = \frac{1}{Z} \int d[U] \, O[U] \prod_f [\text{det} (D_f[U])] \exp (-S[U]) \]

Is measured on stochastically generated representative ensembles of gauge field configurations

- Physical results via an extrapolation to the continuum.
- Cost of the simulation grows when lowering the quark mass.
  \[ \Rightarrow \quad \text{An extrapolation to the physical point is needed in most cases.} \]
- Problem: Momentum is usually introduced by Fourier transformation.
  \[ \Rightarrow \quad \text{Lower momentum cut-off due to finite volume!} \]
Fourier momenta

Example: $a = 0.07$ fm; $L = 32$ $a = 2.3$ fm

solid line: Minimal $q^2$ from Fourier momenta
Twisted boundary conditions

This problem can be cured by the use of partially twisted boundary conditions

[Bedaque (2004); Divitiis, Petronzio, Tantalo (2004)]

- Change of the boundary conditions of the quark fields leads to a shift in the quark momenta:

\[ p_i = \frac{2 \pi}{L} n_i + \frac{\theta_i}{L} \quad \text{\( \theta_i \) : twist angles} \]

- Suitable tuning allows for arbitrarily small (space-like) momentum transfers.

[Boyle et al (2007)]

- Additional finite volume effects are exponentially suppressed for matrix elements with at most one hadronic state in the initial and/or final state (like \( f_\pi, f_\pi \pi, \ldots \))

[Sachrajda, Villadoro (2005)]
3. Results:

Form factor and charge radius
CLS ensembles – Lattice parameters

We use the ensembles generated within the CLS framework (CLS: Coordinated Lattice Simulations)

[https://twiki.cern.ch/twiki/bin/view/CLS/WebHome]

Discretisation: $N_f = 2$; non-perturbatively $O(a)$-improved Wilson

Algorithm: deflation accelerated DD-HMC

[Lüscher (2004-2007)]

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$a[\text{fm}]$</th>
<th>lattice</th>
<th># masses</th>
<th>$m_\pi L$</th>
<th>Labels</th>
<th>Statistic</th>
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<td>5.20</td>
<td>0.08</td>
<td>$64 \times 32^3$</td>
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<td>6.0 – 4.0</td>
<td>A3 – A5</td>
<td>$O(100)$</td>
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<tr>
<td>5.30</td>
<td>0.07</td>
<td>$64 \times 32^3$</td>
<td>2</td>
<td>6.2, 4.7</td>
<td>E4, E5</td>
<td>$O(100)$</td>
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<tr>
<td>5.30</td>
<td>0.07</td>
<td>$96 \times 48^3$</td>
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<td>5.0</td>
<td>F6</td>
<td>233</td>
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<tr>
<td>5.50</td>
<td>0.05</td>
<td>$96 \times 48^3$</td>
<td>3</td>
<td>7.7 – 5.3</td>
<td>N3 – N5</td>
<td>$O(100)$</td>
</tr>
</tbody>
</table>
The pion form factor

Highest precision and lowest momenta ever attained!
The pion form factor
Results

The pion form factor

\[ f_{\pi \pi} \approx (q_0 r_0)^2 \]

- PACS-CS \((N_f=2+1)\) \(m_\pi = 411\) MeV
- PACS-CS \((N_f=2+1)\) \(m_\pi = 296\) MeV
- ETMC \((N_f=2)\) physical point
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\[ N_5, F_6, A_5 \]
The pion form factor

\[ f_{\pi\pi} = \frac{1}{(q^2)^{2N_f}} \]

<table>
<thead>
<tr>
<th>Experiment</th>
<th>( m_\pi ) (MeV)</th>
</tr>
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<tr>
<td>PACS-CS ( (N_f=2+1) )</td>
<td>411</td>
</tr>
<tr>
<td>PACS-CS ( (N_f=2+1) )</td>
<td>296</td>
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<tr>
<td>ETMC ( (N_f=2) )</td>
<td>Physical point</td>
</tr>
<tr>
<td>UKQCD ( (N_f=2+1) )</td>
<td>330</td>
</tr>
</tbody>
</table>

\(- (q^2 r_0^2)\) vs \( f_{\pi\pi} \)
Extraction of the charge radius

- High density of points close to $q^2 = 0$ allows the direct extraction of the slope using a linear fit.
  $\Rightarrow$ No scheme or model dependence!

- Can be checked against polynomial fits.

- Other crosschecks:
  - Vector pole dominance
  - $\chi$Pt to NLO and NNLO
Charge radius: $q^2$ dependence of linear fits

Solid line: $-(q r_0)^2 = 0.15$, maximum $q^2$ used for the linear fits
Charge radius: chiral behavior
Charge radius:
chiral behavior
NLO $\chi_{Pt}$:

**Formulae**

$\chi_{Pt}$ for $f_{\pi}(q^2)$:

\[
f_{\pi\pi}(q^2) = 1 + \frac{m_{\pi}^2}{f_{\pi}^2} \left[ \frac{1}{6} \left( \frac{q^2}{m_{\pi}^2} - 4 \right) \bar{J} \left( \frac{q^2}{m_{\pi}^2} \right) + \frac{q^2}{m_{\pi}^2} \left( -\ell_6^r - \frac{1}{6} L \left( \frac{m_{\pi}^2}{\mu^2} \right) - \frac{1}{288\pi^2} \right) \right]
\]

with $\ell_6^r = -\frac{1}{96\pi^2} \left[ \bar{\ell}_6 + 16\pi^2 L \left( \frac{m_{\pi}^2}{\mu^2} \right) \right]$

$\chi_{Pt}$ for $\langle r_{\pi}^2 \rangle$:

\[
\langle r_{\pi}^2 \rangle = \frac{m_{\pi}^2}{f_{\pi}^2} \left( -6\ell_6^r - L \left( \frac{m_{\pi}^2}{\mu^2} \right) - \frac{1}{16\pi^2} \right)
\]

- only free parameter: $\bar{\ell}_6$
- at the moment $f_{\pi}$ fixed to experimental value
- renormalisation scale: $\mu = m_{\rho}$
NLO $\chi$Pt:

$q^2$ dependence of $\bar{\ell}_6$

Solid line: $-(q r_0)^2 = 0.15$, maximum $q^2$ used for the fits
NLO $\chi$Pt:

pion mass dependence of $\ell_6$
4. Conclusions and outlook:
Conclusions

- State of the art lattice simulations of mesonic and baryonic form factors are still affected by systematic uncertainties.

- Improved simulation techniques are needed to compare lattice results and experiment to high precision to verify the conjectured agreement between the two.

- One interesting case is the charge radius of the pion that, despite other systematic effects, suffers from a model dependent extraction in almost all previous calculations.

- I presented our $N_f = 2$ study of the efficiency of novel techniques for the pion form factor and the extraction of the charge radius.

- Partially twisted boundary conditions are the main ingredient to reduce the model dependence in $\langle r_{\pi}^2 \rangle$, due to a large number of measurements at low $q^2$.

(here we can even do better as experiment)
Outlook

- The accuracy of our data is good enough for lattice artefacts to become visible (eventually) and a more systematic analysis is under way.

- $\chi^2_{\text{Pt}}$ to NLO does not work accurately as can be seen by the behavior of $\ell_6$ with $m^2_\pi$.
  
  $\Rightarrow$ To perform the extrapolation to the physical point we are going to utilise $\chi^2_{\text{Pt}}$ to NNLO.

- The chiral extrapolation will be assisted by additional measurements at smaller pion masses.

- In the end we aim at a model-independent result for the charge radius of the pion from first principles to compare to the experimental value.
Thank you for your attention!