Lorentz completion of effective string action.

Quark Confinement and the Hadron Spectrum X

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Wilson loop and effective string.

A Wilson loop along a closed path $\mathcal{O} \rightarrow$ order parameter for confinement in gauge theories:

$$\langle W(\mathcal{C}) \rangle = \langle \text{Tr } P \exp \left[ - \int_{\mathcal{O}} A_\mu(x) dx^\mu \right] \rangle.$$
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Leading order behavior at strong coupling (rectangular loop $R \times T$):

$$\langle W(R, T) \rangle \sim \left( \frac{1}{g^2} \right)^{RT} \sim e^{-\sigma RT} \quad \text{Area law}.$$ 

Higher orders: random surfaces with boundary on the loop.
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$\downarrow$

Roughening transition and Gaussian fixed point:

$$\langle W(R, T) \rangle \sim e^{-\sigma RT} \int [DX] \exp \left\{ - \frac{\sigma}{2} \int d^2 \xi \partial_a X^i \partial^a X^i \right\}.$$
Long strings beyond the free action.

- All irrelevant couplings allowed by symmetries should appear in the effective action.
- Fields and coordinates rescaling \( \Rightarrow \) Derivative expansion:
  \[
  \partial_a X^i \longrightarrow \frac{1}{\sqrt{\sigma} R} \partial_a X^i.
  \]
- A good guess for first derivative action is the Nambu-Goto action:
  \[
  S_{NG} = -\sigma \int d^2\sigma \sqrt{-\det(\partial_a X^\mu \partial_b X_\mu)}.
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**Spacetime spontaneously broken symmetries** provide constraints.

And shed light on possible higher derivative couplings, But are not sufficient to single out the Nambu-Goto action.
Nonlinear realization for the transverse fluctuations.

- Transverse excitations are Goldstone bosons for translation symmetry breaking,

\[ \delta b_j \epsilon X_i = \epsilon (-\delta_{ij} \xi b_j - X_j \partial b X_i) \]

1 Preserves number of derivatives minus number of fields (scaling);
2 Mixes order in the derivative expansion: recurrence relations.

ISO(1,1) and SO(D-2) invariance \Rightarrow Contraction of indices.

We have a recipe:
1 List all polynomials at lowest order at fixed scaling;
2 Build higher order terms with first derivatives;
3 Fix coefficients through the variation.

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Diagrammatic representation in $2 + 1$ dimensions.

We develop some simple rules:

- Every node is a field derivative, marked by its scaling;
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Translate the variation in the new language:

\[ \delta_{\epsilon}^{12} (\partial_2 X) = -\epsilon (\eta_{a_1} + \partial_a X \partial_{1} X + X \partial_a \partial_{1} X) \rightarrow \delta 0_a = -\eta_{a_1} - 0_a 0_{1} - X 1_{a_1}. \]
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\[
\partial_a \partial_b \partial_c X \partial^a X \partial^b \partial^c X \implies \quad \begin{array}{c}
0 \\
\bigcirc \\
2
\end{array} \\
1
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An invariant corresponds to every graph without first derivatives (scaling zero is the exception).
The case of one transverse direction.

- Independent recursion for each seed $\Rightarrow$ sum up once for all;
- Simple rules to list all seeds belonging to an invariant;
- Numerical factors are simple combinatorial factors in the graph picture.
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- Numerical factors are simple **combinatorial factors** in the graph picture.

For instance:

$$\alpha_1 \left( \partial a X \partial a \partial b X \partial b \partial c X \partial c \right) + \alpha_2 \left( \partial a X \partial a \partial b \partial c X \partial b \partial c \right) \partial d X \partial e X \partial d \partial e X \right)$$

$$= 2 \alpha_2 + 2 \cdot 2 \cdot 1 \frac{1!}{1!} \alpha_1 = 0.$$
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Unrestrained dimensions: scaling zero.

Two more rules:

1. A wavy link stands for a scalar product in the bulk;
2. A dot stands for the matrix of parameters of the transformation.

\[
\sum_{k=1}^{\infty} \left( -1 \right)^{k+1} \frac{1}{k} \left[ \left( \partial X \cdot \partial X \right)^k \right]_{ab} \delta_{ba} = \log \left\{ -\det \left( \eta + \partial X \cdot \partial X \right) \right\}.
\]

The third addend forces to add a new ring. We get a new series:

\[
L_0 = b_0 \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ \frac{1}{2} \log \left\{ -\det \left( \eta + h \right) \right\} \right\}_{n} - b_0 = b_0 \sqrt{-\det \left( \eta + h \right)} - b_0.
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\]
Three moves for higher derivative actions.

First move: substitute the inverse of the induced metric to every solid link\(^2\).

\[
\sum_{k=0}^{\infty} \left[ (\eta + h) - 1 \right]_{ab} = g_{ab}.
\]

Second move: substitute a new metric to every wavy link.

\[
\delta_{ij} \rightarrow t_{ij} = \delta_{ij} - \partial_a X_i g_{ab} \partial_b X_j.
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Third move: split each node according to the pattern of the variation. A seed without 0s becomes invariant under the whole Poincaré group.

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Higher derivative corrections to the DBI lagrangian.

Scaling two:

\[ \mathcal{L}_2^1 \propto \sqrt{-g} \left( \partial_{ab}^2 X_k \partial_{cd}^2 X_k g^{ac} g^{bd} - \partial_{ab}^2 X_k \partial_{cd}^2 X_i \partial_e X^k \partial_f X^i g^{ac} g^{bd} g^{ef} \right) \]

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Scaling four:
- Combinations: \( \mathcal{L}_2^\alpha \mathcal{L}_2^\beta / \sqrt{-g} \), \( \alpha, \beta = 1, 2 \). Among them: \( \mathcal{L} \propto \sqrt{-g} R^2 \).
- Combinations of other seeds generate also \( \mathcal{L} \propto \sqrt{-g} R_{ab} R^{ab} \).
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But More invariants than the geometrical ones.

One with a split vertex:
Non-local coupling and quantization constraints.

Aharony and Dodelson have noticed the importance of the coupling

\[ \mathcal{L}_{AD} = \sqrt{-g} R \frac{1}{\Box} R, \]

where the differential operator is defined such that

\[ g = \frac{1}{\Box} f \quad \Rightarrow \quad \Box g = f + \text{Nambu-Goto e.o.m.} \]
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- This scaling two term is invariant only on-shell, so we don’t find it;
- It is in fact a non-invariant counterterm necessary for the closure of quantum Lorentz algebra\(^3\).

Quantization seems to require more couplings than Lorentz invariant ones.

\(^3\text{Dubovsky et al. (2012), arXiv:1203.1054.}\)
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- First correction to the DBI action for a $p$-brane with $p > 1$ is the HE action.
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- The field strength for an $U(1)$ gauge boson transforms as the induced metric\(^4\): *straightforward generalization* to a photon propagating on a $p$-brane.

Further constraints from quantization?

Thanks for your attention.