Lattice QCD Study of Confinement and Chiral Symmetry Breaking with Dirac-mode Expansion

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Abstract: Using the QCD Dirac-mode expansion, we develop a manifestly gauge-covariant expansion and projection of the QCD operators such as the Wilson loop and the Polyakov loop. With this method, we perform a direct analysis of the correlation between confinement and chiral symmetry breaking in lattice QCD Monte Carlo calculations. Even after removing the low-lying Dirac modes, which are responsible to chiral symmetry breaking, we find that the Wilson loop obeys the area law, and the string tension or the confinement force is almost unchanged. We find also that the Polyakov loop remains to be almost zero even without the low-lying Dirac modes, which indicates the $Z_3$-unbroken confinement phase. These results indicate that one-to-one correspondence does not hold for between confinement and chiral symmetry breaking in QCD.

References:
The relation between Confinement and CSB is not yet known directly from QCD.

Color Confinement and Chiral Symmetry Breaking (CSB) are two of the most important phenomena of Nonperturbative QCD.
Correlation between Confinement and CSB is suggested by Simultaneous Phase Transition of Deconfinement and Chiral Restoration.

Lattice QCD results at finite temperature  

Fig. 2. Deconfinement and chiral symmetry restoration in 2-flavour QCD: Shown is $\langle L \rangle$ (left), which is the order parameter for deconfinement in the pure gauge limit ($m_q \to \infty$), and $\langle \bar{\psi}\psi \rangle$ (right), which is the order parameter for chiral symmetry breaking in the chiral limit ($m_q \to 0$). Also shown are the corresponding susceptibilities as a function of the coupling $\beta = 6/g^2$. 

Polyakov Loop $\langle P \rangle$  
Color Confinement

Chiral Condensate $\langle \bar{q}q \rangle$  
Chiral Symmetry Breaking
Also, similar Coincidence between Deconfinement and Chiral Restoration is found in Finite-Size lattice QCD. In fact, Simultaneous Phase Transitions occur according to the Box Size.

Of course, Finite-Temperature Phase transition is also a kind of Finite-Size effect of Euclidean Lattice in temporal direction.
The close relation between Confinement and CSB has been indicated in terms of Monopoles appearing in Maximally Abelian Gauge in QCD. By removing the Monopoles from the QCD vacuum, the confinement property and chiral symmetry breaking are simultaneously lost.

Important role of Monopole to Chiral Sym Breaking (Lattice QCD)

O. Miyamura, PLB (1995):
First Lattice QCD Study to reveal Important role of Monopoles to CSB

Quark Condensate plotted against $\beta$ in SU(2) QCD on $16^3 \times 4$ lattice

Monopole part (including only monopole): Chiral symmetric
Photon part (after removing monopoles): Chiral Symmetric

If Monopoles are removed from the QCD vacuum, No CSB occurs

In confinement phase, CSB occurs
Hodge Decomposition in Maximally Abelian Gauge

→ Monopole part (including only monopole): Linear Confinement potential
   Photon part (after removing monopole): Coulomb potential

SU(3) Lattice QCD result ($\beta=6.0$, $16^4$)

\[ V_{QQ}(r) \]

- Quark-antiquark potential
- Abelian part ($k_\mu, j_\mu$)
- Monopole part ($k_\mu$ only)
- Photon part ($j_\mu$ only)
Hodge Decomposition in Maximally Abelian Gauge

→ Monopole part (including only monopole): *Instantons* (almost same)

Photon part (after removing monopole): *No Instantons*

Figure 2. Correlations between (a) $Q(Ds)$ and $Q(SU(2))$ at 80 cooling sweeps, (b) $Q(Ph)$ and $Q(SU(2))$ at 10 cooling sweeps.

Fig. 3: Correlation between the total monopole-loop length $L$ and $I_Q$ (the total number of instantons and anti-instantons) in the MA gauge. We plot the data at 10 cooling sweep on $16^3 \times 4$ lattice with various $\beta$. 
Figure 5: An illustration of the relevant role of monopoles to nonperturbative QCD. In the maximally Abelian gauge, QCD becomes Abelian-like due to the large off-diagonal gluon mass of about 1GeV \([19]\), and there appears a global network of the monopole current \([17,18]\). By the Hodge decomposition, the QCD system can be divided into the monopole part and the photon part. The monopole part has confinement \([18]\), chiral symmetry breaking \([15]\) and instantons \([20]\), while the photon part does not have all of them.
Relation between Confinement and Chiral Symmetry Breaking

The lattice QCD studies indicate an important role of monopoles to both Confinement and CSB, and these two nonperturbative phenomena seem to be related through the monopole.

We would like to know the relation between Confinement and CSB in a more direct manner.

Stack-Neiman-Wensley, PRD (1994), ....

O. Miyamura, PLB (1995), R. Woloshyn, PRD (1995), ...

So, we investigate Confinement using the Dirac-mode expansion, because the essential modes for CSB are Low-lying Dirac modes.
Banks-Casher Relation

\[ \Sigma \equiv \left| \langle \bar{q} q \rangle \right| = \lim_{m \to 0} \lim_{V \to \infty} \pi \rho(0) \]

\[ \rho(\lambda) = \frac{1}{V} \left\langle \sum_{k} \delta(\lambda - \lambda_k) \right\rangle \quad \text{: QCD Dirac operator eigenvalue density} \]

Zero-eigenvalue density \( \rho(0) \) of Dirac operator gives Chiral Condensate.
\[ \Rightarrow \text{The essential modes for Chiral Sym Breaking are Low-lying Dirac modes.} \]

\[ \text{The non-zero spectrum is symmetric due to } \{ \gamma_5, \mathcal{D} \} = 0 \]

\[ \therefore \mathcal{D} \psi_n = i \lambda_n \psi_n \to \mathcal{D}(\gamma_5 \psi_n) = -i \lambda_n (\gamma_5 \psi_n) \]
**Eigen-mode of Dirac operator in Lattice QCD**

\[
\mathcal{D}_{xy}^{\text{lat}} = \frac{1}{2a} \sum_{\mu=1}^{4} \gamma^\mu [U_\mu(x) \delta_{y,x+\hat{\mu}} - U_{-\mu}(x) \delta_{y,x-\hat{\mu}}]
\]

: Lattice Dirac operator

\[
\mathcal{D}^{\text{lat}}[U]|n\rangle = i\lambda_n |n\rangle
\]

: Dirac eigen-value, Dirac eigen-state

\[
\sum_y \mathcal{D}_{xy}^{\text{lat}}[U] \psi_n(y) = i\lambda_n \psi_n(x)
\]

: Dirac eigen-function \(\psi_n(x)\)

**Explicit form of eigen-value equation in lattice QCD**

\[
\frac{1}{2a} \sum_{\mu=1}^{4} \gamma^\mu [U_\mu(x) \psi_n(x + \hat{\mu}) - U_{-\mu}(x) \psi_n(x - \hat{\mu})] = i\lambda_n \psi_n(x)
\]

**Gauge trans. property:**

\[
U_\mu(x) \rightarrow V(x) U_\mu(x) V^+(x + \hat{\mu})
\]

\[
\psi_n(x) \rightarrow V(x) \psi_n(x)
\]

same as quark field

apart from an irrelevant phase factor

\[
\langle m | n \rangle = \int d^4x \psi^*_m(x) \psi_n(x) = \delta_{mn}
\]

: normalization
To keep the gauge symmetry manifestly, we take the following “operator formalism”.

- **Link-variable operator** $\hat{U}_\mu$ is defined by the matrix element of
  \[
  \langle x | \hat{U}_\mu | y \rangle = U_\mu (x) \delta_{x+\hat{\mu},y}
  \]

- **Wilson Loop operator** $\hat{W}$ is defined as the product of $\hat{U}_\mu$ along a rectangular loop:
  \[
  \hat{W} \equiv \prod_{k=1}^{L} \hat{U}_{\mu_k} = \hat{U}_{\mu_1} \hat{U}_{\mu_2} \ldots \hat{U}_{\mu_L}
  \]
  For loops $\sum_{k=1}^{L} \mu_k = 0$
Functional Trace of Wilson Loop operator is proportional to ordinary vacuum expectation value of the Wilson loop

- Wilson Loop operator: $\hat{W} \equiv \prod_{k=1}^{L} \hat{U}_{\mu_k} = \hat{U}_{\mu_1} \hat{U}_{\mu_2} \ldots \hat{U}_{\mu_L}$

- Functional Trace of Wilson Loop operator:

$$\text{Tr} \hat{W} = \text{tr} \sum_{x} \langle x | \hat{W} | x \rangle = \text{tr} \sum_{x} \langle x | \hat{U}_{\mu_1} \hat{U}_{\mu_2} \ldots \hat{U}_{\mu_L} | x \rangle$$

$$= \text{tr} \sum_{x_1, x_2, \ldots, x_L} \langle x_1 | \hat{U}_{\mu_1} | x_2 \rangle \langle x_2 | \hat{U}_{\mu_2} | x_3 \rangle \langle x_3 | \hat{U}_{\mu_3} | x_4 \rangle \ldots \langle x_L | \hat{U}_{\mu_L} | x_1 \rangle$$

$$= \text{tr} \sum_{x} \langle x | \hat{U}_{\mu_1} | x + \mu_1 \rangle \langle x + \mu_1 | \hat{U}_{\mu_2} | x + \sum_{k=1}^{2} \mu_k \rangle \ldots \langle x + \sum_{k=1}^{L-1} \mu_k | \hat{U}_{\mu_L} | x \rangle$$

$$= \text{tr} \sum_{x} U_{\mu_1} (x) U_{\mu_2} (x + \mu_1) U_{\mu_3} (x + \sum_{k=1}^{2} \mu_k) \ldots U_{\mu_L} (x + \sum_{k=1}^{L-1} \mu_k)$$

$$= \langle W \rangle \cdot \text{Tr} 1$$

$\text{Tr}$ : functional trace $\quad \text{tr}$ : trace over SU(3) color index
Dirac-mode matrix elements of Link-variable operator:

\[ \langle m | \hat{U}_\mu | n \rangle = \sum_x \langle m | x \rangle \langle x | \hat{U}_\mu | x + \hat{\mu} \rangle \langle x + \hat{\mu} | n \rangle = \sum_x \psi_m^+(x) U_\mu(x) \psi_n(x + \hat{\mu}) \]

**Huge matrix elements: calculable & **Gauge Invariant**

Gauge transformation:

\[
\begin{align*}
U_\mu(x) &\rightarrow V(x) U_\mu(x) V^+(x + \hat{\mu}) \\
\psi_n(x) &\rightarrow V(x) \psi_n(x) \quad \text{(same as quark field)} \\
\end{align*}
\]

**Gauge invariance of the Dirac-mode matrix element** \( \langle m | \hat{U}_\mu | n \rangle \)

\[
\begin{align*}
\langle m | \hat{U}_\mu | n \rangle &= \sum_x \psi_m^+(x) U_\mu(x) \psi_n(x + \hat{\mu}) \\
&\rightarrow \sum_x \psi_m^+(x) V(x) \cdot V^+(x) U_\mu(x) V^+(x + \hat{\mu}) \cdot V^+(x + \hat{\mu}) \psi_n(x + \hat{\mu}) \\
&= \sum_x \psi_m^+(x) U_\mu(x) \psi_n(x + \hat{\mu}) = \langle m | \hat{U}_\mu | n \rangle
\end{align*}
\]

apart from an irrelevant phase factor
Dirac-mode Expansion and Projection

We just use completeness of the Dirac-mode basis:

\[ \sum_n |n\langle n| = 1 \]

\[ \hat{U}_\mu \equiv \sum_m \sum_n |m\langle m| \hat{U}_\mu |n\langle n| \]

Dirac-mode expansion

(This is just insertion of unity!)

In this expansion, Dirac spinor d.o.f. is introduced, and this is of course mathematically correct.

We define Projection operator which restricts the Dirac-mode space.

Projection operator

\[ \hat{P} \equiv \sum_{n \in A} |n\langle n| \]

\[ \hat{P}^2 = \hat{P}, \quad \hat{P}^+ = \hat{P} \]

In this projection, the Dirac-mode sum is done within a subset \( A \).

⇒ Projected Link-variable operator

\[ \hat{U}^P_\mu \equiv \hat{P} \hat{U}_\mu \hat{P} = \sum_{m \in A} \sum_{n \in A} |m\langle m| \hat{U}_\mu |n\langle n| \]
Wilson Loop:
\[ \text{Tr} \hat{W} \equiv \text{Tr} \prod_{k=1}^{L} \hat{U}_{\mu_k} = \text{Tr} \hat{U}_{\mu_1} \hat{U}_{\mu_2} \cdots \hat{U}_{\mu_L} \]

Dirac-mode projection:
\[ \hat{U}_{\mu} \rightarrow \hat{U}_{\mu}^P \equiv \hat{P} \hat{U}_{\mu} \hat{P} \]

Dirac-mode projected Wilson Loop:
\[ \text{Tr} \hat{W}^P \equiv \text{Tr} \prod_{k=1}^{L} \hat{U}_{\mu_k}^P = \text{Tr} \hat{U}_{\mu_1}^P \hat{U}_{\mu_2}^P \cdots \hat{U}_{\mu_L}^P = \text{Tr} \hat{P} \hat{U}_{\mu_1} \hat{P} \hat{U}_{\mu_2} \hat{P} \cdots \hat{P} \hat{U}_{\mu_L} \hat{P} \]
\[ = \sum \text{tr} \langle n_1 | \hat{U}_{\mu_1} | n_2 \rangle \langle n_2 | \hat{U}_{\mu_2} | n_3 \rangle \cdots \langle n_L | \hat{U}_{\mu_L} | n_1 \rangle \]

Gauge Invariant !

Its Gauge Invariance is also checked in lattice QCD calculation.
Dirac-mode projected Wilson Loop

\[ \text{Tr} \hat{W}^P = \text{Tr} \prod_{k=1}^{L} \hat{U}^P_{\mu_k} = \sum_{n_1, n_2, \ldots, n_L} \text{tr} \langle n_1 | \hat{U}_{\mu_1} | n_2 \rangle \langle n_2 | \hat{U}_{\mu_2} | n_3 \rangle \cdots \langle n_L | \hat{U}_{\mu_L} | n_1 \rangle \]

Based on this expression, we investigate the role of specific Dirac modes to the area law of the Wilson loop. In fact, if some Dirac modes are essential to reproduce the area law or the confinement property, the removal of the coupling to these modes leads to a significant change on the area law.

The original Wilson loop couples to all the Dirac modes. The projected Wilson loop couples to restricted Dirac modes.
Dirac-mode projected Inter-Quark Potential

Dirac-mode projected Wilson Loop

\[ \text{Tr} \hat{W}^P \equiv \text{Tr} \prod_{k=1}^L \hat{U}_{\mu_k}^P = \sum_{n_1, n_2, \ldots, n_L \in A} \text{tr} \left\langle n_1 | \hat{U}_{\mu_1} | n_2 \right\rangle \left\langle n_2 | \hat{U}_{\mu_2} | n_3 \right\rangle \cdots \left\langle n_L | \hat{U}_{\mu_L} | n_1 \right\rangle \]

\[ \Rightarrow \text{corresponding Potential} \]

\[ V^P(R) = -\lim_{T \to \infty} \frac{1}{T} \ln \left\{ \text{Tr} \hat{W}^P(R, T) \right\} \]

As a caution, some non-locality appears.

Unprojected case: ordinary inter-quark potential is obtained

\[ \text{cf Trace of Wilson Loop operator is proportional to ordinary vacuum expectation value of the Wilson loop} \]

\[ \text{Tr} \hat{W} = \langle W \rangle \cdot \text{Tr} 1 \]

\[ V(R) = -\lim_{T \to \infty} \frac{1}{T} \ln \left\{ \text{Tr} \hat{W}(R, T) \right\} = -\lim_{T \to \infty} \frac{1}{T} \ln \langle W(R, T) \rangle + \text{irrelevant constant} \]
As a technical difficulty of this formalism, we have to deal with huge dimensional matrix and their products.

Actually, for the matrix $\langle m | \hat{U}_{\mu_1} | n \rangle$, the total matrix dimension is $(\text{Dirac-mode number})^2$. Here, the Dirac-mode number is $(\text{lattice-volume}) \times N_c \times 4$. This number can be reduced to be $(\text{lattice-volume}) \times N_c$, using the Kogut-Susskind technique.

At present, we use a small-size lattice in this calculation.

Lattice Calculation Condition:
SU(3) plaquette action on quenched periodic lattice
$\beta=5.6$ (i.e., $a=0.25\text{fm}$), $6^4$
Eigen-value distribution of QCD Dirac operator

\[ \beta = 5.6 \ (a = 0.25\text{fm for lattice spacing}), \ 6^4 \text{ lattice} \]

Low-lying Dirac modes are responsible to Chiral Symmetry Breaking

(cf. Banks-Casher relation)
Eigen-value distribution of QCD Dirac operator

\[ \beta = 5.6 \ (a = 0.25\text{fm for lattice spacing}), \ 6^4 \text{ lattice} \]

By Removing the Low-lying Dirac modes, Chiral Condensate is Largely Reduced.

(cf. Banks-Casher relation)
Chiral Condensate after removing low-lying Dirac modes

\[ \langle \bar{q}q \rangle_{IR} \propto \sum_{\lambda_n \geq \Lambda_{IR}} \frac{2m}{\lambda_n^2 + m^2} \]

\[ \frac{\langle \bar{q}q \rangle_{IR}}{\langle \bar{q}q \rangle} \approx 0.02 \] for \( m_q \sim 5 \text{ MeV} \)

Chiral Condensate is largely reduced (only 2%!) after removing the low-lying Dirac modes.

FIG. 2: The lattice QCD result of the quark condensate \( \langle \bar{q}q \rangle_{\Lambda_{IR}} \) as the function of the current quark mass \( m \) in the presence of IR cut \( \Lambda_{IR} = 0.5, 1.0, 1.5[a^{-1}] \). The vertical axis is normalized by the original value of \( \langle \bar{q}q \rangle \) without cut. A large reduction is found as \( \langle \bar{q}q \rangle_{\Lambda_{IR}}/\langle \bar{q}q \rangle \approx 0.02 \) for \( \Lambda_{IR} = 0.5a^{-1} \sim 0.4\text{GeV} \) around the physical region of \( m \sim 0.006a^{-1} \sim 5\text{MeV} \).
Wilson Loop obeys Area law with the same slope (confining force), and inter-quark potential is almost the same beside an irrelevant const., even after removing the low-lying Dirac modes, which are responsible to chiral symmetry breaking.
Dirac-mode projected Polyakov Loop and \( Z_3 \) Center Symmetry

**Dirac-mode projected Polyakov Loop**

\[
\text{Tr} \hat{P}^P \equiv \text{Tr}(\hat{U}_4^P)^T = \sum_{n_1, n_2, \ldots, n_T} \text{tr} \langle n_1 | \hat{U}_4 | n_2 \rangle \langle n_2 | \hat{U}_4 | n_3 \rangle \cdots \langle n_T | \hat{U}_4 | n_1 \rangle
\]

**Polyakov Loop**

**Without IR-Dirac modes**

on periodic lattice

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**FIG. 6:** The scatter plot of the Polyakov loop. The left figure shows the original Polyakov loop \( \langle L_P \rangle \). The right figure shows the Polyakov loop \( \langle L_P \rangle_{IR} \) after cutting off the low-lying Dirac modes below the IR-cutoff \( \Lambda_{IR} = 0.5a^{-1} \).

Even after removing the low-lying Dirac modes, Polyakov loop remains to be zero, which means confinement phase and unbroken \( Z_3 \)-center symmetry.
UV-cut case of Dirac modes

Wilson Loop obeys the Area law with the same slope after removing the UV Dirac modes.

FIG. 7: (a) The UV-cut Dirac spectral density $\rho_{UV}(\lambda) \equiv \rho(\lambda)\theta(\Lambda_{UV} - |\lambda|)$ with the UV-cutoff $\Lambda_{UV} = 2a^{-1} \approx 1.6$GeV. (b) The UV-cut Wilson loop $\text{Tr}W^P(R, T)$ (circle) after removing the UV Dirac modes, plotted against $R \times T$. The slope parameter $\sigma^P$ is almost the same as that of the original Wilson loop (square). (c) The corresponding UV-cut inter-quark potential (circle), which is almost unchanged from the original one (square), apart from an irrelevant constant.
Intermediate-cut cases of Dirac modes

Wilson Loop obeys the Area law with the same slope after removing various Dirac modes.
Related Lattice Studies


\[
\langle P \rangle = \frac{1}{8V} \left( 2 \sum_{\lambda} \lambda^N_\lambda - (1 + i)\sum_{\lambda_+} \lambda^N_{\lambda_+} - (1 - i)\sum_{\lambda_-} \lambda^N_{\lambda_-} \right)
\]

Contribution from low-lying Dirac modes to Polyakov loop seems to be strongly suppressed. This seems to be consistent with our result.


They found that confining force is reproduced with low-lying Dirac modes.

Our comment: Their result seems to be consistent with our result on UV-cut case of Dirac modes.


They study Hadron Spectra after cutting off the low-lying Dirac modes.

Our comment: The hadron formation seems to indicate the existence of Confinement Force.
Our Conclusion

- No specific Dirac mode responsible to confinement seems to exist.
- In particular, low-lying Dirac modes would not be essential for confinement.
- We conjecture that the “seed” of confinement is distributed not only in low-lying Dirac modes but also in a wider region of the Dirac-mode space.

For more definite conclusion, large-volume calculation is needed.
**Naive Physical Interpretation:** Instanton and Dirac zero-mode

**Figure 8:** Around each instanton, the Dirac zero-mode is localized, and such low-lying Dirac modes contribute to chiral symmetry breaking. However, the localized objects are hard to contribute to confinement.

Recall that instantons contribute to chiral symmetry breaking, but do not directly lead to confinement [8]. Then, as a thought experiment, if only instantons can be carefully removed from the QCD vacuum, confinement properties would be almost unchanged, but the chiral condensate is largely reduced, and accordingly some low-lying Dirac modes disappear. Thus, in this case, confinement is almost unchanged, in spite of the large reduction of low-lying Dirac modes.
Summary and Concluding Remarks

With the Dirac-mode expansion, we have analyzed the relation between confinement and CSB in SU(3) lattice QCD.

Even after removing the low-lying Dirac modes, which are responsible to chiral symmetry breaking, Wilson loop obeys the Area law with the same slope parameter (confining force), and Polyakov loop remains to be zero, which means confinement phase and unbroken $Z_3$-center symmetry.

These indicate that one-to-one correspondence does not hold for between confinement and chiral symmetry breaking in QCD.
Thank You!
In the previous work, we studied IR/UV-Gluon Contribution to the Ground-State Potential or Confinement.


As a remarkable fact, the string tension is almost unchanged even after cutting off the high-momentum gluon component above 1.5GeV.
The previous method is based on the Fourier expansion.

The Fourier expansion is based on eigen-state of momentum operator. Because of the commutable nature of \([p^\mu, p^\nu] = 0\), all the momentum \(p^\mu\) can be simultaneously diagonalized. This is one of the strong merits of the Fourier expansion.

The Fourier expansion is very useful and keeps Lorentz covariance, but it does not keep gauge invariance in gauge theories.

Therefore, for the use of the Fourier expansion in QCD, one has to select a suitable gauge such as the Landau gauge, where the gauge-field fluctuation is strongly suppressed in Euclidean QCD.

Next, we consider \text{Gauge-Invariant Method}, using a gauge-invariant expansion in QCD instead of the Fourier expansion.
Gauge-invariant expansion in QCD

We consider a generalization of the Fourier expansion or an alternative expansion with keeping the gauge symmetry. A straight generalization is to use covariant derivative operator $D^\mu$ instead of derivative operator $\partial^\mu$. However, due to non-commutable nature of $[D^\mu, D^n] \neq 0$, we cannot diagonalize all the covariant derivative $D^\mu$ simultaneously, but only one of them can be diagonalized.

For example, the expansion by the eigen-state of $D_4$ keeps gauge covariance and is rather interesting, but this type of the expansion inevitably breaks Lorentz covariance.

Then, we consider the Dirac operator $\gamma^\mu D^\mu$ and $D^2 = D^\mu D^\mu$, since the expansion with the eigen-states keeps both gauge symmetry and Lorentz covariance.

In particular, the Dirac-mode expansion is rather interesting because it directly connects with Chiral Sym Breaking and Topological Charge.
We mainly consider manifestly Gauge-Invariant new method using Dirac-mode expansion to examine relevant modes for each QCD phenomenon. 

Here, Dirac operator $\mathcal{D} \equiv \gamma^\mu D_\mu$ is directly related to Chiral Symmetry Breaking, via Banks-Casher relation, and its zero modes are directly related to Topological charge, via Atiyah-Singer Index theorem.
Atiyah-Singer Index Theorem

\[ \text{Index}(D) = Q \]

\[ Q = \frac{g^2}{16\pi^2} \int d^4x \text{Tr} G_{\mu\nu} \tilde{G}^{\mu\nu} \in \mathbb{Z} \quad : \text{Topological charge in QCD} \]

\[ \text{Index}(D) \equiv \dim(\ker(D)) - \dim(\text{co ker}(D)) = \nu_R - \nu_L \quad : \text{index of } D \]

\[ \nu_{R,L} \quad : \text{Right- / Left- handed zero-mode number of } D \]

Zero-mode number asymmetry of Dirac operator
is equal to Topological charge (instanton number) in QCD

※ The non-zero spectrum is symmetric due to \( \{\gamma_5, D\} = 0 \)

\[ \therefore D \psi_n = i\lambda_n \psi_n \rightarrow D(\gamma_5 \psi_n) = -i\lambda_n (\gamma_5 \psi_n) \]
Important role of Monopole to Chiral Sym Breaking (Lattice QCD)

O. Miyamura, PLB (1995) : First Lattice QCD Study to reveal Important role of Monopole to Chiral Sym Breaking

Polyakov loop

Quark Condensate

\[ |\langle \text{Tr}G(0,0) \rangle| \]

Monopole part (including only monopole) : \textbf{Chiral \textit{sym} breaking}

Photon part (after removing monopole) : \textbf{Chial Symmetric}

Fig. 3. Polyakov loop in the SU(2) field (cross), in the U(1) field (open circle), its singular (filled circle) and regular (triangle) components on a $16^3 \times 4$ lattice.

Fig. 4. (a) $|\langle \text{Tr}G(0,0) \rangle|$ for $ma = 0.005$ in the SU(2) field (cross), in the U(1) field (open circle), its singular (filled circle) and regular (triangle) components on a $16^3 \times 4$ lattice. (b) Same for $ma = 0.01$. 