Hadronic contributions to the muon $g - 2$ in lattice QCD

Confinement 2012

TU München

10/2012

Andreas Jüttner
Motivation

- Leptons ($l = e, \mu, \tau$) exhibit a magnetic dipole moment

\[ \vec{\mu}_l = g_l Q \vec{\sigma} \]

- Dirac's (classical) prediction for the gyromagnetic factor: $g_l = 2$. 

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\[ a_l = \frac{g_l - 2}{2} \]
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- deviations from SM? BSM contributions can be sizeable in particular for heavy leptons: \(\delta a_l \propto \frac{m^2_l}{M^2}\)

\((m_e = 0.5\text{MeV}, m_\mu = 106\text{MeV}, m_\tau = 1777\text{MeV})\)
### $a_\mu$ experiment

<table>
<thead>
<tr>
<th>exp.</th>
<th>year</th>
<th>$a_\mu$ result</th>
</tr>
</thead>
<tbody>
<tr>
<td>CERN I</td>
<td>1961</td>
<td>11 450 000(220000)</td>
</tr>
<tr>
<td>CERN II</td>
<td>1962-1968</td>
<td>11 661 600(3100)</td>
</tr>
<tr>
<td>CERN III</td>
<td>1974-1976</td>
<td>11 659 100(110)</td>
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<td>1975-1976</td>
<td>11 659 360(120)</td>
</tr>
<tr>
<td>BNL</td>
<td>1997</td>
<td>11 659 251(150)</td>
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<td>1998</td>
<td>11 659 191(59)</td>
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<tr>
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<td>1999</td>
<td>11 659 202(15)</td>
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<td>11 659 204(9)</td>
</tr>
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<td>2001</td>
<td>11 659 214(9)</td>
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</table>
Experimental result is hopefully going to improve: new muon $g - 2$ experiment at FNAL aiming at a 4-fold improvement over current result.

http://gm2.fnal.gov

If all goes well first data-taking in 2016?
$a_{\mu}$ theoretically

$a_{\mu} \propto \gamma$

$\mu$
$a_\mu$ theoretically

\[ a_\mu \propto \begin{array}{c}
\mu \\
\gamma
\end{array} + \begin{array}{c}
\mu \\
\gamma
\end{array} \]
\( a_\mu \) theoretically

\[ a_\mu \propto \mu + \mu + \ldots + \mu + \ldots + \ldots \]

Hadronic contributions to the muon \( g - 2 \) in lattice QCD

Andreas Jüttner
\( a_\mu \) theoretically

\[ a_\mu \propto \mu + \gamma + \ldots + \gamma, \bar{\mu}, \bar{q} \ldots + \ldots \]

- QED and weak: perturbation theory
Theoretical expression for the anomalous magnetic moment $a_\mu$:

\[ a_\mu \propto \mu + \mu + \ldots + \mu, \bar{q}, \gamma + \ldots + \ldots \]

- QED and weak: perturbation theory
- the QCD coupling constant becomes too large for a reliable perturbative expansion at low energies → non-perturbative methods

(see also talks by C. Fischer, M. Ramsey-Musolf, T. Göcke)

Lattice computation of the $N_f=4$ Schrödinger Functional coupling by ALPHA arXiv:1011.2332
$a_\mu$ status

\[ a_\mu^{\text{exp}} = 1.16592080(63) \times 10^{-3} \]
\[ a_\mu^{\text{SM}} = 1.16591753(53) \times 10^{-3} \]

Jegerlehner, Nyffeler, PR 477 (2009)
the question as to whether the tension is an indication of new physics or not is causing quite a bit of excitement since it turns out to be persistent
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many different ways in which BSM can affect $g - 2$
  - new heavy states
  - extra dimensions
  - discriminate SUSY-scenarios
  - ...
large non-perturbative uncertainties

- hadronic uncertainties dominate the overall uncertainty of the SM prediction

<table>
<thead>
<tr>
<th>Source</th>
<th>$a_\mu /10^{-11}$</th>
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<tbody>
<tr>
<td>Jegerlehner, Nyffeler, PR 477 (2009)</td>
<td>116591753.7</td>
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- leading hadronic contribution
- light-by-light scattering

Hadronic contributions to the muon $g - 2$ in lattice QCD

Andreas Jüttner
Outline

- leading hadronic contribution
- light-by-light scattering
- outlook
Leading hadronic VP from experiment

- the leptonic VP can be computed in PT - for quark-VP PT breaks down at small energies
- current prediction for $a_{\mu}^{\text{LH}}$ from experimental measurement of $e^+e^-$-annihilation, $\tau$-decays
- independent prediction desireable
- a pure theory-prediction would provide a classical test of the SM
the leading hadronic contribution

- vacuum polarisation tensor for Euclidean momenta

\[ \Pi_{\mu\nu}(q) = \int d^4 x e^{iq(x-y)} \langle j_{\text{elm.}}^{\mu}(y) j_{\text{elm.}}^{\nu}(x) \rangle_{\text{QCD}} \]

\[ = (\delta_{\mu\nu} q^2 - q_\mu q_\nu) \Pi(q^2) \]
the leading hadronic contribution

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\Pi_{\mu\nu}(q) = \int d^4x e^{iq(x-y)} \langle j_{\mu}^{elm.}(y) j_{\nu}^{elm.}(x) \rangle_{QCD} = (\delta_{\mu\nu} q^2 - q_{\mu} q_{\nu}) \Pi(q^2)
\]

- \( a_{\mu}^{LH} \propto \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dq^2 f(q^2) \left(\Pi(q^2) - \Pi(0)\right) \)
Lattice QCD

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<tr>
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<td>$N_f$, fundamental $SU(2)$ iso-spin</td>
<td>1+1+1+1+1+1 broken</td>
<td>0, 2, 2+1, 2+1+1 not broken</td>
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<tr>
<td>$m_\pi$</td>
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<td>$m_\pi^{\text{sim}}$</td>
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<td>$V$</td>
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### QCD vs. Lattice QCD

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- $N_f = 2, 2 + 1$ and $2 + 1 + 1$ now standard
- There are now simulations for $m_{\pi}^{\text{sim}} = m_{\pi}^{\text{phys}}$
  - New generation of lattice phenomenology independent of chiral extrapolation
- QCD + EM and QCD with $m_u \neq m_d$ under way
  - To date mostly corrected for within EFT framework
A run through the Mainz-group simulations and problems encountered along the way

Della Morte et al. JHEP 1203 (2012) 055

In principle computing $\langle j^\text{elm.}_\mu j^\text{elm.}_\nu \rangle$ in lattice QCD is a well-defined problem...
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- compute $\Pi_{\mu\nu}(q)$ for several
  - $m_\pi$ ($(\text{pion mass})^2 \propto m_q$)
  - $L$ (lattice-volume)
  - $a$ (lattice spacing)
  - $q$ (Euclidean momenta)
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- structure of VP tensor doesn’t allow to compute $\Pi(0)$
- fit $q^2$-dependence of $\Pi(0)$ and extrapolate to $q^2 = 0$
- integrate fitted function $a_{\mu}^{LH} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dq^2 f(q^2)(\Pi(q^2) - \Pi(0))$
Contributions of quark-disconnected diagrams

\[
\left\langle j^{qq}_\mu(y) j^{qq}_\nu(x) \right\rangle = \left\langle \bar{q}_\mu q(y) \bar{q}_\nu q(x) \right\rangle = \left\langle \text{Tr}\left\{ S_q(y, x) \gamma_\mu S_q(x, y) \gamma_\nu \right\} \right\rangle + \left\langle \text{Tr}\left\{ S_q(y, y) \gamma_\mu \right\} \text{Tr}\left\{ S_q(x, x) \gamma_\nu \right\} \right\rangle
\]
Issues

contributions of quark-disconnected diagrams

\[ \langle j_{\mu}^{qq}(y)j_{\nu}^{qq}(x) \rangle = \langle \bar{q} \gamma_{\mu} q(y) \bar{q} \gamma_{\nu} q(x) \rangle \]
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\[ + \langle \text{Tr}\{S_q(y, y)\gamma_{\mu}\} \text{Tr}\{S_q(x, x)\gamma_{\nu}\} \rangle \]

connected - easy and good signal/noise

disconnected - challenging and bad signal/noise, so far no detailed study available and currently neglected
issues

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connected - easy and good signal/noise

disconnected - challenging and bad signal/noise, so far no detailed study available and currently neglected

• ETM did exploratory numerical study and concluded that disc. contrib. is small [Feng et al. Phys.Rev.Lett. 107 (2011) 081802]

• rough estimate in chiral perturbation theory: connected is -10% of the disconnected contribution, mildly varying with the momentum transfer [Della Morte, Jüttner JHEP 1011 (2010) 154]
compute \( \Pi(q^2) = \frac{1}{\delta_{\mu\nu} q^2 - q_\mu q_\nu} \Pi_{\mu\nu}(q^2) \) (no summation)

- no result at \( q^2 = 0 \)
- discrete momenta
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- partially twisted boundary conditions improve momentum resolution considerably
fit $q^2$-dependence and extrapolate

plots by Benni Jäger
fit $q^2$-dependence and extrapolate

issues: • model-dependent fit
fit $q^2$-dependence and extrapolate

issues:
- model-dependent fit
- no Fourier modes in enhanced region

plots by Benni Jäger
Example: Mainz group \( \text{Della Morte et al., JHEP 1203 (2012) 055} \)

fit \( q^2 \)-dependence and extrapolate

issues:
- model-dependent fit
- no Fourier modes in enhanced region
integrate over phase space (and match to PT at large $q^2$)

$$a_{u}^{\text{LH}} = \left(\frac{\alpha}{\pi}\right)^2 \int_{0}^{\infty} dq^2 K(q^2) \left(\Pi(q^2) - \Pi(0)\right)$$
repeat lattice simulation and analysis for several parameter choices $a, L, m_q$

plots by Benni Jäger
Hadronic contributions to the muon $g - 2$ in lattice QCD

Andreas Jüttner
Looks like there is still a long way to go before we can do better than experiment. . .
A bag full of tricks:

low momenta twisted bcs
result directly at $q^2 = 0$

$q^2$-dependence Padé approximants

$m_q$ dependence physical point or improved extrapolates.

stat. error all-mode-averaging

disc. diagrams face it or estimate it

Della Morte, Jäger, Jüttner, Wittig, JHEP 1203 (2012) 055

de Divitiis, Petronzio, Tantalo, arXiv:1208.5914

Aubin, Blum, Golterman, Peris, Phys.Rev. D86 (2012) 054509


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Model-independent parameterisation of $\Pi(q^2)$

Aubin, Blum, Golterman, Peris, Phys.Rev. D86 (2012) 054509

- once-subtracted dispersion relation

\[ \Pi(q^2) = \Pi(0) - q^2 \Phi(q^2) \quad \text{where} \quad \Phi(q^2) = \frac{1}{\pi} \int_{4m^2_{\pi}}^{\infty} dt \frac{\text{Im}\Pi(t)}{t(t+q^2)} \]

- $\text{Im}\Pi(t) = \rho(t) \geq 0$ for $t \geq 4m^2_{\pi}$

- $\Phi(q^2)$ is analytic for $t > -4m^2_{\pi}$
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- $\Phi$ can be approximated by continued fraction and analyticity provide bounds for coefficients

- the continued fraction is model-independently approximated by the Padé

$$\Pi(q^2) = \Pi(0) - q^2 \left( a_0 + \sum_{n=1}^{M} \frac{a_n}{b_n + q^2} \right)$$

with theory constraints for the coefficients (deriving from analyticity)
Model-independent parameterisation of $\Pi(q^2)$

- poles are free parameters and there is NO a priori reason for identifying them with the mass of vector resonances
- Padés have previously been used in fits but the above work provides the argument that it allows for a model-independent description of the data
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\[ \Pi(q^2) \]
Direct computation of $\Pi(0)$

- naively $\Pi(0)$ not accessible through
  \[ \Pi_{\mu\nu}(q) = (\delta_{\mu\nu}q^2 - q_\mu q_\nu)\Pi(q^2) \]
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- cute idea: de Divitiis, Petronzio, Tantalo, arXiv:1208.5914

\[
\frac{\partial^2}{\partial q_\mu \partial q_\nu} \Pi_{\mu\nu}(q)|_{\mu \neq \nu, q^2=0} = -\frac{\partial^2}{\partial q_\mu \partial q_\nu} (q_\mu q_\nu \Pi(q^2))|_{\mu \neq \nu, q^2=0} = -\Pi(0)
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How can this possibly work in practice?
Direct computation of $\Pi(0)$

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How can this possibly work in practice?

- $\Pi_{12}(q) = \sum_{x,y} \text{Tr} \left\{ S[y, x; U] \Gamma_{V,1}(x, \frac{\vec{q}}{2}) S[x, y; U, \lambda^q] \Gamma_{V,2}(y, \frac{\vec{q}}{2}) \right\}$
Direct computation of $\Pi(0)$

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\[
\Pi(0) = \left(\frac{1}{T L^3}\right)^2 \sum_{x,y} \langle \text{Tr} \left[ S \Gamma_{V,1} \frac{\partial^2 S}{\partial q_1 \partial q_2} \Gamma_{V,2}^2 \right] - \frac{1}{4} \text{Tr} \left[ S \Gamma_{V,1} \Gamma_{V,2}^2 \right] \\
- \frac{i}{2} \text{Tr} \left[ S \Gamma_{V,1} \frac{\partial S}{\partial q_2} \Gamma_{V,2}^2 \right] - \frac{i}{2} \text{Tr} \left[ S \Gamma_{V,1} \frac{\partial S}{\partial q_1} \Gamma_{V,2}^2 \right] \rangle
\]

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Andreas Jüttner
Direct computation of $\Pi(0)$

- they seem to be getting a good signal at $q^2 = 0$
- no more error-prone extrapolation of lattice-data towards $q^2 = 0$
- smaller stat. and syst. uncertainty on $\Pi(q^2) - \Pi(0)$ expected

de Divitiis, Petronzio, Tantalo, arXiv:1208.5914
$m_q$-extrapolation and poles

- the current extrapolation in $m_{\pi}^2$ is clearly unsatisfactory
- simulations very close to the physical point seem to be the only clean way out
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- in the meantime ETM suggests a reparameterisation:

  Feng et al. PRL 107, 081802 (2011)

  - assume pole-form: $\hat{\Pi}(q^2) \propto g^2 V_{q^2} m^2_\pi (m_\pi) + q^2$

  then $a_\mu(m_\pi) \propto \int_0^\infty dq f(q^2) \hat{\Pi}(q^2) \propto g^2 V_{m^2_\pi (m_\pi)}$
**$m_q$-extrapolation and poles**

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- Simulations very close to the physical point seem to be the only clean way out.
- In the meantime ETM suggests a reparameterisation:
  
  Feng et al. PRL 107, 081802 (2011)

- Assume pole-form: 
  \[ \hat{\Pi}(q^2) \propto g_V^2 \frac{q^2}{m_V^2(m_\pi) + q^2} \]

  Then 
  \[ a_\mu(m_\pi) \propto \int_0^\infty dq f(q^2) \hat{\Pi}(q^2) \propto g_V^2 \frac{m_\pi^2}{m_V^2(m_\pi)} \]

  - Assume $g_V$ mass-independent.
  - Reparameterise $f(q^2) \rightarrow f(hq^2)$
  \[ \rightarrow a_\mu(m_\pi) \propto g_V^2 \frac{m_\pi^2}{hm_V^2(m_\pi)} \]
$m_q$-extrapolation and poles

- the current extrapolation in $m_{\pi}^2$ is clearly unsatisfactory
- simulations very close to the physical point seem to be the only clean way out
- in the meantime ETM suggests a reparameterisation:
  
  Feng et al. PRL 107, 081802 (2011)

  - assume pole-form: $\hat{\Pi}(q^2) \propto g_V^2 \frac{q^2}{m_V^2(m_{\pi}) + q^2}$
  - then $a_\mu(m_{\pi}) \propto \int_0^\infty dq f(q^2) \hat{\Pi}(q^2) \propto g_V^2 \frac{m_{\pi}^2}{m_V^2(m_{\pi})}$
    - assume $g_V$ mass-independent
    - reparameterise $f(q^2) \rightarrow f(hq^2)$
      
      $\rightarrow a_\mu(m_{\pi}) \propto g_V^2 \frac{m_{\pi}^2}{hm_V^2(m_{\pi})}$

  - for choice $h = \frac{m_{\pi}^2}{m_V^2(m_{\pi})}$ the dependence of the pion mass cancels
$m_q$-extrapolation and poles

*Hotzl (ETM), talk at Lattice 2012*
$m_q$-extrapolation and poles

Boyle et al. PRD 85, 074504 (2012)
$m_q$-extrapolation and poles

Boyle et al. PRD 85, 074504 (2012)
\( m_q \)-extrapolation and poles

Boyle et al. PRD 85, 074504 (2012)

still need to be careful since vector dominance is a model, extrapolation might still contain structure besides linear term
A bag full of tricks:

$q^2$-dependence  Padé approximants
low momenta  twisted bcs
result directly at $q^2 = 0$
$m_q$ dependence  improved extrapolations
physical point or
improved extrapolations
stat. error  all-mode-averaging
disc. diagrams  face it or estimate it

Aubin, Blum, Golterman, Peris, Phys.Rev. D86 (2012) 054509
Della Morte, Jäger, Jüttner, Wittig, JHEP 1203 (2012) 055
de Divitiis, Petronzio, Tantalo, arXiv:1208.5914
phys. pt. not quite but close: $m_\pi = 166\text{MeV}$ RBC/UKQCD
Feng et al. PRL 107, 081802 (2011)
Blum, Izubuchi, Shintani, arXiv:1208.4349
Della Morte, Jüttner JHEP 1011 (2010) 154
Status of $g - 2$

- leading hadronic VP previous lattice efforts compared to $e^+ e^- \rightarrow$ hadrons

  \[ 688(05) \times 10^{-10} \]

lattice QCD:

- ETM Lattice 2012
  \[ 583(13) \times 10^{-10} \quad N_f = 2 + 1 + 1 \quad u, d \]
- ETM Lattice 2012
  \[ 647(13) \times 10^{-10} \quad N_f = 2 + 1 + 1 \quad u, d, s \]
- ETM Lattice 2012
  \[ 667(14) \times 10^{-10} \quad N_f = 2 + 1 + 1 \quad u, d, s, c \]
  \[ 641(46) \times 10^{-10} \quad N_f = 2 + 1 \quad u, d, s \]
- Mainz JHEP 1203 (2012) 055
  \[ 618(64) \times 10^{-10} \quad N_f = 2 \quad u, d, s \]
- Mainz JHEP 1203 (2012) 055
  \[ 546(66) \times 10^{-10} \quad N_f = 2 \quad u, d \]
- ETM PRL 107, 081802 (2011)
  \[ 572(16) \times 10^{-10} \quad N_f = 2 \quad u, d \]

- given the presented developments this is now a very competitive business
- things still to be studied: iso-spin breaking, charm quark, disconnected diagrams
- watch this place!
Outline

- leading hadronic contribution in lattice QCD
- light-by-light scattering
- outlook
current model estimates $\approx 25 - 30\%$ uncertainty

for now this is not about precision but about feasibility
current model estimates $\approx 25 - 30\%$ uncertainty
for now this is not about precision but about feasibility
one lattice method: brute force 4pt function computation and pert. treatment of QED QCDSF, Rakow
light-by-light

Hadronic contributions to the muon $g - 2$ in lattice QCD

Andreas Jüttner
Blum et al. are now getting a signal (≈ 15% stat error) for unphysical parameters (heavy muon, heavy quark) and are now studying systematics for now strong parameter-dependence but optimistic that results will lie in the expected ballpark.

Further diagrams involving quark-disconnected diagrams need to be studied.

This is really exciting (Blum’s plenary at Lattice 2012) but so far nothing published.
in particular in view of new results coming in from the LHC experiments the anomalous magnetic moment of the muon remains a phenomenologically highly relevant quantity

lattice determinations can potentially play an important role in producing a solid SM prediction

recent surge of interest has produced a long list of new ideas and we can expect real progress in the near future

- twisted boundary conditions for better momentum resolution
- theory for model-independent fit
- quark-disconnected diagrams in effective theory
- computation of $\Pi(0)$
- improved mass-extrapolations / physical point simulations
- …

so far no published results for light-by-light but it will hopefully not take too long
The research leading to these results has received funding from the European Research Council under the European Community’s Seventh Framework Programme (FP7/2007-2013) ERC grant agreement No 279757