Factorization of leading chiral logarithms in the pion form factors

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Outline

- Hard pion ChPT: factorization of leading chiral logarithms
- Pion vector and scalar form factors for $M_{\pi}^2 \ll s$: dispersive representation
- Elastic contributions (intermediate 2 pions): factorization valid to all orders
- Inelastic contributions: factorization violated at 3 loops (intermediate 4 pions)

G. Colangelo, MP, L. Rothen, R. Stucki and J. Tarrús Castellà
Flynn and Sachrajda (2009)

\[ K \to \pi \] form factors in SU(2) ChPT:
predictions for \textbf{leading chiral logarithms} even for \( q^2 = 0 \), when \( E_\pi \sim M_K/2 \)
i.e. the pion is hard

\[ \to \text{improve chiral extrapolations of lattice results for } f^+_+(0) \text{ to enable a precise determination of } |V_{us}| \]

Bijnens and Celis (2009), Bijnens and Jemos (2010, 2011)

predictions for \textbf{leading chiral logarithms} in a variety of processes with hard pions in the final state: \( K \to \pi\pi \) decays, \( B \to \pi \) and \( D \to \pi \) form factors at \( q^2 < q^2_{\text{max}} \), \textbf{pion form factors} \( F_{V,S}(s, M^2_\pi) \) for \( M^2_\pi \ll s \)
Hard pion ChPT predicts that, for $M^2_\pi/s \ll 1$, the leading chiral logarithm factorizes from the energy dependence in the chiral limit:

$$F_{V,S}(s, M^2) = \overline{F}_{V,S}(s) \left[1 + \alpha_{V,S} L \right] + \mathcal{O}(M^2)$$

with

$$L \equiv \frac{M^2}{(4\pi F)^2} \ln \frac{M^2}{s}, \quad M^2_\pi = M^2 + \mathcal{O}(M^4), \quad F^2_\pi = F + \mathcal{O}(M^2)$$

Expanding the two-loop standard SU(2) ChPT result in Bijnens, Colangelo and Talavera (1998) for $M^2_\pi/s \ll 1$ one obtains the factorized form predicted by Hard pion ChPT with

$$\alpha_S = -\frac{5}{2}, \quad \alpha_V = -1$$
Quantitative explanation of this factorization property?

Still valid beyond two loops?
Dispersive representation of the FFs

Analyticity:

\[ F(s) = 1 + \frac{s}{\pi} \int_{4M^2}^\infty ds' \frac{\text{Im} F(s')}{s'(s' - s)} \]

with \[ F_{V,S}(0) = 1 \]

Unitarity:

\[ \text{Im} F(s) = \sigma(s) F(s) t^*(s) + \text{inelastic terms} \]

\[ \sigma(s) = \sqrt{1 - \frac{4M^2}{s}} \]

two-pion phase space

\[ \pi\pi \text{ partial wave with the appropriate quantum numbers} \]
Dispersive representation of the FFs

Analyticity:

\[
F(s) = 1 + \frac{s}{\pi} \int_{4M_{\pi}^2}^{\infty} ds' \frac{\text{Im} F(s')}{s'(s' - s)}
\]

with \( F_{V,S}(0) = 1 \)

Unitarity:

\[
\text{Im} F(s) = \sigma(s) F(s) t^*(s) + \text{inelastic terms}
\]

Our notation (diagrammatic definition):

\[
F(s) = F_{\text{el}}(s) + F_{\text{inel}}(s)
\]

2 pion intermediate states also for the \( \pi\pi \) partial wave

the only contribution up to 2 loops
Analyticity:

\[
F(s) = 1 + \frac{s}{\pi} \int_{\frac{4M^2}{\pi}}^{\infty} ds' \frac{\text{Im} F(s')}{s'(s' - s)}
\]

with \( F_{V,S}(0) = 1 \)

Unitarity:

\[
\text{Im} F(s) = \sigma(s) F(s) t^*(s) + \text{inelastic terms}
\]

ChPT provides a perturbative solution to the dispersion relation, allows us to argue recursively applying the chiral counting:

\[
\text{Im} F^{(2)}(s) = \sigma(s) t^{(2)*}(s)
\]

\[
\text{Im} F^{(4)}(s) = \sigma(s) \left[ t^{(4)*}(s) + F^{(2)}(s) t^{(2)}(s) \right]
\]

\[
\vdots \quad \vdots
\]
Dispersive representation of the FFs

Analyticity:

\[ F(s) = 1 + \frac{s}{\pi} \int_{4M^2/\pi}^{\infty} ds' \frac{\text{Im} F(s')}{s'(s' - s)} \]

with \( F_{V,S}(0) = 1 \)

Unitarity:

\[ \text{Im} F(s) = \sigma(s) F(s) t^*(s) + \text{inelastic terms} \]

ChPT provides a perturbative solution to the dispersion relation, allows us to argue recursively applying the chiral counting:

\[ \text{Im} F^{(2)}(s) = \sigma(s) t^{(2)*}(s) \]

\[ \text{Im} F^{(4)}(s) = \sigma(s) \left[ t^{(4)*}(s) + F^{(2)}(s) t^{(2)}(s) \right] \]

one loop
two loops
Analyticity:

\[
F(s) = 1 + \frac{s}{\pi} \int_{4M_{\pi}^2}^{\infty} ds' \frac{\text{Im} F(s')}{s'(s' - s)}
\]

with \( F_{V,S}(0) = 1 \)

The leading chiral logarithms can arise:

1. from an integrand which does not contain a log of the pion mass and this is produced by the integration over \( s' \)

2. if the integrand itself contains a chiral log
Chiral logs from integration (ELASTIC)

Produced at the lower integration boundary $s' \sim 4M^2_\pi$:

use standard ChPT to analyze the integrand for $s' \sim M^2 \ll s$

$$F(s) = 1 + \frac{s}{\pi} \int_{4M^2_\pi}^{\infty} ds' \, \frac{\text{Im} F(s')}{s'(s' - s)} = 1 + \frac{s}{\pi} \int_{4M^2_\pi}^{\infty} ds' \frac{\sigma(s')}{s'(s' - s)} \left( c_1 M^2 + c_2 s' + O(p^4) \right)$$

The numerical constants $c_1$ and $c_2$ are related to the leading chiral contributions to the $\pi\pi$ scattering lengths and effective ranges

The leading chiral logarithm generated by the dispersive integration is

$$16\pi F^2 (c_1 - 2c_2) L \equiv \alpha_{V,S} L$$

$$\alpha_S = -\frac{5}{2}, \quad \alpha_V = -1$$

in agreement with Bijnens, Colangelo, Talavera (1998), Bijnens and Jemos (2011)
Chiral logs in the integrand (ELASTIC)

**Contribution to the form factors at 2 loops:**

\[
\text{Im } F^{(4)}(s) = \sigma(s) \left[ t^{(4)*}(s) + F^{(2)}(s) t^{(2)}(s) \right]
\]

\[
F^{(2)}(s) t^{(2)}(s) = \left( \overline{F}^{(2)}(s) + \alpha L \right) \overline{t}^{(2)}(s) + O(M^2)
\]

\[
t^{(4)}(s) = \overline{t}^{(4)}(s) + \beta sL + O(M^2)
\]

Using Roy equations for \(\pi\pi\) partial waves, we show that \(\beta = 0\), which implies that factorization is valid up to 2 loops:

\[
F(s) = \left( 1 + \overline{F}^{(2)}(s) \right) (1 + \alpha L) + \overline{F}^{(4)}(s) + O(M^2) + O(p^6)
\]

in agreement with Hard pion ChPT
Chiral logs in the integrand (ELASTIC)

Contribution to the form factors at 2 loops:

\[
\text{Im} F^{(4)}(s) = \sigma(s) \left[ t^{(4)*}(s) + F^{(2)}(s) t^{(2)}(s) \right]
\]

\[
F^{(2)}(s) t^{(2)}(s) = \left( \overline{F}^{(2)}(s) + \alpha L \right) \bar{t}^{(2)}(s) + \mathcal{O}(M^2)
\]

\[
t^{(4)}(s) = \bar{t}^{(4)}(s) + \beta sL + \mathcal{O}(M^2)
\]

Using Roy equations for \(\pi\pi\) partial waves, we show that \(\beta = 0\), which implies that factorization is valid up to 2 loops.

By induction, we prove that all terms \(s^{n-1}L\) in \(t^{(2n)}(s)\) are absent: the elastic part (subclass of diagrams: 2 pion intermediate states) of the form factors factorizes to all orders in the chiral counting.
Inelastic contributions to the DR

Start with 4 intermediate pions (3-loop diagrams)

\[
F_{\text{inel}}(s) = \frac{s}{\pi} \int_{16M^2_\pi}^{\infty} ds' \frac{\text{Im} F_{\text{inel}}(s')}{s'(s' - s)}, \quad \text{Im} F_{\text{inel}}(s) = \frac{1}{2} \int d\Phi_4(s; p_1, p_2, p_3, p_4) F_{4\pi} \cdot T^{*}_{6\pi}
\]

For the scalar form factor:

Chiral logs are produced by integrations over intermediate momenta with pion-mass-dependent boundaries
Inelastic contributions to the DR

Start with 4 intermediate pions (3-loop diagrams)

\[ F_{\text{inel}}(s) = \frac{s}{\pi} \int_{16M^2_\pi}^{\infty} ds' \frac{\text{Im} F_{\text{inel}}(s')}{s'(s' - s)} , \quad \text{Im} F_{\text{inel}}(s) = \frac{1}{2} \int d\Phi_4(s; p_1, p_2, p_3, p_4) F_4 \pi \cdot T_{6\pi}^* \]

For the scalar form factor:

For the **scalar** form factor:

Analytical results for the chiral limit values and coefficients of the leading chiral log for graphs (a), (b), (c), (d) and numerical results for (e), (f) and (g)
Inelastic contributions to the DR

Factorization is not valid at three loops:

\[ F(s) = \left( 1 + F^{(2)}(s) + F^{(4)}(s) \right) (1 + \alpha L) + \alpha_{\text{inel}}(s) L + F^{(6)}(s) + O(M^2) + O(p^8) \]

with

\[ \alpha_{\text{inel}}(s) = \left[ C(\mu^2) + \delta \times \left( \ln \frac{\mu^2}{s} + i\pi \right) \right] \frac{s^2}{(4\pi F)^4}, \quad \delta = -0.53 \pm 0.05 \]

The coefficient of the leading chiral log is not universal

Factorization as conjectured in Hard pion ChPT is not an exact property
The 2-loop scalar FF in the chiral limit is extracted from Bijnens, Colangelo, Talavera (1998)
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Conclusions and outlook

- Factorization of leading chiral logs in the pion form factors for $M_\pi^2 \ll s$

- Dispersion relations and application of chiral counting: recursive analysis

- We show how factorization emerges at two loops and is valid for a whole subclass of diagrams to all orders (with 2 intermediate pions)

- Our calculation at 3 loops shows that factorization is broken by multipion contributions, which generate new leading chiral logs

- Factorization could be valid to a good approximation only if one remains in the low-energy regime, with very small quark masses

- Future work: extension of our analysis to heavy-light form factors
Additional slides
For asymptotically large values of \( s \),

\[
F_V(s) = \frac{F_\pi^2}{s} \int_0^1 dx \, dy \, T(x, y, s) \phi_\pi(x) \phi_\pi(y) \times [1 + \mathcal{O}(\Lambda_{\text{QCD}}^2/s, M_\pi^2/s)]
\]

Brodsky and Lepage (1980)

The leading chiral log is given just by the one in \( F_\pi \)

Chen and Stewart (2004)

Hence the leading chiral log does factorize for \( s \gg \Lambda_{\text{QCD}}^2 \) but \( \alpha_V = -2 \) while Hard pion ChPT predicts \( \alpha_V = -1 \) (valid only in the low-energy regime)