Gauge invariant current and masses in 2D SU(N) Field Theory

Andrew V. Koshelkin
Moscow Institute for Physics and Engineering, Kashirskoye sh., 31, 115409 Moscow, Russia

Cheuk-Yin Wong
Physics Division, Oak Ridge National laboratory, OakRidge, TN 37831, US
1. Introduction

3D (3+1)

“3D” (3+1)

“3D” (2+ (1+1))
2. The main goal

\[ A(4D)[\Psi(4D, x), F_{\mu\nu}^a(4D, x), g_{4D}] \Rightarrow A(2D)[\Psi(2D, x^0, x^3), F_{\mu\nu}^a(2D, x^0, x^3), g_{2D}] \]

\[ A(4D) = \int d^4x \left\{ Tr \left[ \frac{1}{2} \left[ \bar{\Psi}(4D, x) \gamma^\mu(4D) \Pi_\mu(4D) \Psi(4D, x) - \bar{\Psi}(4D, x) \bar{m}(x) \psi(4D, x) \right] \right. \right. \]
\[ - \frac{1}{2} \left[ \bar{\Psi}(4D, x) \gamma^\mu(4D) \bar{\Pi}_\mu(4D) \Psi(4D, x) + \bar{\Psi}(4D, x) \bar{m}(x) \psi(4D, x) \right] \left. \right\} - \frac{1}{4} F_{\mu\nu}^a(4D, x) F_{\mu\nu}^a(4D, x) \]

\( \Pi_\mu(4D) = i\partial_\mu + g_{4D} T_a A_\mu^a(4D, x) = p_\mu + g_{4D} T_a A_\mu^a(4D, x) ; \)

\( \bar{\Pi}_\mu(4D) = i\bar{\partial}_\mu - g_{4D} T_a A_\mu^a(4D, x) = \bar{p}_\mu - g_{4D} T_a A_\mu^a(4D, x) ; \)

\( F_{\mu\nu}^a(4D, x) = \partial_\mu A_\nu^a(4D, x) - \partial_\nu A_\mu^a(4D, x) + g_{4D} f_{bc}^a A_\mu^b(4D, x) A_\nu^c(4D, x) \)

\( \equiv \partial_\mu A_\nu^a(4D, x) - \partial_\nu A_\mu^a(4D, x) - ig_{4D} [A_\mu^b(4D, x), A_\nu^c(4D, x)]^a. \)

Gauge transformation

\[ \delta A_\mu^a(4D, x) = f_{bc}^a \varepsilon^b(x) A_\mu^c(4D, x) - \frac{1}{g_{4D}} \partial_\mu \varepsilon^a(x) \]
3. The key approximation

The longitudinal dominance and transverse confinement

\[
\max \{|A_1^a|; |A_2^a|\} \ll \min \{|A_0^a|; |A_3^a|\} \iff |A_1^a|, |A_2^a| \equiv 0
\]

However!!!

\[
A_0^a = A_0^a(x^0, x), \quad A_3^a = A_3^a(x^0, x)
\]
4. Compactification 4D -> 2D

We transform

(i) a fermion field

\[ \Psi(4D, x) \equiv \begin{pmatrix} \varphi(x_0, x) \\ \varphi_2(x_0, x) \\ \chi_1(x_0, x) \\ \chi_2(x_0, x) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} G_1(r_\perp)(f_+(x_0, x^3) + f_-(x_0, x^3)) \\ -G_2(r_\perp)(f_+(x_0, x^3) - f_-(x_0, x^3)) \\ G_1(r_\perp)(f_+(x_0, x^3) - f_-(x_0, x^3)) \\ G_2(r_\perp)(f_+(x_0, x^3) + f_-(x_0, x^3)) \end{pmatrix}, \]  

(6)

\[ \int dx^1 dx^2 \left( |G_1(r_\perp)|^2 + |G_2(r_\perp)|^2 \right) = 1. \]  

(7)

(ii) a gauge field

\[ A_\mu^a(4D, x_0, x^3, r_\perp) = \sqrt{|G_1(r_\perp)|^2 + |G_2(r_\perp)|^2} A_\mu^a(2D, x_0, x^3). \]  

(8)

The coupling constant and gauge transformation become

\[ g_{2D} = \int dx^1 dx^2 g_{4D} \left( |G_1(r_\perp)|^2 + |G_2(r_\perp)|^2 \right)^{3/2} \]  

(9)

\[ \delta A_\mu^a(2D, x_0, x^3) = f_{bc}^a \varepsilon^b(x_0, x^3) A_\mu^c(2D, x_0, x^3). \]  

(10)
5. 2D - action functional

\[ A(2D) = \int d^2 X \left\{ \text{Tr} \left[ \frac{1}{2} \left[ \Psi(2D, X) \gamma^k(2D) \Pi_k(2D) \Psi(2D, X) - \Psi(2D, X)m_{qT} \Psi(2D, X) \right] - \frac{1}{2} \left[ \Psi(2D, X) \gamma^k(2D) \Pi_k(2D) \Psi(2D, X) + \Psi(2D, X)m_{qT} \Psi(2D, X) \right] \right] - \frac{1}{4} F^{a}_{\mu \nu}(2D) F^{\mu \nu}_{a}(2D) + \frac{1}{2} m^2_{gT}[A^{a}_{a}(2D)A^{a}_{\mu}(2D)] \right\}, \]

where \( \{\mu, \nu\} = 0, 3 \), and

\[ \Pi_{\mu}(2D) = i \partial_{\mu} + g_{2D} T_{a} A^{a}_{\mu}(2D, x) = p_{\mu} + g_{2D} T_{a} A^{a}_{\mu}(42, x), \]

\[ \Pi^{\mu}(2D) = i \partial^{\mu} - g_{2D} T_{a} A^{a}_{\mu}(2D, x) = \overline{p}^{\mu} - g_{2D} T_{a} A^{a}_{\mu}(2D, x). \]

\[ \Psi(X) = \Psi(2D, X) = \left( \begin{array}{c} f_{+}(X) \\ f_{-}(X) \end{array} \right), \quad X = (x^{0}; x^{3}), \]

\[ \gamma^{0}(2D) = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \quad \gamma^{3}(2D) = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \quad g^{\mu \nu}(2D) = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \]

\[ m_{qT} = \int dx^{1} dx^{2} \left\{ m(r_{\perp}) \left( |G_{1}(r_{\perp})|^{2} - |G_{2}(r_{\perp})|^{2} \right) + (G_{1}^{*}(r_{\perp})(p_{1} - ip_{2})G_{2}(r_{\perp})) - (G_{1}(r_{\perp})(p_{1} + ip_{2})G_{2}^{*}(r_{\perp})) \right\}. \]

\[ m^2_{gT} = \frac{1}{2} \int dx^{1} dx^{2} \left[ \left\{ \partial_{1} [G_{1}(r_{\perp})]^{2} + [G_{2}(r_{\perp})]^{2} \right\}^{1/2} \right]^{2} \left[ \left\{ \partial_{2} [G_{1}(r_{\perp})]^{2} + [G_{2}(r_{\perp})]^{2} \right\}^{1/2} \right]^{2}. \]
6. The Dirac fields in (1+1) space-time

\[ \{i\gamma^\mu \left( \partial_\mu - ig_\text{D} \cdot A_\mu^a(x)T_a \right) - m_{qT} \} \Psi(2D, X) = 0, \quad \mu = 0, 3, \tag{15} \]

\[ A_\mu^a(2D, X) = (A_0^a(X), -A_3^a(X)), \quad X = (x^0; x^3). \]

(i) Formal solution

\[ \Psi(2D, X) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\sqrt{L}} \sum_p \frac{m_{qT}}{\sqrt{\omega^2 + p\omega}} \exp(-iP_\mu x^\mu) a(p, \omega) \left[ \delta(\omega - \varepsilon(p)) + \delta(\omega + \varepsilon(p)) \right] \left( \frac{\omega + p}{m_{qT}} + 1 \right) \left( \frac{\omega + p}{m_{qT}} - 1 \right) \]

\[ \times \left\{ T_{i(M_0; M)} \exp \left\{ ig_\text{D} T_a \int dx^\mu A_\mu^a \right\} , \right. \]

where \( a(p, \omega) \) is an operator of creation of any particle (a particle or antiparticle).
(ii) Fermion current

\[ J^\mu_a = g_{2D} \, Tr \left\{ \bar{\Psi}(x) \gamma^\mu T_a \Psi(x') \right\} ; \quad x' \to x. \]  

(17)

\[ J^\mu_a (x) = \frac{g_{2D}^2}{2\pi} \, N_f \left[ A^\mu_a - \partial^\mu \frac{1}{\partial\lambda \partial\lambda} \partial_\nu A^\nu_a \right] \]  

(18)

Gauge transformation

\[ \delta J^\mu_a = \varepsilon_b f^{bc}_a J^\mu_c \]  

(19)
7. Equation of motion for the 2D Gauge Fields

(In the Lorentz gauge)

\[
A(2D) = \int d^2X \left\{ \frac{i}{2} \left[ \bar{\Psi}(2D, X) \gamma^k(2D) \partial_k(2D) \Psi(2D, X) - \bar{\Psi}(2D, X)m_{qT}\Psi(X) \right] \\
- \frac{i}{2} \left[ \bar{\Psi}(2D, X) \gamma^k(2D) \bar{\partial}_k(2D) \Psi(2D, X) + \bar{\Psi}(2D, X)m_{qT}\Psi(X) \right] \\
- \frac{1}{4} F_{\mu\nu}^a(2D) F_{\alpha}^{a\mu\nu}(2D) + \frac{1}{2} M_{gT}^2 A_{\nu}(2D)A_{a}(2D) \right\},
\]

\[
M_{gT}^2 = \frac{1}{2} \int d^1 x d^2 x \left[ \{ \partial_1 \left[ |G_1(r_\perp)|^2 + |G_2(r_\perp)|^2 \right]^{1/2} \}^2 + \{ \partial_2 \left[ |G_1(r_\perp)|^2 + |G_2(r_\perp)|^2 \right]^{1/2} \}^2 \right] + \frac{g_{2D}^2 N_f}{2\pi} \\
\equiv m_{gT}^2 + m_{gfT}^2 \geq 0.
\]
(i) The motion equation

\[ A_\alpha^\nu(2D, X) = M_{gT}^2 A_\alpha^\nu(2D, X) \]  \hspace{1cm} (22)

(ii) Its solution

\[ A_\alpha^\nu(2D, X) = \sum_k \frac{e_\alpha^\nu M_{qT}}{\sqrt{(k^2 + M_{qT}^2)}} \left\{ \exp(-ikX) b_\alpha(k, \nu) + \exp(+ikX) \bar{b}_\alpha(k, \nu) \right\} \]

\[ k^\mu = (k^0; k), \quad E(k) \equiv k^0 = +\sqrt{k^2 + M_{qT}^2} \]

(iii) Colorless particles

\[ A_{\text{color-singlet}}^\nu = \frac{1}{\sqrt{8}} \sum_\alpha A_\alpha^\mu |8, \alpha\rangle \]

\[ A_{\text{color-singlet}}^\nu(2D, X) = M_{gT}^2 A_{\text{color-singlet}}^\nu(2D, X). \]
8. Transverse motion in a tube

\[ \mathcal{F} = A(4D) + \frac{\lambda}{2} \int dx^1 dx^2 \left( |G_1(\vec{r}_\perp)|^2 + |G_2(\vec{r}_\perp)|^2 \right) \int dx^0 dx^3 (\bar{\psi}(2D, X) \psi(2D, X)) \]  \hspace{1cm} (25)

\[ (p_1 + ip_2)G_1(\vec{r}_\perp) = (m(\vec{r}_\perp) + \lambda)G_2(\vec{r}_\perp), \]
\[ (p_1 - ip_2)G_2(\vec{r}_\perp) = (\lambda - m(\vec{r}_\perp))G_1(\vec{r}_\perp), \]
\[ (p_1 + ip_2)G_2^*(\vec{r}_\perp) = (m(\vec{r}_\perp) - \lambda)G_1^*(\vec{r}_\perp), \]
\[ (p_1 - ip_2)G_1^*(\vec{r}_\perp) = -(m(\vec{r}_\perp) + \lambda)G_2^*(\vec{r}). \]  \hspace{1cm} (26)

\[ \lambda = \lambda^* = m_q T \]  \hspace{1cm} (!)  \hspace{1cm} (27)
9. The Gaussian tube

\[ G_1(\vec{r}_\perp) = e^{-i\frac{\vec{r}_\perp^2}{2R_T^2}} \frac{e^{-\frac{r_1^2}{2R_T^2}}}{R_T \sqrt{2\pi}}; \quad G_2(\vec{r}_\perp) = -i e^{+i\frac{\vec{r}_\perp^2}{2R_T^2}} \frac{e^{-\frac{r_1^2}{2R_T^2}}}{R_T \sqrt{2\pi}} \]

\[ g_{2D} = \frac{g_{AD}}{R_T} \sqrt{\frac{2}{9\pi}} \]

\[ m_{fgT}^2 = \frac{2 g_{AD}^2}{9\pi R_T^2} \]

\[ m_{gT}^2 = 2 g_{AD}^2 R_T^{-2} \]

\[ m_{gT} = \frac{g_{AD}}{R_T} \sqrt{2 + \frac{2}{9\pi}} \]
10. Conclusion

1. The $4D \rightarrow 2D$ compactification is realized in terms of the action principle in the SU(N) gauge field theory.

2. The exact gauge invariant 2D-Lagrangian is derived. On a basis of the obtained Lagrangian, the gauge invariant fermion current and the transverse mass of a fermion are calculated.

3. The derived current is found to be proportional to the amplitude of the gauge field, similar to the case of 2D QED (J. Schwinger, 1962).

4. The obtained mass for SU(N) field theory depends strongly on the 2D coupling constant, number of flavors, as well as on the transverse mass of the fermion.
12. References


and so on...
Thank you very much for your attention!