Lorentz Invariance in Heavy Particle Effective Theories

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If you just arrived, you missed this:
... but you just arrived in time for this:

Outline

- Motivation: NRQCD/HQET and Reparametrization Invariance (RPI)
- Lorentz Invariance using Wigner’s Little Group
- Invariant operator method
- Conclusion
Example: NRQCD

NRQCD \[\text{[Caswell, Lepage '86]}\]

\[\mathcal{L}_v = \bar{\psi}_v \left\{ iv \cdot D - c_2 \frac{D^2}{2M} - c_F g \frac{\sigma_{\alpha\beta} G^{\alpha\beta}}{4M} + ic_S g \frac{\nu_\lambda \sigma_{\alpha\beta} \{ D^\alpha_\bot, G^{\lambda\beta} \}}{8M^2} + \ldots \right\} \psi_v\]

with \(\bar{\psi}_v = \psi_v\) and mass term removed by \(\psi_v \rightarrow e^{iMv \cdot x} \psi_v\).

- We demand invariance under Reperametrization Invariance (RPI) \[\text{[Luke, Manohar '92]}\]

\[v \rightarrow v + q/M\]

\[\psi_v \rightarrow e^{iq \cdot x} \left[1 + \frac{q}{2M} + \frac{\sigma_{\alpha\beta} q^\alpha D^\beta_\bot}{4M^2} + \ldots \right] \psi_v\]

⇒ RPI - Relations

\[c_2 = 1\]

\[c_S = 2c_F - 1\]
What has this to do with Lorentz invariance?

- QCD is invariant under

\[ \phi(x) \rightarrow U(\Lambda)\Phi(\Lambda^{-1}x) = (1 + \eta_A(\Lambda)L^A + \ldots)\Phi(x) \]

with Poincaré generators \( L^A = (h, p, j, k) \):

\[
\begin{align*}
  h &= i\partial_t, \\
  p &= -i\partial, \\
  j &= r \times p + \Sigma, \\
  k &= rh - tp + i\Sigma,
\end{align*}
\]

Poincaré commutation relations (PCR)

\[
\begin{align*}
  [h, p^i] &= [h, j^i] = 0 \\
  [j^i, j^j] &= i\epsilon^{ijk}j^k \\
  [h, k^i] &= -i p^i \\
  [j^i, K^j] &= i\epsilon^{ijk}k^k \\
  [p^i, p^j] &= 0 \\
  [j^i, p^j] &= i\epsilon^{ijk}p^k \\
  [p^i, k^j] &= -i\delta^{ij} h \\
  [k^i, k^j] &= -i\epsilon^{ijk}j^k
\end{align*}
\]
The origin of RPI - III

- The conserved charges in NRQCD should also obey these PCRs. [Brambilla, Gromes, Vairo ’03]

\[
H = \int d^3x \psi^\dagger \left( M - c_2 \frac{D^2}{2M} - c_F g \frac{\sigma \cdot B}{2M} - \frac{c_D g}{8M^2} [D_\nu, \Pi] - i c_S g \frac{\sigma \cdot [D \times, \Pi]}{8M^2} \right) \psi
\]

\[
P = \int d^3x \left( \psi^\dagger (-iD) \psi + \frac{1}{2} [\Pi^a \times, B^a] \right),
\]

\[
J = \int d^3x \left( \psi^\dagger \left( x \times (-iD) + \frac{\sigma}{2} \right) \psi + \frac{1}{2} x \times [\Pi^a \times, B^a] \right),
\]

\[
K = -t \mathbf{P} + \int d^3x \frac{\{x, h\}}{2} - k^{(1)} \int d^3x \left( \frac{1}{2M} \psi^\dagger \frac{\sigma}{2} \times (-iD) \psi \right),
\]

\[k^{(1)} \text{ is a coefficient to be determined.}\]

+ Quantize canonically:

\[
[\Pi(x, t), A(y, t)] = i\delta^{(3)}(x - y), \{\psi_\alpha(x, t), \psi_\beta^\dagger(y, t)\} = \delta_{\alpha\beta} \delta^{(3)}(x - y), \ldots
\]

+ Enforce Poincaré algebra:

Relations for coefficients \(\sim\) RPI relations

\[k^{(1)} = 1, \quad \text{and} \quad c_2 = 1, \quad c_S = 2c_F - 1, \ldots\]
RPI \sim \text{Lorentz} !?

Conjecture/ Theorem?

RPI is enforcing Lorentz invariance of the Lagrangian

⇒ Forget about NRQCD and RPI for a moment and start from scratch

How can we make a non-relativistic EFT Lorentz invariant?
Wigner’s little group in QM

How do QM states transform under Lorentz transformation?

- Pick a reference vector $k^\mu$, e.g. $k^\mu = (M, 0, 0, 0)$, and a standard Lorentz transformation $L(p)$, s.t. $L(p)k = p$.
- Define the one particle states: $|p, m\rangle \sim L(p) |k, m\rangle$

**Unitary Representations of the Lorentz group** [Wigner '39]

(Infinite dim.) Unitary representations of the Lorentz group are given by

$$U(\Lambda) |p, m\rangle \sim D[W(\Lambda, p)] |\Lambda p, m\rangle$$

where $D$ is a rep. of the little group element $W(\Lambda, p) = L(\Lambda p)^{-1} \Lambda L(p)$.

- **Wigner’s little group** = Subgroup of Lorentz that leaves $k$ invariant: $Wk = k$

⇒ The little group determines the unitary rep’s of the Lorentz group.
- For a massive particle, e.g. $k = (M, 0, 0, 0)$ → Little group is $SO(3)$
Wigner’s little group in field theories

Apply this to field theories

Demand invariance under

\[ \phi_a(x) \rightarrow D[W(\Lambda, i\partial)]_{ab}\phi_b(\Lambda^{-1}x) \]

Compute \( W(\Lambda, p) \) for \( k = M\nu \) and \( L(p) = \exp[-i\theta J_{\alpha\beta} \frac{p^\alpha}{M} \nu^\beta] \):

1. For “rotations” with \( \mathcal{R}\nu = \nu \)

\[ W(\mathcal{R}, i\partial) = \mathcal{R} \]

No dependence on \( \partial \) and get \( \mathcal{R} \) back.

2. For infinitesimal “boosts” with \( \mathcal{B}\nu = \nu - q/M \) (and \( q \cdot \nu = 0 \))

\[ W(\mathcal{B}, i\partial) = 1 - \frac{i}{2} \left[ \frac{q^\alpha i\partial^\beta - i\partial^\alpha q^\beta}{M(M + \nu \cdot i\partial)} \right] J_{\alpha\beta} + \mathcal{O}(q^2). \]

with \( \partial^\alpha_\perp = \partial^\alpha - (\nu \cdot \partial)\nu^\alpha \).
What about Lorentz invariance?

For Simplicity, choose \( v = (1, 0, 0, 0) \). The corresponding generators are

### Rotations:

\[
\phi(x) \rightarrow \mathcal{R} \phi(\mathcal{R}^{-1} x)
\]

\[
j = r \times i \partial + \Sigma
\]

### Boosts:

\[
\phi(x) \rightarrow W(B, i \partial) \phi(B^{-1} x)
\]

\[
k = r i \partial_t - t i \partial + \frac{\Sigma \times i \partial}{M + \sqrt{M^2 - \partial^2}}
\]

- With \( h = i \partial_t \) and \( p = -i \partial \) these satisfy the PCR on fields which obey

\[
i \partial_t \phi = \sqrt{M^2 - \partial^2} \phi.
\]

⇒ Theory seems to be Lorentz invariant.

**BUT:** Appearance of \( \partial \) in spin part of \( k \) violates gauge invariance!
Lorentz invariance of the interacting theory

Appearance of $\partial$ in $W(B, i\partial)$ (= spin part of $k$) violates gauge invariance.

- Obvious fix: Replace $\partial$ by covariant $D = \partial - igA$.

$$k = ri\partial_t - ti\partial + \frac{\Sigma \times iD}{M + \sqrt{M^2 - D^2}} + \mathcal{O}(G_{\mu\nu})^*$$

- Spoils the PCR, but still describes a Lorentz invariant theory!
- Skip proof, most important ingredients:
  1. $\mathcal{L}$ is already invariant under translations and rotations.
  2. $k$ Reduces to free from in the free case ($g \to 0$).
  3. $k$ has the form $k = -tp + \ldots$.

Lorentz invariance

1. The charges obey the Poincaré algebra in the interacting theory.
2. $S$ matrix is Lorentz invariant.
How to build a Lorentz invariant theory

\[ \mathcal{L} \sim \bar{\phi}_v \left\{ \ldots v^\mu \ldots D^\mu \ldots \gamma^\mu \ldots \epsilon_{\mu \nu \alpha \beta} \ldots \right\} \phi_v \]

1. For generalized rotations \( W(R, i\partial) = R \)
   \( \rightarrow \mathcal{L} \) is invariant, since \( R v = v \).

2. For boosts with \( B v = v - q/M + \mathcal{O}(q^2) \)

Demand invariance of \( \mathcal{L} \) under boosts

\[ \phi_v(x) \rightarrow W(B, iD)\phi_v(x') \]

\[ \partial^\mu \rightarrow B^\mu_\nu \partial^\nu \quad \text{and} \quad A^\mu(x) \rightarrow B^\mu_\nu A^\nu(x') \]
Connection of Lorentz invariance to RPI

Demand invariance of $\mathcal{L}$ under boosts

$$
\phi_\nu(x) \rightarrow W(B, iD)\phi_\nu(x')
$$

$$
\partial^\mu \rightarrow B^\mu_\nu \partial^\nu \quad \text{and} \quad A^\mu(x) \rightarrow B^\mu_\nu A^\nu(x')
$$

$$
\nu^\mu \rightarrow \nu^\mu
$$
Demand invariance of $\mathcal{L}$ under boosts

$$\phi_v(x) \rightarrow \mathcal{B} \mathcal{B}^{-1} \mathcal{W}(\mathcal{B}, i\mathcal{D}) \phi_v(x')$$

$$\partial^\mu \rightarrow \mathcal{B}^\mu_\nu \partial'^\nu \quad \text{and} \quad A^\mu(x) \rightarrow \mathcal{B}^\mu_\nu A^\nu(x')$$

$$v^\mu \rightarrow \mathcal{B} \mathcal{B}^{-1} v^\mu$$
Connection of Lorentz invariance to RPI

Demand invariance of $\mathcal{L}$ under boosts

$$\phi_v(x) \rightarrow B B^{-1} W(B, iD) \phi_v(x')$$

$$\partial^\mu \rightarrow B_{\nu}^\mu \partial^{\nu} \quad \text{and} \quad A^\mu(x) \rightarrow B_{\nu}^\mu A^{\nu}(x')$$

$$\nu^\mu \rightarrow B B^{-1} \nu^\mu$$

This transformation is equivalent to

**Lorentz invariance = RPI**

$$\phi_v(x) \rightarrow B^{-1} W(B, iD) \phi_v(x), \quad \text{and} \quad \nu \rightarrow B^{-1} \nu = \nu + q/M$$

However, $\nu$ is fixed! This is just a mathematical equivalence.
For a spin 1/2 spinor:

\[ \mathcal{B}^{-1} W(\mathcal{B}, iD) = \left\{ 1 + \frac{q}{2M} \right\} \left[ 1 + \frac{\sigma_{\alpha\beta} q^\alpha D^\beta}{2M(M + i(\nu \cdot D))} \right] \]

Together with \( \Phi_\nu \rightarrow e^{iM \nu \cdot x} \Phi_\nu \), one obtains the RPI transformation.
Obtaining coefficient constraints

- Can implement Lorentz invariance on the field level, and obtain relations between coefficients, e.g. at order $1/M^3$:

$$\delta L_3 = \psi^\dagger \left[ \frac{e}{8} c_D [D_t, q \cdot E] + \frac{e}{8} (c_F - c_D + 2c_M) q \cdot [\partial \times B] + \frac{i}{4} (c_2 - c_4) \{q \cdot D, D^2\} \right. $$

$$+ \frac{ie}{8} c_S \{D_t, \sigma \times q \cdot E\} + \frac{ie}{8} (c_2 + 2c_F - c_S - 2c_{W1} + 2c_{W2}) \{q \cdot D, \sigma \cdot B\}$$

$$+ \frac{ie}{8} (-c_2 + c_F - c_{p'p}) \{\sigma \cdot D, q \cdot B\} + \frac{ie}{8} (-c_F + c_S - c_{p'p}) q \cdot \sigma (D \cdot B + B \cdot D) \bigg] \psi.$$ 

- Lower order relations: $c_2 = 1$, $c_S = 2c_F - 1$

- Field redefinition to change $W(B, iD)$:

$$\psi(x) \rightarrow e^{-iq \cdot x} \left\{ 1 + \frac{iq \cdot D}{2M^2} - \frac{\sigma \times q \cdot D}{4M^2} + \frac{ic_D}{8M^3} eq \cdot E + \frac{c_S}{8M^3} eq \cdot \sigma \times E + \ldots \right\} \psi(B^{-1}x)$$

⇒ We get a vanishing variation iff

$$c_4 = 1, \quad 2c_M = c_D - c_F, \quad c_{W2} = c_{W1} - 1, \quad c_{p'p} = c_F - 1$$

→ Does this work to any order? (Is there an invariant Lagrangian to all orders in $1/M$?)
Invariant operator method

- Learn from RPI

Find a redefinition $\Gamma(\nu, iD)$

\[ \psi_\nu := \Gamma(\nu, iD)\psi_\nu \quad \xrightarrow{\mathcal{B}} \quad \mathcal{B} e^{iq \cdot x} \psi_\nu \]

- The field $\psi_\nu := \Gamma(\nu, iD)\psi_\nu$ is invariant under Lorentz, iff

Invariance Equation

\[ \Gamma(\nu + q/M, iD - q) \ast \mathcal{B}^{-1} W(\mathcal{B}, iD + M\nu) = \Gamma(\nu, iD). \]

→ Build invariants using $\gamma^\mu$ and $\nu^\mu = \nu^\mu + iD^\mu / M$:

  e.g. $\bar{\psi}_\nu (i\slashed{\partial} + M\slashed{\gamma})\psi_\nu$, $\bar{\psi}_\nu i\sigma^{\mu\nu} [D_\mu, D_\nu] \psi_\nu$, …
Solving the invariance equation - free case

Invariance Equation

\[ \Gamma ( \nu + q/M, iD - q) \ast B^{-1} W(B, iD + M\nu) = \Gamma (\nu, iD). \]

- Note that in the free case \( (D \rightarrow \partial) \) we can write

\[
B^{-1} W(B, i\partial + M\nu) = \Lambda (\hat{V}_{\text{free}}, \nu + q/M)^{-1} \ast \Lambda (\hat{V}_{\text{free}}, \nu) \\
= 1 + \frac{\partial}{2M} + \frac{1}{4M^2} \sigma_{\perp}^{\mu\nu} q_{\mu} \partial_{\nu} \left[ 1 - \frac{i\nu \cdot \partial}{M} \right] + \ldots
\]

Comparing to the invariance equation one can see:

Closed, all-order solution

\[ \Gamma (\nu, i\partial) = \Lambda (\hat{V}_{\text{free}}, \nu) = 1 + \frac{i\partial_{\perp}}{2M} + \frac{1}{M^2} \left[ -\frac{1}{8} (i\partial_{\perp})^2 - \frac{1}{2} i\partial_{\perp} i\nu \cdot \partial \right] + \ldots \]
Solution to the invariance equation - interacting case

Why not proceed as before? Just replace \( \partial \rightarrow D \)

"RPI invariant operator method"

\[
\Gamma_{\text{naive}}(\nu, iD) = 1 + \frac{i\mathcal{D}_{\perp}}{2M} + \frac{1}{M^2} \left[ -\frac{1}{8}(iD_{\perp})^2 - \frac{1}{2}i\mathcal{D}_{\perp}i\nu \cdot D \right] \\
+ \frac{1}{M^3} \left[ \frac{1}{4}(iD_{\perp})^2i\nu \cdot D + \frac{i\mathcal{D}_{\perp}}{2} \left( -\frac{3}{8}(iD_{\perp})^2 + (i\nu \cdot D)^2 \right) \right] + \mathcal{O}(1/M^4)
\]

Unfortunately, the interacting case is more complicated!

\( \rightarrow \) This is NOT a solution to the invariance equation, starting at \( 1/M^3 \).

(Cannot get \( G_{\mu\nu} \sim [D_{\mu}, D_{\nu}] \) terms from \( \partial \rightarrow D \))

\( \Rightarrow \) The "RPI invariant operator method" starts to fail to build a Lorentz invariant Lagrangian at order \( 1/M^4 \)!

One needs to solve the invariance equation explicitly (order-by-order either in pedestrian way or systematically)
Invariance Equation

\[ \Gamma(v + q/M, iD - q)X(v, iD) = \Gamma(v, iD) \].

- The systematic solution shows, that we may choose

\[ X(v, iD) = 1 + \frac{q}{2M} \frac{1}{M^2} \sigma_{\mu\nu} q^\mu D^\nu \left(1 - \frac{i v \cdot D}{M}\right) + \ldots \]

- Expanding \( \Gamma \) in powers of \( 1/M \)

\[ \Gamma = 1 + \frac{1}{M} \Gamma^{(1)} + \frac{1}{M^2} \Gamma^{(2)} + \frac{1}{M^3} \Gamma^{(3)} + \ldots , \]

- Find the solution order by order (remove terms by field redef.)

\[ \Gamma^{(1)} = \frac{1}{2} i \slashed{D}_\perp , \quad \Gamma^{(2)} = -\frac{1}{8} (iD_\perp)^2 - \frac{1}{2} i \slashed{D}_\perp iv \cdot D \]

\[ \Gamma^{(3)} = \frac{1}{4} (iD_\perp)^2 iv \cdot D + \frac{i \slashed{D}_\perp}{2} \left[ -\frac{3}{8} (iD_\perp)^2 + (iv \cdot D)^2 \right] - \frac{g}{8} G_{\mu\nu} v^\mu D_\perp^\nu - \frac{g}{16} \sigma_{\mu\nu} G_{\mu\nu} i \slashed{D}_\perp \]

→ Last two terms are essential in solving the invariance equation
Relations for the coefficients

- Methods differ at $1/M^3$ in $\Gamma(\nu, iD)$
- Coefficient relations are different at order $1/M^4$
- E.g. NRQED at order $1/M^4$ \[ \text{[Hill, Lee, Paz, Solon (in progress)]} \]

\[ \mathcal{L} \sim \psi \left\{ iD_t + c_F e \frac{\sigma \cdot B}{2M} + c_D e \frac{[\partial \cdot E]}{8M^2} - c_{A2} e^2 \frac{E^2}{16M^3} + ic_{X4} e^2 \frac{\{D^i, [E \times B]\}^i}{32M^4} + \ldots \right\} \psi. \]

- Enforce Lorentz (field transformations or invariant operator method)

**correct**

\[ c_{X4} = 4c_F^2 + 4c_D - 2c_{A2} - 1 - 4c_F \]

**Using the incorrect RPI invariant operator method**

**wrong**

\[ c_{X4} = 4c_F^2 + 4c_D - 2c_{A2} - 1 - 6c_F \]
Summary and conclusion

- Can be easily extended to any spin
  - e.g. spin $3/2$ $\psi^\mu_v$ with invariant constraints $\gamma^\mu \psi^\mu_v = \psi^\mu_v$ and $\gamma_\mu \psi^\mu_v = 0$, and self-conjugate fields
  - e.g. a Majorana spinor $\psi_M = \psi^c_M = C \psi^*_M$, and less trivially: Massless particles (little group is $\mathbb{E}_2$).
  - Obtain the same as RPI in SCET. [Manohar, Mehen, Pirjol, Stewart ’02]

⇒ Applications to HQETs, dark matter EFTs, atomic bound states, ...

Wigner’s little group allows the construction of Lorentz invariant EFTs by transformations on the field level.

- Clears up the connection between Lorentz and RPI.
- The “RPI invariant operator method” fails at $1/M^4$ in $\mathcal{L}$.
- No need to refer to UV theory, which might be unknown (dark matter) or not even exist (protons).
...(almost) the end of a long day

Thanks for your attention
Appendix: Lorentz invariance of the $S$ matrix
Lorentz invariance of the interacting theory

Condition for a Lorentz invariant theory

Lorentz invariance of the $S$ matrix $\iff S$ commutes with $H_0, P_0, J_0$ and $K_0$.

- Note, the free charges $H_0, P_0, J_0, K_0$ obey the PCR, since the generators $h_0, p_0, j_0, h_0$ do.
- The charges in the interacting theory are $(H, P, J, K)$:
  - $H = H_0 + V$, while $P = P_0$ and $J = J_0$

$\Rightarrow$ Thus $[K^i, P^j] = -i\delta^{ij} H$ implies $K \neq K_0$!

- How do we find the right form of the generator $k$?
  I claim we already did.

Lorentz invariance of $\mathcal{L}$

Demand invariance under rotations & $\phi_v(x) \rightarrow W(B, iD)\phi_v(x' = B^{-1}x)$

$\rightarrow$ Now prove the Lorentz invariance of the $S$ matrix.
Lorentz invariance of the $S$ matrix - II

Follow the standard proof of Lorentz invariance for the $S$ matrix [e.g. Weinberg Vol. I]

- Invariance of the $S$ matrix $\Leftrightarrow S$ commutes with $H_0, P_0, J_0$ and $K_0$

Need to show two things

1. $[P_0, V] = [J_0, V] = 0$
2. $[K^i, H] = -iP^i$, with “smooth” $\Delta K = K - K_0$.

Then

- All free Lorentz charges commute with the $S$ matrix

$\Rightarrow S$ matrix is Lorentz invariant.

- In addition, charges in the interacting theory are similarity transforms of the free ones, e.g. $K\Omega = \Omega K_0$ with $\Omega \Phi_0 = \Phi_{\text{int}}$.

$\Rightarrow (H, P, J, K)$ obey the Poincaré algebra.
to 1) \([P_0, V] = 0: \mathcal{L}\) is invariant under translations ✓
\([J_0, V] = 0: \mathcal{L}\) is invariant under rotations ✓

to 2) \(k\) is of the form

\[
k = ri\partial_t - ti\partial + \frac{\Sigma \times iD}{M + \sqrt{M^2 - D^2}} + \mathcal{O}(g)^*
\]

Thus the conserved charge has the form

\[
K = -tP + (...)
\]

with no explicit time dependence in the dots.

⇒ In the Heisenberg picture this then yields

\[
0 = \frac{d}{dt}K = \frac{\partial}{\partial t}K + i[H, K] = -P + i[H, K]
\]

hence \([H, K^i] = -iP^i\). ✓
Appendix: Application to massless particles
For massless particles: SCET

Soft-Collinear-Effective theory (SCET)

- Build an effective Lagrangian for massless quarks, separating hard, soft and collinear/anti-collinear degrees of freedom.

⇒ There are two fixed light-like vectors in the Lagrangian:
  - Collinear: \( n = (1, \vec{n}) \)
  - Anti-collinear \( \bar{n} = (1, -\vec{n}) \)

The Little Group for a null vector \( n \)

\[ E_2 = \text{Translations and rotations in 2d} \]

→ Every \( W \) can be brought into the form \( W = S(\alpha, \beta)R(\theta) \)
  - \( S(\alpha, \beta) \) must act trivially on physical states: \( S(\alpha, \beta)\Phi_k = \Phi_k \)

\[ U(\Lambda)\Phi_k \sim e^{i\theta(\Lambda, n)}\Phi_{\Lambda k} \]
For massless particles: SCET - II

- Calculate \( W(\Lambda, p) = L(\Lambda p)^{-1} \Lambda L(p) \) for:
  1. \([n \to n, \bar{n} \to \bar{n}]\) Rotation: 1 dof,
  2. \([n \to n, \bar{n} \to \bar{n} + s_\perp]\) Parabolic LT: 2 dof,
  3. \([n \to n + t_\perp, \bar{n} \to \bar{n}]\) Parabolic LT: 2 dof,
  4. \([n \to (1 + \eta)n, \bar{n} \to (1 - \eta)\bar{n}]\) Boost: 1 dof.

- Decompose \( W = S(\alpha_{\Lambda,k}, \beta_{\Lambda,k})R(\theta_{\Lambda,k}) \)
  \( \to S(\alpha_{\Lambda,k}, \beta_{\Lambda,k}) \) gives a constraint, \( R(\theta_{\Lambda,k}) \) the transformation.

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<td>( W = \Lambda = R(\theta_{\Lambda,k}) )</td>
<td>( W = S(\alpha_{\Lambda,k}, \beta_{\Lambda,k})R(\theta_{\Lambda,k}) )</td>
<td>( W \equiv 1 )</td>
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For massless particles: SCET - III

- In the non-trivial case, one obtains the constraints
  - Scalar $\phi$: -
  - Spinor $\xi^\alpha$: $\phi \xi = 0 \Leftrightarrow \phi \bar{\eta} \xi = \xi$
  - Vector $A^\mu$: Does not exist/reduces to scalar!
    As it must, $\exists$ only gauge theories for vectors!
  - $T_{\mu \nu}$: Decomposes into a scalar $T = T^\mu_\mu$ and $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.
  - ...

- The corresponding transformation for $\xi$ is

$$\xi \rightarrow \left(1 + \frac{1}{2} \frac{\$}{\bar{n} \cdot \Phi} \right) \xi$$

This agrees with the one previously obtained from RPI in SCET.
[Manohar, Mehen, Pirjol, Stewart ’02]
Appendix: How to solve the invariance equation
Solving the invariance equation

\[ \Gamma(v + q/M, iD - q)B^{-1}W(B, iD + Mv) = \Gamma(v, iD) \]

- Expand everything in orders of M

\[ X \equiv B^{-1}W = 1 + q^\mu X_\mu = 1 + q^\mu \left[ \frac{1}{M} X_\mu^{(1)} + \frac{1}{M^2} X_\mu^{(2)} + \ldots \right], \]

\[ \Gamma = 1 + \frac{1}{M} \Gamma^{(1)} + \frac{1}{M^2} \Gamma^{(2)} + \ldots. \]

- In \( \Gamma \), find term linear in \( q \)

\[ \Gamma(v + q/M, iD - q) - \Gamma(v, iD) = q^\mu \left( - \frac{\partial}{\partial iD^\mu} \Gamma + \frac{1}{M} \frac{\partial}{\partial v^\mu} \Gamma \right) + \ldots \]

Need to solve

\[ \frac{\partial}{\partial iD^\mu} \Gamma^{(n)} = \frac{\partial}{\partial v^\mu} \Gamma^{(n-1)} + \Gamma^{(n-1)} X_\mu^{(1)} + \Gamma^{(n-2)} X_\mu^{(2)} + \ldots + \Gamma^{(0)} X_\mu^{(n)} \equiv Y_\mu^{(n)} \]

- Looks like \( \nabla \phi = E \) in electrodynamics.
Solving the invariance equation - II

Solution for \( \Gamma^{(n)} \)

\[
\Gamma^{(n)} = \imath D_{\mu} Y_{\mu}^{(n)} - \frac{1}{2!} \imath D_{\mu} iD_{\nu} \frac{\partial}{\partial iD_{\mu}} Y_{\nu}^{(n)} + \ldots
\]

- In EM: Need \( \nabla \times \mathbf{E} = 0 \) to solve \( \nabla \phi = \mathbf{E} \).
  
Here:

**Constraint for \( Y^{(n)} \)**

\[
\frac{\partial}{\partial iD_{\nu}} Y_{\mu}^{(n)} = 0
\]

- Let’s solve the constraint for \( Y^{(n)} \).
Solving the invariance equation - III

- Plug in definition of $Y$ and solution for lower order $\Gamma$’s

$$\Rightarrow X = B^{-1} \mathcal{W}$$ must obey a constraint.

**Constraint for $X^{(n)}$**

$$\frac{\partial}{\partial D[\nu]} X^{(n)}_{\mu} = - \frac{\partial}{\partial \nu[\mu]} X^{(n-1)}_{\nu} + X^{(n-1)}_{[\mu} X^{(1)}_{\nu]} + X^{(n-2)}_{[\mu} X^{(2)}_{\nu]} + \cdots + X^{(1)}_{[\mu} X^{(n-1)}_{\nu]} \equiv Z^{(n)}_{\mu\nu}.$$

- Again EM: This looks like $\nabla \times A = B$.

$$\rightarrow B$$ needs to obey $\nabla \cdot B = 0$.

**Constraint for $Z^{(n)}$**

$$0 = \epsilon^{\mu\nu\rho\sigma} \frac{\partial}{\partial D[\rho]} Z^{(n)}_{\mu\nu}.$$
Can show that constraint for $Z^{(n)}$ is obeyed by induction, i.e. if we have $X^{(1)}, \ldots, X^{(n-1)}$ obeying all constraints, then so does $Z^{(n)}$.

⇒ We can “solve” for $X^{(n)}$, but what does that mean?

Before: we can add terms to $X = B^{-1} W$ that vanish in the free case.

→ Constraint for $X^{(n)}$: we must add some such terms to solve for $\Gamma$.

**Solution for $X^{(n)}$**

$$X^{(n)}_{\mu} = \hat{X}^{(n)}_{\mu} + 2 \sum_{m=1}^{n-1} \frac{(-1)^m}{(m+1)!} iD^{\nu_1}_{\mu} \cdots iD^{\nu_m}_{\mu} \frac{\partial}{\partial iD^{\nu_1}_{\nu_1}} \cdots \frac{\partial}{\partial iD^{\nu_{m-1}}_{\nu_{m-1}}} \left( Z^{(n)}_{\nu m \mu} - \hat{Z}^{(n)}_{\nu m \mu} \right)$$

where $\hat{X}^{(n)} = \text{naive covariantization of given } X(i\partial)$, and $\hat{Z}$ corresponding to it.

Now use this to solve for $Y$ and $\Gamma$.
Since $Z^{(n)}$ has mass dimension $n - 2$, the first field strength dependent terms can appear at $n = 4$.

⇒ We find

\[
X^{(1)}_{\mu} = \frac{\gamma^\perp_{\mu}}{2} = \hat{X}^{(1)}_{\mu} \\
X^{(2)}_{\mu} = \frac{1}{4} \sigma^\perp_{\mu\nu} D^\nu = \hat{X}^{(2)}_{\mu} \\
X^{(3)}_{\mu} = -\frac{1}{4} \sigma^\perp_{\mu\nu} D^\nu \mathbf{i} \mathbf{v} \cdot D = \hat{X}^{(3)}_{\mu}, \\
X^{(4)}_{\mu} = \sigma^\perp_{\mu\nu} D^\nu \left[ \frac{1}{4} (\mathbf{i} \mathbf{v} \cdot D)^2 - \frac{1}{16} (iD^\perp)^2 \right] + \frac{g}{32} iD^\perp_\nu \left( -iG^\perp_{\mu\nu} + \sigma^\perp_{\mu\sigma} G^\perp_\nu^\sigma - \sigma^\perp_\nu^\sigma G^\perp_{\mu\sigma} \right) = \hat{X}^{(4)}_{\mu}
\]

⇒ Find extra terms, which are needed to obtain the correct $\Gamma (\nu, iD)$. 