Transverse momentum-dependent parton distribution functions
from lattice QCD

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“What is the probability of finding a quark with a given momentum $k$ in a nucleon?”

Light cone coordinates:

$$w^\pm = \frac{1}{\sqrt{2}}(w^0 \pm w^3)$$

Nucleon with large momentum along 3-axis: $P^+ \text{ large, } P_T = 0$

Quark momentum components: $k^+ \sim P^+ / m_N$, $k_T \sim 1$, $k^- \sim m_N / P^+$

Ask for distribution of quarks $f(x, k_T)$

- longitudinal momentum fraction $x = k^+ / P^+$
- transverse momentum $k_T$
Definition of TMDs

Heuristically,

\[ \Phi[\Gamma](x, k_T, P, S, \ldots) \equiv \int dk^- \langle P, S| \bar{q}(k) \Gamma q(k) |P, S\rangle |_{k^+ = xP^+} \]

Decompose in terms of TMDs, for example

\[ \Phi[\gamma^+](x, k_T, P, S, \ldots) = f_1(x, k_T^2, \ldots) - \frac{\epsilon_{ij} k_i S_j}{m_N} f_{1T}^+(x, k_T^2, \ldots) \]
Definition of TMDs

More precisely, in terms of local operators,

\[
\Phi^{[\Gamma]}(x, k_T, P, S, \ldots) \equiv \int \frac{d^2 b_T}{(2\pi)^2} \int \frac{d(b \cdot P)}{(2\pi)P^+} \exp \left( i x (b \cdot P) - i b_T \cdot k_T \right) \Phi^{[\Gamma]}_{\text{unsubtr.}}(b, P, S, \ldots) \left|_{b^+ = 0} \right. 
\]

\[
\Phi^{[\Gamma]}_{\text{unsubtr.}}(b, P, S, \ldots) \equiv \frac{1}{2} \langle P, S | \bar{q}(0) \Gamma U[0, \ldots, b] q(b) | P, S \rangle 
\]

- "Soft factor" \( \mathcal{S} \) required to subtract divergences of Wilson line \( U \)
- \( \mathcal{S} \) is typically a combination of vacuum expectation values of Wilson line structures
- Here, will consider only ratios in which soft factors cancel
Definition of TMDs

All leading twist structures:

\[
\Phi[\gamma^+] = f_1 - \left[ \frac{\epsilon_{ij} k_i S_j}{m_N} f_{1T}^\perp \right]_{\text{odd}}
\]

\[
\Phi[\gamma^+ \gamma^5] = \Lambda g_1 + \frac{k_T \cdot S_T}{m_N} g_{1T}
\]

\[
\Phi[i\sigma^i + \gamma^5] = S_i h_1 + \frac{(2k_i k_j - k^2_T \delta_{ij}) S_j}{2m_N^2} h_{1T}^\perp + \frac{\Lambda k_i}{m_N} h_{1L}^\perp + \left[ \frac{\epsilon_{ij} k_j}{m_N} h_{1T}^\perp \right]_{\text{odd}}
\]
### Definition of TMDs

All leading twist structures:

<table>
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<tr>
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$\downarrow$ Sivers (T-odd)

$\leftarrow$ Boer-Mulders (T-odd)
Fourier-transformed TMDs

\[ \tilde{f}(x, b_T^2, \ldots) \equiv \int d^2 k_T \exp(i b_T \cdot k_T) f(x, k_T^2, \ldots) \]

\[ \tilde{f}^{(n)}(x, b_T^2, \ldots) \equiv n! \left( -\frac{2}{m_N^2} \partial_{b_T^2} \right)^n \tilde{f}(x, b_T^2, \ldots) \]

In limit \(|b_T| \to 0\), recover \(k_T\)-moments:

\[ \tilde{f}^{(n)}(x, 0, \ldots) \equiv \int d^2 k_T \left( \frac{k_T^2}{2 m_N^2} \right)^n f(x, k_T^2, \ldots) \equiv f^{(n)}(x) \]

ill-defined for large \(k_T\), so will not attempt to extrapolate to \(b_T = 0\), but give results at finite \(|b_T|\).

Also, we can only access limited range of \(b \cdot P\), so cannot Fourier-transform to obtain \(x\)-dependence. Therefore, consider only first \(x\)-moments (accessible at \(b \cdot P = 0\)):

\[ f^{[1]}(k_T^2, \ldots) \equiv \int_{-1}^{1} dx \ f(x, k_T^2, \ldots) \]
Relation to physical processes

Context: All this is largely academic if we can’t connect it to a physical measurement.

Factorization framework which allows one to separate cross section into hard amplitude, fragmentation function, TMD?

In general, no! (e.g., processes with multiple hadrons in both initial and final state)

Factorization arguments have been given only for selected processes (SIDIS, Drell-Yan).

Physical process should also inform appropriate choice of gauge link $U[0, \ldots, b]$
SIDIS (Semi-inclusive deep inelastic scattering)

\[ l + N(P) \rightarrow l' + h(P_h) + X \]

Gauge link structure:

In matrix element \( \Phi_{\text{unsubtr.}}(b, P, S, \ldots) \equiv \frac{1}{2} \langle P, S | \bar{q}(0) \Gamma \mathcal{U}[0, \ldots, b] q(b) | P, S \rangle \)

Staple-shaped gauge link \( \mathcal{U}[0, \eta v, \eta v + b, b] \)

incorporates SIDIS final state effects
Relation to physical processes

Staple-shaped links incorporate SIDIS final state effects:

• Gauge link roughly follows direction of ejected quark, (close to) light cone
• Effective resummed description of gluon exchanges between ejected quark and remainder of nucleon in evolving final state
• Beyond tree level: Rapidity divergences force taking staple direction slightly off the light cone. Approach of Aybat, Collins, Qiu, Rogers makes $v$ space-like. Parametrize in terms of Collins-Soper parameter

$$\hat{\zeta} \equiv \frac{P \cdot v}{|P||v|}$$

Light-like staple for $\hat{\zeta} \to \infty$. Perturbative evolution equations for large $\hat{\zeta}$.

• In this approach, have “modified universality”, $f^{T\text{-odd}}$, SIDIS = $-f^{T\text{-odd}}$, DY (initial state interactions in DY case). SIDIS: $\eta v \cdot P \to \infty$, DY: $\eta v \cdot P \to -\infty$.

Without initial/final state effects, T-odd Sivers and Boer-Mulders functions would vanish!
Invariant amplitudes

Return to correlator

\[ \Phi_{\text{unsubtr.}}^{[\Gamma]}(b, P, S, \hat{\zeta}, \mu) \equiv \frac{1}{2} \langle P, S| \bar{q}(0) \Gamma U[0, \eta v, \eta v + b, b] q(b) |P, S\rangle \]

Decompose in terms of invariant amplitudes; at leading twist,

\[ \frac{1}{2P^+} \Phi_{\text{unsubtr.}}^{[\gamma^+]} = \bar{A}_{2B} + im_N \epsilon_{ij} b_i S_j \bar{A}_{12B} \]

\[ \frac{1}{2P^+} \Phi_{\text{unsubtr.}}^{[\gamma^+\gamma^5]} = -\Lambda \bar{A}_{6B} + i[(b \cdot P)\Lambda - m_N (b_T \cdot S_T)] \bar{A}_{7B} \]

\[ \frac{1}{2P^+} \Phi_{\text{unsubtr.}}^{[i\sigma^i + \gamma^5]} = im_N \epsilon_{ij} b_j \bar{A}_{4B} - S_i \bar{A}_{9B} \]

\[ -im_N \Lambda b_i \bar{A}_{10B} + m_N[(b \cdot P)\Lambda - m_N (b_T \cdot S_T)] b_i \bar{A}_{11B} \]
Invariant amplitudes and TMDs

Conversely, invariant amplitudes directly give selected $x$-integrated TMDs in Fourier ($b_T$) space (showing just the ones relevant for Sivers, Boer-Mulders shifts):

\[
\tilde{f}_1^{[1](0)}(b_T^2, \hat{\zeta}, \ldots, \eta v \cdot P) = 2\tilde{A}_2B(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P)/\tilde{S}(b^2, \ldots)
\]

\[
\tilde{f}_{1T}^{[1](1)}(b_T^2, \hat{\zeta}, \ldots, \eta v \cdot P) = -2\tilde{A}_{12}B(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P)/\tilde{S}(b^2, \ldots)
\]

\[
\tilde{h}_1^{[1](1)}(b_T^2, \hat{\zeta}, \ldots, \eta v \cdot P) = 2\tilde{A}_4B(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P)/\tilde{S}(b^2, \ldots)
\]
**Generalized shifts**

Form ratios in which soft factors, ($\Gamma$-independent) multiplicative renormalization factors cancel

**Sivers shift:**

$$\langle k_y \rangle_{TU} \equiv m_N \frac{f_{1T}^{[1](1)}}{f_{1}^{[1](0)}} = \frac{\int dx \int d^2 k_T k_y \Phi^{[\gamma^+]}(x, k_T, S_T = (1, 0))}{\int dx \int d^2 k_T \Phi^{[\gamma^+]}(x, k_T, S_T = (1, 0))}$$

Average transverse momentum of unpolarized ("U") quarks orthogonal to the transverse ("T") spin of nucleon; normalized to the number of valence quarks. "Dipole moment" in $b_T^2 = 0$ limit, "shift".

**Issue:** $k_T$-moments in this ratio singular; generalize to ratio of Fourier-transformed TMDs at nonzero $b_T^2$,

$$\langle k_y \rangle_{TU}(b_T^2, \ldots) \equiv m_N \frac{\tilde{f}_{1T}^{[1](1)}(b_T^2, \ldots)}{\tilde{f}_{1}^{[1](0)}(b_T^2, \ldots)}$$

(remember singular $b_T \to 0$ limit corresponds to taking $k_T$-moment). “Generalized shift”.
Generalized shifts from amplitudes

Now, can also express this in terms of invariant amplitudes:

\[
\langle k_y \rangle_{TU}(b_T^2, \ldots) \equiv m_N \frac{\tilde{f}_{1T}^{[1](1)}}{f_{1}^{[1](0)}}(b_T^2, \ldots) = -m_N \frac{A_{12B}(-b_T^2, 0, \zeta, \eta v \cdot P)}{A_{2B}(-b_T^2, 0, \zeta, \eta v \cdot P)}
\]

Analogously, Boer-Mulders shift:

\[
\langle k_y \rangle_{UT}(b_T^2, \ldots) = m_N \frac{A_{4B}(-b_T^2, 0, \zeta, \eta v \cdot P)}{A_{2B}(-b_T^2, 0, \zeta, \eta v \cdot P)}
\]
Lattice setup

- Evaluate directly
  \[ \Phi_{\text{unsubtr.}}^{[\Gamma]}(b, P, S; \hat{\zeta}, \mu) \equiv \frac{1}{2} \langle P, S | \bar{q}(0) \Gamma U[0, \eta v, \eta v + b, b] q(b) | P, S \rangle \]

- Euclidean time: Place entire operator at one time slice, i.e., \( b, \eta v \) purely spatial

- Since generic \( b, v \) space-like, no obstacle to boosting system to such a frame!

- Parametrization of correlator in terms of \( \bar{A}_i \) invariants permits direct translation of results back to original frame

- Form desired ratios of \( \bar{A}_i \) invariants

- Extrapolate \( \eta \to \infty, \hat{\zeta} \to \infty \) numerically
Lattice setup

Use three MILC 2+1-flavor gauge ensembles with $a \approx 0.12$ fm:

$m_\pi = 369 \text{ MeV} ; \ 28^3 \times 64 ; \ 2184 \text{ samples}$

$m_\pi = 369 \text{ MeV} ; \ 20^3 \times 64 ; \ 5264 \text{ samples}$

$m_\pi = 518 \text{ MeV} ; \ 20^3 \times 64 ; \ 3888 \text{ samples}$

Sink momenta $P$: $(0, 0, 0), (-1, 0, 0), (-2, 0, 0), (1, -1, 0)$

Variety of $b, \eta v$; note $b \perp P, b \perp v$ (lowest $x$-moment, kinematical choices/constraints)

Largest $\hat{\zeta} = 0.78$
Results: Sivers shift

Dependence on staple extent; sequence of panels at different $|b_T|$
Results: Sivers shift

Dependence on staple extent; sequence of panels at different $|b_T|$

\[ \zeta = 0.39, \]
\[ |b_T| = 0.24 \text{ fm}, \]
\[ m_{N} = 518 \text{ MeV} \]
Results: Sivers shift

Dependence on staple extent; sequence of panels at different $|b_T|$
**Results: Sivers shift**

Dependence on staple extent; sequence of panels at different $|b_T|$
Results: Sivers shift

Dependence of SIDIS limit on $|b_T|$
Results: Sivers shift

Dependence on staple extent; sequence of panels at different $\hat{\zeta}$

\begin{align*}
\hat{f}_{1T}^{(I)(Q)} & \approx f_{1T}^{(I)(Q)} \\
\zeta & = 0.39, \\
|b_{T}| & = 0.36 \text{ fm}, \\
m_{\pi} & = 518 \text{ MeV}
\end{align*}
Results: Sivers shift

Dependence on staple extent; sequence of panels at different $\hat{\zeta}$.

![Graph showing Sivers shift, $u$–$d$ quarks]

- $\hat{\zeta} = 0.78$
- $|b_T| = 0.36$ fm
- $m_\pi = 518$ MeV

Label: Sivers–Shift, $u$–$d$ quarks

Axis labels:
- $m_N f_{1T}$ (GeV)
- $\eta |v|$ (lattice units)
Results: Sivers shift

Dependence of SIDIS limit on $\hat{\zeta}$

$Sivers Shift_{LParen1SIDISRParen1}, u - d \quad quarks$
Results: Sivers shift

dependence of SIDIS limit on $\zeta$, all three ensembles

\[ \zeta/m_\Pi = \text{Equal } 369 \text{ MeV} \]

\[ \zeta/m_\Pi = \text{Equal } 518 \text{ MeV} \]

$\alpha_{\text{Sivers}} = |b_T| = 0.36 \text{ fm}$
Results: Boer-Mulders shift

Dependence on staple extent

\[ \hat{\tau} = 0.39, \quad |b_T| = 0.36 \text{ fm}, \quad m_\pi = 518 \text{ MeV} \]
Results: Boer-Mulders shift

Dependence of SIDIS limit on $|b_T|$
Results: Boer-Mulders shift

Dependence of SIDIS limit on $\hat{\zeta}$

$\text{Boer-Mulders Shift (SIDIS), } u-d \text{ – quarks}$

| $|b_T|$ | $m_\pi$ |
|-------|---------|
| 0.36 fm | 518 MeV |

$m_{N, h_1}^{\gamma}(t)/f_1^{(0)}$ (GeV)

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<td>1.4</td>
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</table>

$\hat{\zeta}$
Results: Boer-Mulders shift

Dependence of SIDIS limit on $\hat{\zeta}$, all three ensembles

\[ \frac{m_N h_1^{I-0|I-1|0}}{f_1^{I-0|I-1|0}} \] (GeV)

\[ 0.0 \quad 0.1 \]

\[ -0.3 \quad -0.2 \quad -0.1 \quad 0.0 \quad 0.1 \]

Boer–Mulders Shift (SIDIS), $u-d$ quarks

\[ |b_T| = 0.36 \text{ fm} \]

\[ m_\pi = 518 \text{ MeV} \ 20^3 \]

\[ m_\pi = 369 \text{ MeV} \ 20^3 \]

\[ m_\pi = 369 \text{ MeV} \ 28^3 \]
Conclusions

- Exploratory study of TMDs using staple-shaped gauge link structures
- Accessed T-odd Sivers, Boer-Mulders observables; SIDIS, DY limits distinguished by sign of $v \cdot P$. For u-d quark combination, SIDIS Sivers and Boer-Mulders TMDs both sizeable and negative.
- To avoid soft factors, multiplicative renormalization constants, constructed appropriate ratios of Fourier-transformed TMDs (“shifts”).
- $v$ taken off light cone: Dependence on Collins-Soper parameter $\hat{\zeta}$. In addition to $\eta v \to \infty$, need to also consider $\hat{\zeta} \to \infty$.
- $\eta v \to \infty$ seems under good control; plateaux reached at moderate values.
- $\hat{\zeta} \to \infty$ remains a challenge. No clear trends seen in the data sets available. Need much larger $\hat{\zeta}$. Presently investigating pion with this in mind.
- No significant volume dependence, pion mass dependence detected within the limited set of (three) cases considered