The meson spectrum in large $N$ QCD

Gunnar Bali with Luca Castagnini, Sara Collins (Regensburg)
Francis Bursa, Biagio Lucini (Swansea)
Luigi Del Debbio (Edinburgh)
Marco Panero (Helsinki)
- Large $N$ QCD: motivation
- Lattice simulation: details and techniques
- Quark and pion masses
- The meson spectrum
- Conclusions
Example: scalar field theory with $N$-component field $\phi^a$, $a = 1, \ldots, N$

\[ \mathcal{L} = \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{2} \mu^2 \phi^a \phi^a - \frac{1}{8} g^2 (\phi^a \phi^a)^2. \]

We define the 't Hooft coupling $\lambda = g^2 N$:

\[
\begin{align*}
g^2 &= \frac{\lambda}{N} \\
g^4 N &= \frac{\lambda^2}{N} \\
g^4 &= \frac{\lambda^2}{N^2}
\end{align*}
\]

Now we take the limit $g^2 \rightarrow 0$ and $N \rightarrow \infty$ at fixed $\lambda$ ('t Hooft limit). Obviously, this leads to simplifications!
Large $N$ QCD

\[ A_\mu \mapsto \sqrt{N} A_\mu , \quad \psi \mapsto \sqrt{N} \psi . \]

SU($N$) Lagrangian:

\[ \mathcal{L} = N \left[ \frac{1}{4\lambda} F_{\mu\nu}^a F_{\mu\nu}^a + \bar{\psi} (\not{D} + m) \psi \right] . \]

Counting rules:

- corner each vertex $\propto N$
- edge each propagator $\propto 1/N$
- face closed color loop $\propto N$

\[ \Rightarrow \langle \cdot \rangle \propto N^{V-E+F} = N^\chi \]

$\chi = V - E + F = 2 - 2h($andles$) - b($oundaries, holes$)$ is the Euler characteristic.

sphere: $h = b = 0 \Rightarrow \chi = 2$, torus: $h = 1, b = 0 \Rightarrow \chi = 0.$
Consequences of counting rules

- Only “planar” diagrams survive at large $N$.

- The leading connected vacuum diagrams are of order $N^2$ (planar graphs made of gluons only).
- The leading connected vacuum diagrams with quark lines are of order $N$.
- Corrections are suppressed by factors $1/N^2$ in the pure gauge theory and by $N_f/N$ in the theory with $N_f$ fermions.
Properties of large $N$ QCD

- Sea quark effects $\propto 1/N \Rightarrow$ The $N = \infty$ limit is quenched.
- Mixing glueballs-mesons $\propto 1/\sqrt{N} \Rightarrow$ No mixing between glueballs and mesons at $N = \infty$.
- Meson decay widths $\propto 1/N \Rightarrow$ mesons do not decay at $N = \infty$.
- OZI rule exact at $N = \infty$.

Is $N = \infty$ close to $N = 3$ QCD?

AdS/QFT starts from $N = \infty$. Also many simplifications in chiral EFT.

But $N = \infty$ QCD is far from being solved!
Light hadrons: $1/N^2 = 1/9 \approx 3/3 = N_f/N = 3/3$

If $1/9 \approx 0$ then $\exists$ evidence that $1 \approx 0$:

Full SU(3) QCD BMW-c: S Dürr et al 08
Quenched SU(3) QCD PACS-CS: S Aoki et al 02

Obviously cannot work for flavour singlets ($f_0(500), \eta', \omega$) but still …
Glueballs at large $N$

from B Lucini, A Rago, E Rinaldi 10

What about mesons?

$a^{-1} \approx 1.5 \text{ GeV}$
Lattice parameters

- Number of colours $N = 2, 3, 4, 5, 6, 7, 17$
- Volumes:

<table>
<thead>
<tr>
<th>$N$</th>
<th>vol</th>
</tr>
</thead>
<tbody>
<tr>
<td>2,3</td>
<td>$16^3 \times 32$, $24^3 \times 48$, $32^3 \times 64$</td>
</tr>
<tr>
<td>4,5,6,7</td>
<td>$24^3 \times 48$</td>
</tr>
<tr>
<td>17</td>
<td>$12^3 \times 24$</td>
</tr>
</tbody>
</table>

- 200 configs for each $N$ and volume (80 configs for $N = 17$)
- Lattice spacing $a \approx 0.093$ fm
- Pion mass as low as $m_\pi \approx 230$ MeV
- Wilson gluon and quark action
Setting the scale

- Inverse coupling $\beta = 2N/g^2 = 2N^2/\lambda$ is fixed by imposing $a\sqrt{\sigma} \approx 0.2093$ for all $SU(N)$. (Lattice spacing $a \approx 0.093$ fm is kept constant in units of the string tension $\sigma \approx 1$ GeV/fm).

- Other possible choices include $aT_c = \text{const}$, $a/r_0 = \text{const}$.

- The $\kappa$-parameter ($2am_q = \kappa^{-1} - \kappa_c^{-1}$) is adjusted so that our set of pseudoscalar masses matches between different $N$ (achieved by exploratory simulations).

Plan:

- Vary $\kappa$ to study $m_A(m_q, N)$ for each particle $A$.

- Extrapolate to $N = \infty$ and study $1/N^2$ corrections.
Setting the scale for SU(17)

Fit estimated $\Lambda_L$ parameter in $1/N^2$ for $N \leq 8 \Rightarrow \beta_{17} = 208.45$

$$a\Lambda_L = \exp\left(-\frac{1}{2b_0\alpha}\right) (b_0\alpha)^{-\frac{b_1}{2b_0}} \left(1 + \frac{1}{2b_0^3} \left(b_1^2 - b_2 b_0\right) \alpha + \ldots\right)$$

with $b_{0,1,2} = f(N)$

Other approaches are compatible within 0.3 %:
- linear fit vs. quadratic fit
- fit of $\Lambda_{\overline{MS}}$
- fit of ’t Hooft $\lambda$
Meson interpolators

- A generic meson interpolator is a bilinear:
  \[ O(x, t) = \bar{\psi}(x, t) \Gamma \psi(x, t) \]

  \( \Gamma \) can be a product of Dirac matrices and covariant derivatives.

- Zero momentum projected two point function:
  \[ C_{\Gamma, \Gamma}(t) = \sum_x \left\langle O(x, t) \bar{O}(0) \right\rangle = \mp \sum_x \text{Tr} \left\langle \Gamma \gamma_5 G(x, t) \gamma_5 \Gamma G(x, 0) \right\rangle \]

  \[ C(t) \propto e^{-mt} \quad \text{for} \quad t \to \infty \]

- We study mesons with \( m_u = m_d \):

<table>
<thead>
<tr>
<th>Particle</th>
<th>( \pi )</th>
<th>( \rho )</th>
<th>( a_0 )</th>
<th>( a_1 )</th>
<th>( b_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bilinear</td>
<td>( \bar{u} \gamma_5 d )</td>
<td>( \bar{u} \gamma_i d )</td>
<td>( \bar{ud} )</td>
<td>( \bar{u} \gamma_5 \gamma_i d )</td>
<td>( \frac{1}{2} \epsilon_{ijk} \bar{u} \gamma_i \gamma_j d )</td>
</tr>
<tr>
<td>( J^{PC} )</td>
<td>0--</td>
<td>1--</td>
<td>0++</td>
<td>1++</td>
<td>1+-</td>
</tr>
</tbody>
</table>
Variational method

- APE smearing on the links and Wuppertal smearing on sources/sinks.
- Study of the cross-correlation matrix

\[ C_{ij}(t) = \langle O_i(t) \bar{O}_j(0) \rangle \]

where \( i, j \) correspond to different numbers of smearing iterations (0, 20, 80 and 180 steps).
- Solve the generalized eigenvalue problem

\[ C(t) \mathbf{v}^\alpha = \lambda^\alpha(t) C(t_0) \mathbf{v}^\alpha \]

and extract the masses from the large \( t \) behaviour of the eigenvalues:

\[ \lambda(t) = A \left( e^{-mt} + e^{-m(N_\tau a_t)} \right) \]

- Ground and first excited states.
Axial and vector Takahashi-Ward identity masses

Partially conserved axial current (PCAC):

\[
\sum_x \partial_4 \langle 0 | A_4(x, t) | \pi \rangle = 2m_{PCAC} \sum_x \langle 0 | j_5(x, t) | \pi \rangle \\
\text{where} \quad \begin{cases} 
A_\mu(x) &= \bar{u}(x) \gamma_\mu \gamma_5 d(x) \\
j_5(x) &= \bar{u}(x) \gamma_5 d(x) \\
m_{PCAC} &= Z_P / (Z_A Z_S) m_q 
\end{cases}
\]

\[m_q = (\kappa^{-1} - \kappa_c^{-1})/(2a)\] is the vector quark mass and \(Z_i(\lambda)\) are renormalization constants.

We fit

\[
\frac{m_{PCAC}}{\sqrt{\sigma}} = A \left( \frac{1}{\kappa} - \frac{1}{\kappa_c} \right) + B \left( \frac{1}{\kappa} - \frac{1}{\kappa_c} \right)^2
\]

with dimensionless fit parameters \(A, B, \kappa_c\) that are expected to have \(O(1/N^2)\) corrections.
Determination of $\kappa_C(N)$

\[
\frac{m_{\text{PCAC}}}{\sqrt{\sigma}} = A \left( \frac{1}{\kappa} - \frac{1}{\kappa_C} \right) + B \left( \frac{1}{\kappa} - \frac{1}{\kappa_C} \right)^2, \quad \kappa_C = 0.15980(4) - 0.0275(4) \frac{1}{N^2}
\]
Pion mass vs. PCAC mass

\[
\left( \frac{m_\pi}{\sqrt{\sigma}} \right)^2 = A + B \frac{m_{PCAC}}{\sqrt{\sigma}} + C \left( \frac{m_{PCAC}}{\sqrt{\sigma}} \right)^2
\]
Pion mass: $1/N^2$ fit of the parameters

\[
\left( \frac{m_\pi}{\sqrt{\sigma}} \right)^2 = \left( -0.0139(40) + 0.706(97) \frac{1}{N^2} \right) + \left( 12.129(39) - 1.13(87) \frac{1}{N^2} \right) \frac{m_{\text{PCAC}}}{\sqrt{\sigma}} \\
+ \left( 1.904(54) - 5.712(12) \frac{1}{N^2} \right) \left( \frac{m_{\text{PCAC}}}{\sqrt{\sigma}} \right)^2
\]
Pion mass: quenched chiral logs

In the **quenched** theory the pion behaviour should be modified as,

\[
\left( \frac{m_\pi}{\sqrt{\sigma}} \right)^2 = A \left( \frac{1}{\kappa} - \frac{1}{\kappa_c} \right)^{ \frac{1}{1+\delta} } + B \left( \frac{1}{\kappa} - \frac{1}{\kappa_c} \right)^2
\]

where \( \delta \propto 1/N \) goes to zero at large \( N \) (S Sharpe PRD 46 (92) 3146)

- \( \delta \) compatible with zero for \( N \geq 5 \)
- not sensible to chiral logs at \( m_\pi > \sqrt{\sigma} \)
- linear fit to \( 3 \leq N \leq 5 \):

\[
\delta = 0.108(27) \frac{1}{N}
\]
\[ \frac{m_\rho}{\sqrt{\sigma}} = A + B \frac{m_{\text{PCAC}}}{\sqrt{\sigma}} + C \left( \frac{m_{\text{PCAC}}}{\sqrt{\sigma}} \right)^{3/2} \]
ρ: $1/N^2$ fit of the parameters

\[
\frac{m_\rho}{\sqrt{\sigma}} = \left(1.5212(76) + 0.60(12) \frac{1}{N^2}\right) + \left(3.387(63) - 5.40(10) \frac{1}{N^2}\right) \frac{m_{\text{PCAC}}}{\sqrt{\sigma}} \\
+ \left(-1.043(68) + 4.71(11) \frac{1}{N^2}\right) \left(\frac{m_{\text{PCAC}}}{\sqrt{\sigma}}\right)^{3/2}
\]
$a_1$: fit vs. PCAC mass

\[
\frac{m_{a_1}}{\sqrt{\sigma}} = A + B \frac{m_{\text{PCAC}}}{\sqrt{\sigma}}
\]
The $\rho$ mass issue

Disagreement: L Del Debbio et al JHEP 03 (08) 062, G Bali, F Bursa JHEP 09 (08) 110, T DeGrand PRD 86 (12) 034508 vs. A Hietanen et al PLB 674 (09) 80

- $N \to \infty$ from bigger lattices and smaller $N$
- $SU(17)$ volume: $12^3 \times 24$
- $\beta = 208.45 \ (b = \frac{1}{\lambda} = 0.3606)$
- Wilson fermions
- Zero momentum correlators:
  \[ C(t) = e^{-m_\rho t} \]

- $\rho$ mass in chiral limit:
  \[ m_\rho(N = 17) = 1.54(3)\sqrt{\sigma} \]

- $N \to \infty$ from smaller lattices and bigger $N$
- $SU(17)$ volume: $11^4$
- $\beta = 208.08 \ (b = 0.360)$
- Overlap fermions
- Momentum space propagator:
  \[ M_{\mu\nu} = \frac{A(p_\mu p_\nu - p^2 \delta_{\mu\nu})}{p^2 + m_\rho^2} \]

- $\rho$ mass in chiral limit:
  \[ m_\rho(N = 17) = 3.50(22)\sqrt{\sigma} \]
Meson spectrum and decay constants at $m_q = 0$
Comparison with phenomenology

\( N = \infty \) is a good starting point for studies of strong decays and mixing between different sectors: glueballs, mesons, (meson)\(^2\) etc.

Prediction for full \( N \geq 3 \) QCD from unitarized \( \chi PT \) from J Peláez, G Ríos PRL 97 (06) 242002
See also J Nieves et al, PRD 84 (11) 096002
Comparison with AdS/QFT

AdS/CFT

J Babington et al
PRD 69 (04) 066007

\[
\frac{M_\rho(M_\pi)}{M_\rho(0)} \simeq 1 + 0.307 \left[ \frac{M_\pi}{M_\rho(0)} \right]^2
\]

Holographic model

T Sakai, S Sugimoto
hep-th/0412141

\[
\frac{M_\rho^2(a_1(1260))}{M_\rho^2} \simeq 2.4
\]
\[
\frac{M_\rho^2(a_0(1450))}{M_\rho^2} \simeq 4.3
\]
\[
\frac{M_\rho^2(1450)}{M_\rho^2} \simeq 4.9
\]

Our lattice results

\[
\frac{M_\rho(M_\pi)}{M_\rho(0)} \simeq 1 + 0.369(4) \left[ \frac{M_\pi}{M_\rho(0)} \right]^2
\]

\[
\frac{M_\rho^2(a_1(1260))}{M_\rho^2} \simeq 3.3
\]
\[
\frac{M_\rho^2(1450)}{M_\rho^2} \simeq 5.4
\]
\[
\frac{M_\rho^2(a_0(1450))}{M_\rho^2} \simeq 7.5
\]
Conclusions

- We computed the quenched meson spectrum of SU($N$) for degenerate quark masses.
- The isovector SU(3) masses as well as decay constants are close to the $N = \infty$ limit. Even $N = 2$ is described on a qualitative level.
- $1/N^2$ corrections are small.
- Differences smaller than 10% at $N = 3$ between the $N_f = 2 + 1$ theory and the quenched approximation indicate that $N_f/N$ corrections may also be small.
- Results may help improving AdS backgrounds or holographic models.
- Results may constrain large $N$ based models and EFTs of low energy strong interactions. Impact on the phenomenology of light scalars?