Definition and Evolution of TMDs at NNLL

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Introduction; Questions..

- Transverse Momentum distributions are fundamental in the factorization of DY at small qT and SIDIS
- Can we formulate their definition independently of the IR/collinear regulators that we use?
- How do we write the evolution of TMDs? Up to which order do we know their evolution?
- Are TMDs universal?
- Is the evolution of all quark TMDs the same?
- Can we have a model independent evolution of the TMDs?
Transverse Momentum distributions are fundamental in the factorization of DY at small qT and SIDIS
Can we formulate their definition independently of the IR/collinear regulators that we use? **YES**
How do we write the evolution of TMDs? Up to which order do we know their evolution? **We can up to NNLL**
Are TMDs universal? **YES**
Is the evolution of all quark TMDs the same? **YES**
Can we have a model independent evolution of the TMDs? **YES, no effective strong coupling is necessary**
DY at small $q_T$

We have multiscale problem!

$$q^2 = Q^2 \gg q_T^2 \gg \Lambda_{QCD}^2$$

$$\lambda_1 \sim \frac{q_T}{Q} \quad \text{and} \quad \lambda_2 \sim \frac{\Lambda_{QCD}}{q_T}$$

$$\sigma = L^{\mu\nu}M_{\mu\nu}$$

$$M = |H(Q^2)|^2 \frac{f_n}{\phi} \otimes \frac{f_{\bar{n}}}{\phi} \otimes \Phi$$  \text{Factorization}  

$$M = |H(Q^2)|^2 \hat{f}_n \otimes \hat{f}_{\bar{n}} \otimes \phi^{-1}$$  \text{Overlapping}  

$$M = |H(Q^2)|^2 j_n \otimes j_{\bar{n}}$$  \text{Final definition}  

LIGHT-CONE DIVERGENCES

In PDF mixed UV-IR divergences cancel between virtual and real diagrams! This is not the case for TMD: there is no integration over $q_T$
Drell-Yan at small $q_T$

\[
d\Sigma = \frac{4\pi\alpha}{3q^2s} \frac{d^4q}{(2\pi)^4} \frac{1}{4} \sum_{\sigma_1\sigma_2} \int d^4y e^{-iqy} (-g_{\mu\nu}) \langle N_1(P, \sigma_1) N_2(\bar{P}, \sigma_2) \left| F^{\mu +}(y) F^{\nu -}(0) \right| N_1(P, \sigma_1) N_2(\bar{P}, \sigma_2) \rangle
\]

\[
F^{\mu} = \sum_q e_q \bar{\psi} \gamma^{\mu} \psi \rightarrow F^{\mu} = H(Q^2, \mu) \sum_q e_q \bar{\chi}_n S^T_{\mu} \gamma^{\mu} S^T_n \chi_n \left| y^+, 0^-, y_\perp \right\
\]

Collinear, anti-collinear and soft act on different Hilbert spaces!! (SCET)

\[
d\Sigma = \frac{4\pi\alpha^2}{3N_c q^2 s} \frac{d^4q}{(2\pi)^4} \left| H(Q^2, \mu) \right|^2 \int d^4y e^{-iqy} \sum_q e_q^2 F_n(0^+, y^-, y_\perp) F_{\bar{n}}(y^+, 0^-, y_\perp) \Phi(0^+, 0^-, y_\perp)
\]

\[
F_n(0^+, y^-, y_\perp) = \frac{1}{2} \sum_{\sigma_1} \langle N_1(P, \sigma_1) | \bar{\chi}_n(0^+, y^-, y_\perp) \frac{n_{\mu} \gamma^{\mu}}{2} \chi_n(0) | N_1(P, \sigma_1) \rangle
\]

\[
F_{\bar{n}}(y^+, 0^-, y_\perp) = \frac{1}{2} \sum_{\sigma_1} \langle N_2(\bar{P}, \sigma_2) | \bar{\chi}_{\bar{n}}(0) \frac{n_{\mu} \gamma^{\mu}}{2} \chi_{\bar{n}}(y^+, 0^-, y_\perp) | N_2(\bar{P}, \sigma_2) \rangle
\]

\[
\Phi(0^+, 0^-, y_\perp) = \langle 0 | \text{Tr} S_n^T S_{\bar{n}}^T (0^+, 0^-, y_\perp) T[S_{\bar{n}}^T S_{n}^T](0) | 0 \rangle, \quad \chi = W^{T+} \xi
\]

However this is not the end of the story…
Light-cone (Rapidity) Divergences

- When we take the collinear limit of QCD diagrams new type of divergences arise

\[
\hat{f}_{n_1} = -2ig^2 C_F \delta(1-x) \delta^{(2)}(\vec{k}_{n_1}) \mu^2 \epsilon \int \frac{d^dk}{(2\pi)^d} \frac{p^+ + k^+}{[k^+ - i0^+] [(p + k)^2 + i0^-] [k^2 + i0]}
\]

\[
\rightarrow \frac{1}{\epsilon_{UV}} \int_{0}^{1} \frac{dt}{t}
\]

\[
\phi_1 = -2ig^2 C_F \delta^{(2)}(\vec{k}_{n_1}) \mu^2 \epsilon \int \frac{d^dk}{(2\pi)^d} \frac{1}{[k^+ - i0^+] [k^- + i0^-] [k^2 + i0]}
\]

\[
\rightarrow \frac{2}{\epsilon_{UV}} \int_{0}^{1} \frac{dt}{t}
\]

In the PDFs there is a cancelation between real and virtuals, but it is not the case now

The rapidity divergence of the soft part is the double of the one of the collinear part

IN QCD THE HADRONIC TENSOR (M) HAS NO RAPIDITY DIVERGENCES!!
Double Counting

- Taking the soft limit of collinear graphs one can get the soft contribution

\[
\hat{f}_{n_1} = -2ig^2C_F\delta(1-x)\delta^{(2)}(\vec{k}_{n_\perp})\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d} \frac{p^+ + k^+}{[k^+ - i0^+][(p+k)^2 + i0^+][k^2 + i0^+]}
\]

\[
\phi_1 = -2ig^2C_F\delta^{(2)}(\vec{k}_{n_\perp})\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d} \frac{1}{[k^+ - i0^+]\epsilon^+ i\epsilon^0 \epsilon^2 + i0^+][k^2 + i0^+]}
\]

WE HAVE TO SUBTRACT THE SOFT PART FROM NAIVE COLLINEAR FUNCTION

\[
f_n(0^+, r^-, \vec{r}_\perp) = \frac{\hat{f}_n(0^+, r^-, \vec{r}_\perp)}{\phi(0^+, 0^-, \vec{r}_\perp)}, \quad f_{\bar{n}}(r^+, 0^-, \vec{r}_\perp) = \frac{\hat{f}_{\bar{n}}(r^+, 0^-, \vec{r}_\perp)}{\phi(0^+, 0^-, \vec{r}_\perp)}
\]
Definition Of The TMDPDF

• To Cancel The Mixed Divergences, Avoid Double Counting Among Soft And Collinear And As Imposed By SCET Power Counting

We Define The TMDPDF

\[ j_n(x, \vec{k}_{n\perp}) = \frac{1}{2} \int \frac{dr^- d^2 \vec{r}_\perp}{(2\pi)^3} e^{-i(\frac{1}{2} \vec{r} \cdot \vec{p} - \vec{r}_\perp \cdot \vec{k}_{n\perp})} f_n(0^+, r^-, \vec{r}_\perp) \phi(0^+, 0^-, \vec{r}_\perp) \]

\[ j_{\bar{n}}(z, \vec{k}_{n\perp}) = \frac{1}{2} \int \frac{dr^z d^2 \vec{r}_\perp}{(2\pi)^3} e^{-i(\frac{1}{2} \vec{r} \cdot \vec{p} - \vec{r}_\perp \cdot \vec{k}_{n\perp})} f_{\bar{n}}(r^+, 0^-, \vec{r}_\perp) \phi(0^+, 0^-, \vec{r}_\perp) \]

• At least Seven Different Definitions of TMDPDF
1.-2. Collins 82, Collins 2011 (off-the-light-cone)
3. Ji, Ma and Yuan 03 (off-the-light-cone)
4. Cherednikov and Stefanis 08 (collinear with subtraction of complete soft function in LC gauge)
5. Mantry and Petriello, 2010 (fully unintegrated collinear matrix element)
7. Chiu, Jain, Neil, Rothstein 2011 (Rapidity Regulator)

THE SCET POWER COUNTING AND LIGHT CONE GAUGE SYMMETRY ARE EXPLICITLY BROKEN OFF –THE-LIGHT-CONE.
Rapidity Divergences

- All properties of TMDPDF are regulator independent (each regulator has its subtleties).
- We did our calculations staying on-the-light-cone and using the \( \Delta \)-regulator (Chiu, Fuhrer, Hoang, Manohar,’09), and we have checked that our results are consistent with all other regulators.

\[
\frac{i(p + k)}{(p + k)^2 + i\Delta} \rightarrow \frac{1}{k^- \pm i\delta}, \quad \delta = \frac{\Delta}{Q}
\]

- THE TMD THAT WE PROPOSE IS FREE FROM RAPIDITY DIVERGENCES (They cancel between the collinear and the square root of the soft matrix elements)
Universality of the unpolarized TMDPDF

- The collinear and soft matrix element are the same in DY and SIDIS

- The definition of Wilson lines in DY and SIDIS is different

\[ W_n(x) = \bar{P} \exp \left[ ig \int_{-\infty}^{0} ds \bar{n} \cdot A_n(x + s\bar{n}) \right] \]

\[ S_n(x) = P \exp \left[ ig \int_{-\infty}^{0} ds \bar{n} \cdot A_s(x + s\bar{n}) \right] \]

\[ W_n(x) = \bar{P} \exp \left[ -ig \int_{-\infty}^{0} ds \bar{n} \cdot A_n(x + s\bar{n}) \right] \]

\[ S_n(x) = P \exp \left[ -ig \int_{-\infty}^{0} ds \bar{n} \cdot A_s(x + s\bar{n}) \right] \]
Universality of the unpolarized TMDPDF

- Universality of the Soft Function

\[ \Phi_{1,DIS}^r = \Phi_{1,DY}^r + \frac{\alpha_s C_F}{2\pi} \delta^{(2)}(\vec{k}_T)\pi^2; \quad \Phi_{1,DIS}^v = \Phi_{1,DY}^v - \frac{\alpha_s C_F}{2\pi} \delta^{(2)}(\vec{k}_T)\pi^2; \]

- Universality of the Collinear Function

\[ f_{n1,DIS}^r = f_{n1,DY}^r - \frac{\alpha_s C_F}{2\pi} \delta^{(2)}(\vec{k}_T)\pi^2; \quad f_{n1,DIS}^v = f_{n1,DY}^v + \frac{\alpha_s C_F}{2\pi} \delta^{(2)}(\vec{k}_T)\pi^2; \]

Both Naive Collinear And Soft ME Are Universal!

_Ergo_, the TMDPDF Is Universal
The hadronic tensor is RG scale independent

\[ \tilde{M} = H \left( \frac{Q^2}{\mu^2} \right) \tilde{j}_n (x; \bar{b}_\perp, Q, \mu) \tilde{j}_n (z; \bar{b}_\perp, Q, \mu) \]

\[ \frac{d \ln \tilde{M}}{d \ln \mu} = 0 = \gamma_H + \gamma_n + \gamma_{\tilde{n}} = \gamma_H + 2\gamma_n = \gamma_H + 2\gamma_n \]

\[ \gamma_H = A(\alpha_s) \ln \left( \frac{Q^2}{\mu^2} \right) + B(\alpha_s); \quad \tilde{j}_n (x; \bar{b}_\perp, Q, \mu) = \exp \left[ \int_{\mu_\perp}^{\mu} \frac{d \mu'}{\mu'} \gamma_{\tilde{n}} \right] \tilde{j}_n (x; \bar{b}_\perp, Q, \mu_\perp) \]

The hard coefficient is the same as for inclusive DY! Ergo, we know the AD of the TMDPDF up to 3-loops
OPE of the TMDPDF on to the PDF

- When $qT$ is in the perturbative region the TMDPDF can be factorized in a Wilson coefficient and a PDF like in OPE

$$\tilde{j}_f(x; \bar{b}_\bot, Q, \mu) = \sum_{j=q, g} \int_x^1 \frac{dx'}{x'} \tilde{C}_{f/j} \left( \frac{x}{x'}; b, Q, \mu \right) Q_{j/P}(x'; \mu)$$

The coefficient $C$ works as any other Wilson coefficient

IT IS INDEPENDENT OF IR-SCALES

BUT THERE IS STILL A $Q^2$ DEPENDENCE

$$\tilde{C}_n(x; b, Q, \mu) = \delta(1-x) + \frac{\alpha_s C_F}{2\pi} \left[ -P_{q/q} L_T + (1-x) - \delta(1-x) \left( \frac{1}{2} L_T^2 - \frac{3}{2} L_T + \ln \frac{Q^2}{\mu^2} L_T + \frac{\pi^2}{12} \right) \right]$$

THESE TERMS HAVE TO BE RESUMMED!!

$$L_T = \ln \frac{\mu^2 b^2}{4e^{-2\gamma_E}}$$
Q^2-Resummation

- Using Lorentz invariance and dimensional analysis

\[
\ln \tilde{j}_n = \ln f_n - \frac{1}{2} \ln \phi
\]

\[
\ln f_n = R_n \left( x; \alpha_s, L_T, \ln \frac{\Delta}{Q^2} \right), \quad \ln \phi = R_\phi \left( \alpha_s, L_T, \ln \frac{\Delta^2}{Q^2 \mu^2} \right)
\]

Since the TMDPDF (Wilson coefficients and PDFs) is free from rapidity divergencies to all orders in perturbation theory:

\[
\frac{d}{d \ln \Delta} \ln \tilde{j}_n = 0
\]

Which is equivalent to Collins-Soper-Sterman
**Q^2-Resummation**

- The Q^2 dependence can be exponentiated (NO need of CSS evolution equation!) (Becher, Neubert '11)

\[
\ln \tilde{j}_n = \ln \tilde{j}^{\text{sub}}_n - D(\alpha_s, L_T) \left( \ln \frac{Q^2}{\mu^2} + L_T \right)
\]

\[
\tilde{j}_n(x; b_\perp, Q, \mu) = \left( \frac{Q^2 b^2 e^{2\gamma_E}}{4} \right)^{-D(\alpha_s, L_T)} C_n(x; b_\perp, \mu) \otimes Q_n(x; \mu)
\]

\[
\frac{dD(\alpha_s, L_T)}{d \ln \mu} = \Gamma_{\text{cusp}}(\alpha_s)
\]

\[
D(\alpha_s, L_T) = \sum_{n=1}^{\infty} d_n(L_T) \left( \frac{\alpha_s}{4\pi} \right)^n
\]

\[
d'_n(L_\perp) = \frac{1}{2} \Gamma_{n-1} + \sum_{m=1}^{n-1} m\beta_{n-1-m} d_m(L_\perp)
\]

The cusp AD is known at 3-loops!!

*Ergo,* The function D is known up to order \(\alpha^2\)
Resumming!

\[ \tilde{j}_{f/P}(x; \vec{b}_\perp, Q^2, \mu = Q) = \sum_{j=q,g} \exp \left[ \int_{\mu_i}^{\mu} \frac{d\mu'}{\mu'} \gamma_n \right] \left( \frac{Q^2}{\mu^2} \right)^{-D(b,\mu_i)} C_{f/I}(x; \vec{b}_\perp, \mu_I) \otimes Q_{j/P}(x; \mu_I) \]

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Aybat, Collins, Qiu, Rogers; Aybat, Rogers; Anselmino, Boglione, Melis

Our Group

Known pieces. C from Gehrmann et al. '12
The Evolution of all quark TMDs

• The hard matching coefficient $H$ does not depend on spin! And its AD governs all evolution of the TMDs and also the evolution of the $D$-function!

$$F_{\alpha\beta}(x, \vec{k}_\perp) = \frac{1}{2} \int \frac{dr^- d^2\vec{r}_\perp}{(2\pi)^3} e^{-i\frac{1}{2}r^-xP^+ - \vec{r}_\perp \cdot \vec{k}_\perp} \Phi^q_{\alpha\beta}(0^+, r^-, \vec{r}_\perp) \sqrt{S(0^+, 0^-, \vec{r}_\perp)}$$

$$\Phi^q_{\alpha\beta}(0^+, r^-, \vec{r}_\perp) = \langle P\bar{S} \left[\xi_{n\alpha} W_n^T\right](0^+, y^-, \vec{y}_\perp)[W_n^{T\dagger} \xi_{n\beta}](0)\right| P\bar{S}\rangle$$

$$S = \langle 0\right| \text{Tr} \left[S_{n}^{T\dagger} S_{n}^{T}\right](0^+, 0^-, \vec{y}_\perp)[S_{n}^{T\dagger} S_{n}^{T}](0)\left| 0\rangle, \quad \alpha, \beta = \text{Dirac indeces}$$

THIS IS SPIN INDEPENDENT

$$\gamma_F = \frac{-1}{2} \gamma_H$$
In order to use a TMDPDF it is fundamental to transport it from one Q to another. Again the evolutor is spin independent!!

\[ \tilde{F}(x, b; Q_f) = \tilde{F}(x, b; Q_i) \tilde{R}(b; Q_i, Q_f) \]

\[ \tilde{R}(b; Q_i, Q_f) = \exp \left\{ \int_{Q_i}^{Q_f} \frac{d\mu}{\mu} \gamma_F \left( \alpha_s(\mu), \ln \frac{Q_f^2}{\mu} \right) \right\} \left( \frac{Q_f^2}{Q_i^2} \right)^{-D(b; Q_i)} \]

Now in the function D there can be large logs, because of small qT. WE MUST RESUM THE PERTURBATIVE SERIES OF D WITH THE COUNTING \( \alpha_s L_T \sim 1 \)

\[ \frac{d}{dL_\perp} d_n(L_\perp) = \frac{1}{2} \Gamma_{n-1} + \sum_{m=1}^{n-1} m \beta_{n-1-m} d_m(L_\perp) \]
Results

• In the literature people uses \( \mu = 1/b \) and an effective coupling constant to avoid the Landau pole. With the resummed D we can perform the Fourier transform directly, \( \mu \neq 1/b \), avoiding the Landau pole.

\[
\alpha_s(b) \rightarrow \alpha_s(b^*) , \quad b^* = b / \sqrt{1 + (b / b_{\text{max}})^2}
\]

We compare with previous approaches. \( b_{\text{max}} = 0.5 \) is Collins ideal \( b_{\text{max}} = 1.5 \) is fitted from Phenomenology (Konychev, Nadolsky’06)
Results

All graph show an agreement With the $b_{\text{max}}=1.5$ choice
CONCLUSIONS

- Collins approach to factorization is of limited use. We can get a better formulation of factorization on-the-light-cone (no parameters on any matching coefficient!)
- We can relate the AD of the hard matching coefficient to the one of the TMDPDF WE KNOW THE EVOLUTION OF ALL TMDPDF UP TO NNLL
- We can build an evolutor for TMDPDF removing the problem of the Landau pole in a model independent way (agreement with fits that use bmax=1.5)
- We need experiments to get a mapping of TMDs as precise as for PDFs
BACKUP SLIDES
There are a whole bunch of unphysical parameters whose values is fix by ansatz. Moreover up to now this evolutot is implemented with an effective coupling constant which introduces even more parameters.
Resummation of the Evolution Kernel

\[ 2d_n = \frac{\Gamma_0}{\beta_0} \frac{(\beta_0 L_\perp)^n}{n} + (\beta_0 L_\perp)^{n-1} \left( \frac{\Gamma_0 \beta_1}{\beta_0^2} \left( -1 + H_{n-1}^{(1)} \right) \right)_{n \geq 3} + \frac{\Gamma_1}{\beta_0} \left|_{n \geq 2} \right) + (\beta_0 L_\perp)^{n-2} \left( (n - 1) d_2^0 \right)_{n \geq 2} + (n - 1) \frac{\Gamma_2}{2\beta_0} \left|_{n \geq 3} \right) + \beta_1 \Gamma_1 \beta_0^2 s_n \left|_{n \geq 4} \right) + \frac{\beta_1^2 \Gamma_0}{\beta_0^3} t_n \left|_{n \geq 5} \right) + \frac{\beta_2 \Gamma_0}{2\beta_0^2} (n - 3) \left|_{n \geq 4} \right) + \ldots \]

\[ s_n = (n - 1) H_{n-2}^{(1)} + \frac{1}{2} (5 - 3n), \]

\[ t_n = \frac{1}{2} \left[ (1 - n) H_{n-1}^{(2)} + n + 1 \right. \]

\[ \left. + (n - 1)(\psi(n) + \gamma_E - 2)(\psi(n) + \gamma_E) \right] \]
Resummation of the Evolution Kernel

- We can perform the resummation when $\alpha_s L_\perp \sim 1$

\[
2D = -\frac{\Gamma_0}{\beta_0} \ln(1 - X) + a \left[ -\frac{\beta_1 \Gamma_0}{\beta_0^2} \frac{X + \ln(1 - X)}{(1 - X)} + \frac{\Gamma_1}{\beta_0} \frac{X}{1 - X} \right] + a^2 \left[ \frac{d_2^0}{(1 - X)^2} + \frac{\Gamma_2}{2\beta_0} \frac{X(2 - X)}{(1 - X)^2} + \frac{\beta_1 \Gamma_1}{\beta_0^2} \frac{X(X - 2) - 2\log(1 - X))}{2(1 - X)^2} + \frac{\beta_2 \Gamma_0}{2\beta_0^2} \frac{X^2}{(1 - X)^2} \right]
\]

\[
X = a \beta_0 L_\perp
\]

\[
\beta_1^2 \Gamma_0 \beta_0^3 \frac{121X^6 + 188X^5 - 13X^4 - 30X^3}{24(X - 1)^2 X^2}
\]

\[
\beta_1^2 \Gamma_0 \beta_0^3 \frac{12X^2 (1 - \text{Li}_2(X)) - 12X(X + 1)\log(1 - X)}{24(X - 1)^2 X^2}
\]

\[
+ \frac{\beta_1^2 \Gamma_0}{2 \beta_0^3} \sum_{n=5}^{\infty} X^{n-2} (n - 1) \left[ H_{n-1}^{(1)} \right]^2 + ...
\]

Resummation Borel convergence for $X < 1$. 