τ hadronic spectral function moments: perturbative expansions and $\alpha_s$

Diogo Boito

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in collaboration with Martin Beneke and Matthias Jamin (work in progress)
introduction
A new determination of $\alpha_s$ from $\tau$ decays

$\alpha_s (m_{\tau}^2) = 0.36 \pm 0.04$  
Braaten, Narison, and Pich '92

Spread in the results reflect (mainly) details of the theoretical input.

see also talk by Matthias Jamin on Tuesday
A new determination of $\alpha_s$ from $\tau$ decays

Preparation for the RG improvement

$\alpha_s(m_{\tau}^2) = 0.36 \pm 0.04$  
Braaten, Narison, and Pich ‘92

Spread in the results reflect (mainly) details of the theoretical input.

There are still open questions (Renormalization Group Improvement, duality violations, ...)

see also talk by Matthias Jamin on Tuesday
Sum rules for the spectral functions

\[ \int_0^{s_0} ds \, w(s) \frac{1}{\pi} \text{Im} \tilde{\Pi}(s) = \frac{-1}{2\pi i} \int \frac{dz}{|z|=s_0} w(z) \tilde{\Pi}(z) \]

experiment

(OPAL and ALEPH)

theory

\[ J_\mu = \bar{u} \gamma_\mu (\gamma_5) q(x) \]
Sum rules for the spectral functions (in tau decays) Braaten, Narison, and Pich, 1992

\[ \int_0^{s_0} ds \, w(s) \, \frac{1}{\pi} \text{Im} \tilde{\Pi}(s) = \frac{-1}{2\pi i} \int \frac{dz}{|z|=s_0} \, w(z) \, \tilde{\Pi}(z) \]

Contributions to the sum rule (theory side)

\[ R_{V/A}^{w_i}(s_0) = \frac{N_c}{2} S_{EW} |V_{ud}|^2 \left[ \delta_{w_i}^{\text{tree}} + \delta_{w_i}^{(0)}(s_0) + \sum_{D\geq 2} \delta_{w_i,V/A}^{(D)}(s_0) + \delta_{w_i,V/A}^{\text{DV}}(s_0) \right] \]

\[ \delta_{w_i}^{\text{tree}} + \left( \frac{\alpha_s^2}{\alpha_s^4} \right) + \left( \frac{\alpha_s^3}{\alpha_s^4} \right) + \cdots \]

our focus is on \( \delta_{w_i}^{(0)} \) (moment dependence)

\[ \alpha_s^4 : \text{Baikov, Chetyrkin, Kühn 2008} \]
In the literature several weight functions are used

DB et al ’11, ’12, Davier et al ’08, Maltman and Yavin ’08, ALEPH ’98, ‘05, OPAL ‘99

One often employs \( w_\tau(x) = (1 - x)^2(1 + 2x) \) (gives \( R_\tau = \frac{\Gamma[\tau \to \text{hadrons } \nu_\tau]}{\Gamma[\tau \to e^-\bar{\nu}_e \nu_\tau]} \))

and many others:

\[
\begin{align*}
 w(x) &= 1, \quad w(x) = 1 - x^2, \quad w(x) = x(1 - x)^2, \quad w^{(k)}(x) = (1 - x)^3x^k(1 + 2x), \\
 w^{(n)}(x) &= 1 - \frac{n}{n-1} x + \frac{n}{n-1} x^n \ldots
\end{align*}
\]

Different emphasis on the experimental spectrum. Change the relative contributions on the theory side (pert., OPE, DVs)

\( \alpha_s \) depence comes mainly from \( \delta_{w_i}^{(0)} \)

Open questions in \( \delta_{w_i}^{(0)} \):

- Renormalization group improvement: what is the best prescription?
  CIPT vs FOPT
- Renormalization group improvement: moment dependence
- Are there better moments to determine \( \alpha_s \)?
A new determination of $\alpha_s$ from $\tau$ decays

Diogo Boito

spectral function moments: pt. exp. and $\alpha_s$

renormalization group
A new determination of $\alpha_s$ from $\tau$ decays

$$D^{(1+0)}_{\text{pert}}(s) = \frac{N_c}{12\pi^2} \sum_{n=0}^{\infty} a_n^\mu \sum_{k=1}^{n+1} k \, c_{n,k} \left( \log \frac{-s}{\mu^2} \right)^{k-1}$$

$$a_\mu = \alpha(\mu)/\pi$$

- only $c_{n,1}$ are independent (known up to $c_{4,1}$). $c_{n,k}$ depend on $c_{n,1}$ and $\beta_m$.

Prescriptions for the RG improvement

**FOPT**

\[ \mu = s_0 \]

\[ \delta^{(0)}_{\text{FO},w_i} = \sum_{n=1}^{\infty} a(s_0)^n \sum_{k=1}^{n} k \, c_{n,k} \, J^{\text{FO},w_i}_{k-1} \]

\[ J^{\text{FO},w_i}_n \equiv \frac{1}{2\pi i} \int_{|x|=1} dx \, W_i(x) \log^n(-x) \]

**CIPT**

\[ \mu = -s_0 x \]

\[ \delta^{(0)}_{\text{CI},w_i} = \sum_{n=1}^{\infty} c_{n,1} \, J^{\text{CI},w_i}_n(s_0) \]

\[ J^{\text{CI},w_i}_n(s_0) \equiv \frac{1}{2\pi i} \int_{|x|=1} dx \, W_i(x) a^n(-s_0 x) \]

Le Diberder and Pich '92

\[ w(x) = 1 \]

\[ \alpha_s(m_\tau) = 0.3186 \]

\[ w_\tau(x) = (1-x)^2(1+2x) \]

\[ w(x) = (1-x)^3x^3(1+2x) \]
higher orders
A new determination of $\alpha_s$ from $\tau$ decays

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Diagonal spectral function moments: pt. exp. and $\alpha_s$

General structure of large-order behavior (believed to be) known

Borel transformed Adler function

$$B[\hat{D}](t) \equiv \sum_{n=0}^{\infty} \frac{c_{n,1}}{\pi^n} \frac{t^n}{n!}$$

Borel sum:

$$\hat{D}(\alpha) \equiv \int_0^{\infty} dt \ e^{-t/\alpha} B[\hat{D}](t)$$

Singularities in the $t$ plane

UV renormalons

- sign alternating
- leading sing. in the Adler function at $u = -1$
- no-sing alternation in known coeff.: small residue for the leading UV pole

$$B[D_p] = \frac{c_p}{(p-u)^\gamma} \left[ 1 + \tilde{b}_1 (p-u) + \cdots \right]$$

Structure of each singularity in principle calculable (up to $c_p$)

IR renormalons

- fixed sign
- sing. at $u = 2, 3, 4...$ related to dim-4, dim-6, dim-8... contributions
- $u = 2$ related to the gluon condensate

(review) Beneke 1999

$\tau$ spectral function moments: pt. exp. and $\alpha_s$
With the leading singularities one can reproduce the known coefficients and study the higher-order behavior of the Adler function.

Powerful tool to discuss FOPT vs CIPT

**Reference model (RM)**

\[ B[\hat{D}](u) = B[\hat{D}^{\text{UV}}](u) + B[\hat{D}^{\text{IR}}_1](u) + B[\hat{D}^{\text{IR}}_2](u) + d^\text{PO}_0 + d^\text{PO}_1 u \]

- Model with the leading UV and the first two IR singularities.
- Small polynomial terms to fix \( c_{1,1} \) and \( c_{2,1} \).
- Favors FOPT (related to the presence of \( u = 2 \) sing).
With the leading singularities one can reproduce the known coefficients and study the higher-order behavior of the Adler function.

- Powerful tool to discuss FOPT vs CIPT

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- Model with the leading UV and the first two IR singularities.
- Small polynomial terms to fix \( c_{1,1} \) and \( c_{2,1} \).
- Favors FOPT (related to the presence of \( u = 2 \) sing).

**Alternative model (AM)**

\[ B[\hat{D}](u) = B[\hat{D}^{\text{UV}}_1](u) + B[\hat{D}^{\text{IR}}_3](u) + B[\hat{D}^{\text{IR}}_4](u) + d_0^{\text{PO}} + d_1^{\text{PO}} u \]

- No IR singularity at \( u = 2 \).
- Favors CIPT.

\( \tau \) spectral function moments: pt. exp. and \( \alpha_s \)
why FOPT is better in the reference model

Reference model

Beneke & Jamin ‘08

Separating the contributions in FOPT

\[
\delta_{\text{FO},w_i}^{(0)} = \sum_{n=1}^{\infty} \left[ c_{n,1} \delta_{w_i}^{\text{tree}} + g_n^{[w_i]} \right] a(s_0)^n
\]

\[
g_n^{[w_i]} = \sum_{k=2}^{n} k c_{n,k} J_{k-1}^{\text{FO},w_i}
\]

Result at \(\alpha_s^n\). FOPT sums the first \(n\) rows. Important cancellations.

\[
\begin{array}{cccccccccc}
\alpha_s^n & c_{1,1} & c_{2,1} & c_{3,1} & c_{4,1} & c_{5,1} & c_{6,1} & c_{7,1} & c_{8,1} & g_n & \frac{c_n + g_n}{c_n} \\
1 & 1 & & & & & & & & & 1 \\
2 & g_2 & 3.56 & + & 1.64 & & & & & & 3.56 \\
3 & g_3 & 8.31 & + & 11.7 & + & 6.37 & & & & 20.0 \\
4 & g_4 & -20.6 & + & 30.5 & + & 68.1 & + & 49.1 & & 78 \\
\vdots & & & & & & & & & & \vdots \\
6 & g_6 & -2924 & -2858 & -2280 & 2214 & 5041 & & 3275 & & -807 \\
\vdots & & & & & & & & & & \vdots \\
8 & g_8 & 14652 & -29552 & -145846 & -502719 & -393887 & 260511 & 467787 & 388442 & -329054 \\
\end{array}
\]

Fixed Order

CIPT sums the first \(n\) columns to all orders. Misses the cancellations.

\(n\) spectral function moments: pt. exp. and \(\alpha_s\) Xth Quark Confinement
moment analysis
Moments studied can have their behavior separated in classes

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<tr>
<th>$k$</th>
<th>$w_k(x)$</th>
<th>Reference</th>
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<td>[Boito et al]</td>
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<td>$1 - x^3$</td>
<td>[Boito et al]</td>
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<td>9</td>
<td>$1 - \frac{3x}{2} + \frac{x^3}{2}$</td>
<td>[Maltman &amp; Yavin]</td>
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<td>$(1 - x)^2$</td>
<td>[Maltman &amp; Yavin]</td>
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<td>11</td>
<td>$(1 - x)^3$</td>
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<td>12</td>
<td>$w_\tau$ $(1 - x)^2(1 + 2x)$</td>
<td>[All recent works]</td>
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<td>13</td>
<td>$(1 - x)^3(1 + 2x)$</td>
<td>[Davier et al, ALPEH, OPAL]</td>
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<td>14</td>
<td>$(1 - x)^2x$</td>
<td>[Maltman &amp; Yavin]</td>
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<td>$(1 - x)^3x(1 + 2x)$</td>
<td>[Davier et al, ALPEH, OPAL]</td>
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<td>16</td>
<td>$(1 - x)^3x^2(1 + 2x)$</td>
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<td>17</td>
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</tbody>
</table>

Monomials

Pinched ($w(1) = 0$) with a “1”

Pinched with no “1”

Note: moments with a term “x” form a separate class.
- Pinched moments with a “1”
- At least one of the methods approach the Borel result at relatively low orders

\[ w_\tau(x) = (1 - x)^2(1 + 2x) \]

\[ w(x) = 1 - x^2 \]

\[ w(x) = 1 - x^3 \]
Moments with the term $x$

Very sensitive to $D = 4$. Unstable results if the $u = 2$ singularity is sizable.

$$w(x) = 1 - x$$

$$w(x) = (1 - x)^3 x (1 + 2x)$$
Pinched moments starting at $x^2$ (or higher)

Borel results are never well reproduced at low orders.

$$w(x) = (1 - x)^3 x^2 (1 + 2x)$$

$$w(x) = (1 - x)^3 x^3 (1 + 2x)$$
consequences for $\alpha_s$

(exploratory study: power corrections and DVs as external inputs)
A new determination of $\alpha_s$ from $\tau$ decays

Data: Updated ALEPH [Davier et al (2008)]. Warning: Correlations due to unfolding missing in this data set. Experimental errors potentially underestimated!
conclusions
Some moments are more suitable for the extraction of $\alpha_s$.

The pinched moments with a “1” and without an “x” are ideal:
- Good convergence of FOPT (RM) or CIPT (AM) at low orders

Moments composed only by powers of “x” should be avoided:
- problems in the convergence of both FOPT and CIPT,
- power corrections are too important.

Some of the recent extractions of $\alpha_s$ employed moments that are not optimal. (see also) [Maltman & Yavin 2008]
A new determination of $\alpha_s$ from $\tau$ decays

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$\tau$ spectral function moments: pt. exp. and $\alpha_s$  Xth Quark Confinement
A new determination of $\alpha_s$ from $\tau$ decays
Seperating the contributions in FOPT

\[
\delta_{FO,w_i}^{(0)} = \sum_{n=1}^{\infty} \left[ c_{n,1}^{\text{tree}} + g_{n}^{[w_i]} \right] a(s_0)^n
\]

\[
g_{n}^{[w_i]} = \sum_{k=2}^{n} k c_{n,k} J_{k-1}^{FO,w_i}
\]

Re-expanding CIPT

\[
\delta_{CI,w_i}^{(0)} = \sum_{n=1}^{\infty} c_{n,1} J_{n}^{CI,w_i}(s_0) \quad J_{n}^{CI,w_i}(s_0) \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} W_i(x) a^n(-s_0x)
\]

\[
a(-s_0x) = a(s_0) - \frac{1}{2} a(s_0)^2 \beta_1 \log(-x) + \cdots
\]

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<tr>
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<th>(c_{8,1})</th>
<th>(g_n)</th>
<th>(\frac{c_{n}+g_{n}}{c_{n}})</th>
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Separating the contributions in FOPT

\[
\delta_{\text{FO}, w_i}^{(0)} = \sum_{n=1}^{\infty} \left[ c_{n,1} \delta_{w_i}^{\text{tree}} + g_n^{[w_i]} \right] a(s_0)^n
\]

\[
g_n^{[w_i]} = \sum_{k=2}^{n} k c_{n,k} J_{k-1}^{\text{FO}, w_i}
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Re-expanding CIPT

\[
\delta_{\text{CI}, w_i}^{(0)} = \sum_{n=1}^{\infty} c_{n,1} J_{n, w_i}^{\text{CI}, w_i} (s_0)
\]

\[
J_{n, w_i}^{\text{CI}, w_i} (s_0) \equiv \frac{1}{2\pi i} \int_{|x|=1} dx \frac{W_i(x) a^n (-s_0 x)}{x} 
\]

\[
a(-s_0 x) = a(s_0) - \frac{1}{2} a(s_0)^2 \beta_1 \log(-x) + \cdots
\]

Result at \(\alpha\frac{n}{s}\). FOPT sums the first \(n\) rows.

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<th>(c_{1,1})</th>
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\(\tau\) spectral function moments: pt. exp. and \(\alpha\frac{s}{n}\) Xth Quark Confinement

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A new determination of $\alpha_s$ from $\tau$ decays

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$\tau$ spectral function moments: pt. exp. and $\alpha_s$ Xth Quark Confinement

$\delta_{\text{FO},w_i}^{(0)} = \sum_{n=1}^{\infty} \left[ c_{n,1} \delta_{w_i}^{\text{tree}} + g_{n}^{[w_i]} \right] a(s_0)^n$

$g_{n}^{[w_i]} = \sum_{k=2}^{n} k c_{n,k} J_{k-1}^{\text{FO},w_i}$

$\delta_{\text{CI},w_i}^{(0)} = \sum_{n=1}^{\infty} c_{n,1} J_{n}^{\text{CI},w_i}(s_0)$

$J_{n}^{\text{CI},w_i}(s_0) \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} W_i(x) a^n(-s_0x)\uparrow$

$a(-s_0x) = a(s_0) - \frac{1}{2} a(s_0)^2 \beta_1 \log(-x) + \cdots$

Result at $\alpha_s^n$. CIPT sums the first $n$ columns to all orders.

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Contour Improved