Carving Out the Space of Conformal Field Theories

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Xth Quark Confinement and the Hadron Spectrum
Many reasons to study Conformal Field Theories:

- QFTs often flow to conformal fixed points
- They describe quantum gravity via AdS/CFT
- They describe condensed matter systems at phase transitions
- They could play a role in physics beyond the Standard Model
Why Study Conformal Field Theories?

- In most CFT applications, one is interested in knowing the *scaling dimensions* of local operators: 
  \[ [D, \mathcal{O}(0)] = \Delta_\mathcal{O} \mathcal{O}(0) \]

- These control how the theory behaves when you perturb it:

  \[ \mathcal{L}_{CFT} + \epsilon \mathcal{O} \to \mathcal{L}_{CFT} + \epsilon \left( \frac{\mu}{\Lambda} \right)^{\Delta_\mathcal{O} - d} \mathcal{O} \]
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\( \Delta_\mathcal{O} \)'s can be calculable for:

- Perturbative theories (e.g., Banks-Zaks or Wilson-Fischer)
- Large N theories (via AdS/CFT)
- Protected/chiral operators in supersymmetric theories
- 2D CFTs (much bigger symmetry group)
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\]

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If we aren’t handed one of these situations, is the lattice our only hope???
Another Path Forward...

[Rattazzi, Rychkov, Tonni, Vichi '08]:

Crossing Symmetry + Unitarity leads to \emph{bounds} on operator dimensions!

- Concrete realization of \textit{Conformal Bootstrap} in $D > 2$ [Polyakov '74]
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Idea was then extended to:

- $\mathcal{N} = 1$ Superconformal Theories [DP, Simmons-Duffin ’10]
- CFTs with global symmetries [Rattazzi, Rychkov, Vichi ’10; Vichi ’11]
- Bounds on 3pt function coefficients
  - Scalar 3pt functions [Caracciolo, Rychkov ’09]
  - Flavor Symmetry Currents [DP, Simmons-Duffin ’10]
  - Stress Tensor [DP, Simmons-Duffin ’10; Rattazzi, Rychkov, Vichi ’10]
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- Best 4D results (using new methods) in [DP, Simmons-Duffin, Vichi '11]
- 3D bounds in [El-Showk, Paulos, DP, Rychkov, Simmons-Duffin, Vichi, '12]
Outline

1. CFT Review
2. Bounds from Crossing Relations
3. Best Results
The conformal algebra $SO(d, 2)$ contains:

- Translations $P^a$ and rotations $M^{ab}$
- Dilatations $D$ (scale transformations)
- Special conformal generators $K^a$ (inv. $\rightarrow$ trans. $\rightarrow$ inv.)

\[
[K^a, P^b] = 2\eta^{ab}D - 2M^{ab}
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\[
[K^a, P^b] = 2\eta^{ab}D - 2M^{ab}
\]

- Primary operators $O(0)$ are defined by $[K^a, O(0)] = 0$
- Descendants obtained using $[P^a, O(0)] = \partial^a O(0)$
Conformal symmetry fixes primary 2pt and 3pt functions in terms of dim’s and spins, up to coefficients $\lambda_O$ [Polyakov '70; Osborn, Petkou '93]

$$\langle O^{a_1..a_\ell}(x_1)O^{b_1..b_\ell}(x_2) \rangle = \frac{I^{a_1b_1}..I^{a_\ell b_\ell}}{x_{12}^{2\Delta}}$$

$$\left[ I^{ab} \equiv \eta^{ab} - 2\frac{x_{12}^a x_{12}^b}{x_{12}^2} \right]$$

$$\langle \phi(x_1)\phi(x_2)O^{a_1..a_\ell}(x_3) \rangle = \lambda_O \frac{Z^{a_1}..Z^{a_\ell}}{x_{12}^{2\Delta_{\phi}-\Delta+\ell} x_{23}^{\Delta-\ell} x_{13}^{\Delta-\ell}}$$

$$\left[ Z^a \equiv \frac{x_{31}^a}{x_{31}^2} - \frac{x_{32}^a}{x_{32}^2} \right]$$
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$$\langle O^{a_1 \ldots a_\ell}(x_1)O^{b_1 \ldots b_\ell}(x_2) \rangle = \frac{I^{a_1 b_1 \ldots I^{a_\ell b_\ell}}}{x^{2\Delta}_{12}} \left[ I^{ab} \equiv \eta^{ab} - 2\frac{x^{a}_{12}x^{b}_{12}}{x^{2}_{12}} \right]$$

$$\langle \phi(x_1)\phi(x_2)O^{a_1 \ldots a_\ell}(x_3) \rangle = \lambda_O \frac{Z^{a_1} \ldots Z^{a_\ell}}{x^{2\Delta_{\phi}-\Delta+\ell}_{12}\frac{x^{\Delta-\ell}_{23}x^{\Delta-\ell}_{13}}{x^{2}_{12}}\left[ Z^a \equiv \frac{x^{a}_{31}}{x^{2}_{31}} - \frac{x^{a}_{32}}{x^{2}_{32}} \right]}$$

Unitary CFTs have a lower bound $\Delta \geq \ell + d - 2 - \frac{(d-2)}{2}\delta_{\ell,0}$ [Mack '77]

- Requirement that 2pt functions of descendants are $\geq 0$
CFT Review: Correlation Functions

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\[
\langle O^{a_1 \ldots a_\ell}(x_1) O^{b_1 \ldots b_\ell}(x_2) \rangle = \frac{I^{a_1 b_1} \ldots I^{a_\ell b_\ell}}{x_1^{2\Delta}} \left[ I^{ab} \equiv \eta^{ab} - 2x_{12}^a x_{12}^b \right]
\]

\[
\langle \phi(x_1) \phi(x_2) O^{a_1 \ldots a_\ell}(x_3) \rangle = \lambda_O \frac{Z^{a_1} \ldots Z^{a_\ell}}{x_{12}^{2\Delta_\phi-\Delta+\ell} x_{23}^\Delta x_{13}^{-\Delta-\ell}} \left[ Z^a \equiv \frac{x_{31}^a}{x_{31}^2} - \frac{x_{32}^a}{x_{32}^2} \right]
\]

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  - Requirement that 2pt functions of descendants are $\geq 0$

- Higher $n$-pt functions *not* fixed by conformal symmetry alone, but are determined once spectrum and $\lambda_O$’s are known...
Let $\phi$ be a scalar primary in a 4D CFT:

$$\phi(x)\phi(0) = \sum_{\mathcal{O} \in \phi \times \phi} \lambda_{\mathcal{O}} C_I(x, \partial) \mathcal{O}^I(0)$$  \hspace{1cm} \text{(OPE)}$$

- Sum runs over primary $\mathcal{O}$’s
- $\mathcal{O}^I = \mathcal{O}^{a_1...a_\ell}$ any spin-$\ell$ Lorentz rep with $\ell = 0, 2, ...$
- $C_I(x, \partial)$ fixed by conformal symmetry
  - E.g., for scalars $C(x, \partial) \sim x^{\Delta-2\Delta_{\phi}} \left[ 1 + \frac{1}{2} x^a \partial_a + ... \right]$
CFT Review: Conformal Block Decomposition

Use OPE to evaluate 4-point function [Ferrara, Gatto, Grillo '73; ...]

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle$$

$$= \sum_{O \in \phi \times \phi} \lambda_O^2 C_I(x_{12}, \partial_2) C_J(x_{34}, \partial_4) \langle O^I(x_2)O^J(x_4) \rangle$$

$$\equiv \frac{1}{x_{12}^{2\Delta} x_{34}^{2\Delta}} \sum_{O \in \phi \times \phi} \lambda_O^2 g_{\Delta, \ell}(u, v)$$

- $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$ conformally-invariant cross ratios.
- $g_{\Delta, \ell}(u, v)$ conformal block ($\Delta = \dim O$ and $\ell = \text{spin of } O$)
  - Power series expansions known since 70’s, now known in terms of hypergeometric functions [Dolan, Osborn '00; Dolan, Osborn '03]
CFT Review: Conformal Blocks

Explicit formula in 4D [Dolan, Osborn '00]

\[ g_{\Delta,l}(u, v) = \frac{z\bar{z}}{z - \bar{z}}[k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z}) - z \leftrightarrow \bar{z}] \]

\[ k_{\beta}(x) = x^{\beta/2} \, _2F_1(\beta/2, \beta/2, \beta; x), \]

where \( u = z\bar{z} \) and \( v = (1 - z)(1 - \bar{z}) \).

- Similar closed-form expressions in other even dimensions, recursion relations known in odd dimensions
- Can be viewed as eigenfunctions of the quadratic Casimir of the conformal group [Dolan, Osborn '03]
CFT Review: Crossing Relations

- $\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle$ is symmetric under permutations of $x_i$
- Switching $x_1 \leftrightarrow x_3$ after OPE gives the "crossing relation":

$$\sum \quad \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
\end{array}
\end{array} = \sum \quad \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
\end{array}
\end{array}$$

$$v^{\Delta_\phi} \sum_{\mathcal{O} \in \phi \times \phi} \lambda_{\mathcal{O}}^2 g_{\Delta,\ell}(u, v) = u^{\Delta_\phi} \sum_{\mathcal{O} \in \phi \times \phi} \lambda_{\mathcal{O}}^2 g_{\Delta,\ell}(v, u)$$
CFT Review: Crossing Relations

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\]

This is a constraint on the spectrum of \( \Delta \)'s, \( \ell \)'s, and \( \lambda_{\mathcal{O}} \)'s:

- Many insights about CFTs just waiting to be extracted...
CFT Review: Crossing Relations

Convenient to write as a sum rule (separating out $\phi \times \phi \sim 1 + \ldots$)

$$1 = \sum \lambda_{O}^{2} F_{\Delta, \ell}(u, v)$$

where

$$F_{\Delta, \ell}(u, v) \equiv \frac{v^{\Delta_{\phi}} g_{\Delta, \ell}(u, v) - u^{\Delta_{\phi}} g_{\Delta, \ell}(v, u)}{u^{\Delta_{\phi}} - v^{\Delta_{\phi}}}. $$
CFT Review: Crossing Relations

Convenient to write as a sum rule (separating out $\phi \times \phi \sim 1 + \ldots$)

$$1 = \frac{\lambda^2}{2} F_{\Delta,\ell}(u, v)$$

where

$$F_{\Delta,\ell}(u, v) \equiv \frac{u^{\Delta_\phi} g_{\Delta,\ell}(u, v) - u^{\Delta_\phi} g_{\Delta,\ell}(v, u)}{u^{\Delta_\phi} - v^{\Delta_\phi}}.$$ 

This discussion can also be generalized to CFTs with global symmetries:

- E.g., crossing of $\langle \phi_i \phi_j \phi_k \phi_l \rangle$ for $SO(N)$ gives a system of sum rules [Rattazzi, Rychkov, Vichi '10]

- In SUSY theories, $\lambda_\mathcal{O}$'s related: $g_{\Delta,\ell} \rightarrow \mathcal{G}_{\Delta,\ell}$ (Superconformal Blocks) [Dolan, Osborn '01; DP, DSD '10; Fortin, Intriligator, Stergiou '11]
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2. Bounds from Crossing Relations
3. Best Results
How Does Crossing Symmetry Lead to CFT Bounds?

Crossing relation for real scalar $\phi$:

$$1 = \sum \lambda_O^2 F_{\Delta, \ell}(u, v)$$

unit op.  everything else
How Does Crossing Symmetry Lead to CFT Bounds?

Crossing relation for real scalar $\phi$:

$$1 = \sum \lambda^2 O F_{\Delta, \ell}(u, v)$$

unit op.  everything else

- Make an assumption: all scalars have dimension $\Delta > \Delta_{\text{min}}$
How Does Crossing Symmetry Lead to CFT Bounds?

Crossing relation for real scalar $\phi$:

$$1 = \sum \lambda_O^2 F_{\Delta, \ell}(u, v)$$

- Make an assumption: all scalars have dimension $\Delta > \Delta_{\text{min}}$
- Search for a linear functional $\alpha$ such that
  $$\alpha(1) < 0, \quad \text{and} \quad \alpha(F_{\Delta, \ell}) \geq 0,$$
  for all other $O \in \phi \times \phi$.

- If you find one, the assumption is ruled out!
CFT Bounds

Convenient to phrase search as a convex optimization problem:

Minimize $\alpha(1)$ subject to $\alpha(F_{\Delta,\ell}) \geq 0$
CFT Bounds

Convenient to phrase search as a convex optimization problem:

\[
\begin{align*}
\text{Minimize } & \alpha(1) \text{ subject to } \alpha(F_{\Delta, \ell}) \geq 0
\end{align*}
\]

Simplest Approach: [Rattazzi, Rychkov, Tonni, Vichi ’08]

- Impose \( \alpha(F_{\Delta_i, \ell_i}) \geq 0 \) on a finite lattice \( \{(\Delta_i, \ell_i)\} \)
  (verify positivity on intermediate values later)
- Take \( \alpha \) to be linear combinations of \( \partial^n \partial^m F_{\Delta, \ell} \) at some point
- Implement as a linear programming problem that can be solved numerically (e.g., by Mathematica, GLPK, CPLEX, ...)

Bounds from Crossing Relations

Best Results

CFT Review
CFT Bounds

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- Impose \(\alpha(F_{\Delta_i,\ell_i}) \geq 0\) on a finite lattice \(\{(\Delta_i, \ell_i)\}\) (verify positivity on intermediate values later)
- Take \(\alpha\) to be linear combinations of \(\partial^n z \partial^m z F_{\Delta,\ell}\) at some point
- Implement as a \textit{linear programming} problem that can be solved numerically (e.g., by Mathematica, GLPK, CPLEX, ...)

Can also be made more efficient (esp. w/ global symmetries) using an approach based on \textit{semi-definite programming} [DP, Simmons-Duffin, Vichi ’11]
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Singlet Dimension Bounds

- Bound on lowest dim scalar ("$\phi^2$") in $\phi \times \phi$ OPE
- Best bound: 66-dimensional space of derivatives
SO(4) or SU(2) Singlet Dimension Bounds

- Upper bound on $\phi^\dagger \phi$ for SO(4) or SU(2)

- Lowest dim singlet in $\phi_i^\dagger \times \phi_j$, where $\phi_i$ is SU(2) fundamental

- LHC aside, theory constraint on Conf. Technicolor [e.g., Luty, Okui '04]
SO($N$) or SU($N/2$) Singlet Dimension Bounds

Upper bound on $\phi^\dagger \phi$ for SO($N$) or SU($N/2$), $N = 2..15$

- Bounds get weaker as $N$ increases
- SO($N$) bounds and SU($N/2$) bounds are identical
Superconformal Operator Dimension Bounds

- Bound on lowest dimension scalar in $\Phi \times \Phi^\dagger$ OPE, where $\Phi$ is a chiral superconformal primary in an $\mathcal{N} = 1$ SCFT
- See a kink near $\{\Delta_\Phi, \Delta_{\Phi^\dagger\Phi}\} \sim \{1.4, 3.2\}$: maybe an SCFT lives there?
For Comparison: 2D Dimension Bounds

- Kink at 2D Ising model, exact solution: $\Delta_\sigma = 1/8$, $\Delta_\epsilon = 1$
- Bound saturated by operators in unitary minimal models
For Comparison: 3D Dimension Bounds

See Alessandro’s Talk!
Future Directions

Some directions for this program:

- Understand 3D Bounds (see next talk…)
  [El-Showk, Paulos, DP, Rychkov, Simmons-Duffin, Vichi, ’12]
- Explore effects of gaps in the spectrum
- Explore kink in 4D $\Phi^+\Phi$ bound $\rightarrow$ known SCFT or something new?
- Learn to extract full spectrum of theories living on boundary
- Study 4pt functions of containing $\phi^2$ or operators with spin
  (for conformal blocks see [Costa, Penedones, DP, Rychkov ’11])
- Bounds in other dimensions (e.g., 6D, 8D, $4 - \epsilon$, …) or more SUSY
- Improve analytic understanding
Backup Slides
Generalization to Global Symmetries

Suppose $\phi_i$ is an $\text{SO}(N)$ fundamental. The OPE is

$$\phi_i \times \phi_j \sim \sum_{S^+} \delta_{ij} O + \sum_{T^+} O_{(ij)} + \sum_{A^-} O_{[ij]},$$

and the 4pt function can be expanded in various tensor structures

$$x_{12}^{2d} x_{34}^{2d} \langle \phi_i(x_1) \phi_j(x_2) \phi_k(x_3) \phi_l(x_4) \rangle = \sum_{S^+} \lambda_{O}^2 (\delta_{ij} \delta_{kl}) g_{\Delta,\ell}(u, v)$$

$$+ \sum_{T^+} \lambda_{O}^2 \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{2}{N} \delta_{ij} \delta_{kl} \right) g_{\Delta,\ell}(u, v)$$

$$+ \sum_{A^-} \lambda_{O}^2 (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) g_{\Delta,\ell}(u, v).$$
Generalization to Global Symmetries

Symmetry under $x_1 \leftrightarrow x_3$ and $i \leftrightarrow k$ leads to the triple-sum rule: [Rattazzi, Rychkov, Vichi '10]

$$\sum_{S^+} \lambda_\mathcal{O}^2 \begin{pmatrix} 0 \\ F_\Delta,\ell \\ H_\Delta,\ell \end{pmatrix} + \sum_{T^+} \lambda_\mathcal{O}^2 \begin{pmatrix} F_\Delta,\ell \\ (1 - \frac{2}{N})F_\Delta,\ell \\ -(1 + \frac{2}{N})H_\Delta,\ell \end{pmatrix} + \sum_{A^-} \lambda_\mathcal{O}^2 \begin{pmatrix} -F_\Delta,\ell \\ F_\Delta,\ell \\ -H_\Delta,\ell \end{pmatrix} = 0$$

(Here $H_\Delta,\ell(u,v)$ is $F_\Delta,\ell(u,v)$ with $- \rightarrow +$)

- 3 sum rules $\leftrightarrow$ 3 tensor structures
Generalization to Global Symmetries

Symmetry under $x_1 \leftrightarrow x_3$ and $i \leftrightarrow k$ leads to the triple-sum rule:
[Rattazzi, Rychkov, Vichi '10]

$$\sum_{S^+} \lambda_O^2 \begin{pmatrix} 0 \\ F_{\Delta,\ell} \\ H_{\Delta,\ell} \end{pmatrix} + \sum_{T^+} \lambda_O^2 \begin{pmatrix} F_{\Delta,\ell} \\ (1 - \frac{2}{N}) F_{\Delta,\ell} \\ -(1 + \frac{2}{N}) H_{\Delta,\ell} \end{pmatrix} + \sum_{A^-} \lambda_O^2 \begin{pmatrix} -F_{\Delta,\ell} \\ F_{\Delta,\ell} \\ -H_{\Delta,\ell} \end{pmatrix} = 0$$

(Here $H_{\Delta,\ell}(u,v)$ is $F_{\Delta,\ell}(u,v)$ with $- \rightarrow +$)

- 3 sum rules $\leftrightarrow$ 3 tensor structures

Similar rules for other global symmetries:

- $SU(N) \rightarrow 6$ sum rules
- $N = 1$ SCFTs $\rightarrow 3$ sum rules (since $U(1)_R \sim SO(2)$)
  - $\mathcal{O}$’s in same SUSY multiplet have related $\lambda$’s: $g_{\Delta,\ell} \rightarrow G_{\Delta,\ell}$
  - (superconformal blocks) [DP, DSD '10; Fortin, Intriligator, Stergiou '11]
$\mathcal{N} = 1$ Superconformal Algebra

\[
\begin{array}{cccc}
\dim & +1 & P_a & \overline{Q}_{\dot{\alpha}} \\
+1/2 & M_{\alpha\beta} & Q_{\alpha} & \{Q, \overline{Q}\} = P \\
0 & D, R & S_{\alpha} & \{S, \overline{S}\} = K \\
-1/2 & -1 & K_{\alpha}, & \\
-1 & & & \\
\end{array}
\]

- Superconformal primary means $[S, \mathcal{O}(0)] = [\overline{S}, \mathcal{O}(0)] = 0$
- Descendants obtained by acting with $P, Q, \overline{Q}$
- Chiral means $[\overline{Q}, \Phi(0)] = 0$
Superconformal Block Decomposition

Φ: scalar chiral superconformal primary of dimension $d$ in an SCFT

$$\langle \Phi(x_1)\Phi^\dagger(x_2)\Phi(x_3)\Phi^\dagger(x_4) \rangle = \frac{1}{x_1^{2d}x_2^{2d}} \sum_{\mathcal{O} \in \Phi \times \Phi^\dagger} |\lambda_\mathcal{O}|^2 \mathcal{G}_{\Delta,\ell}(u,v)$$

- Sum over s.c. primaries $\mathcal{O}$ with $R = 0$ and $\ell = 0, 1, 2 \ldots$
- $x_1 \leftrightarrow x_3$ gives crossing relation only involving $\mathcal{O} \in \Phi \times \Phi^\dagger$
- Additional constraints come from relation to $\Phi \times \Phi$ OPE

Note: $\mathcal{G}_{\Delta,\ell}(u,v)$ is a finite sum of conformal blocks, since $\mathcal{O}$ has finite number of descendants that are conformal primaries!
Superconformal Block Derivation

Multiplet built from $\mathcal{O}$ (generically) contains four conformal primaries with vanishing $R$-charge and definite spin:

<table>
<thead>
<tr>
<th>name</th>
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<tbody>
<tr>
<td>$\mathcal{O}$</td>
</tr>
<tr>
<td>$J, N$</td>
</tr>
<tr>
<td>$D$</td>
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</tbody>
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<tr>
<th>operator</th>
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<tr>
<td>$\mathcal{O}$</td>
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<tr>
<td>$Q\bar{Q}\mathcal{O} + #P\mathcal{O}$</td>
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<tr>
<td>$Q^2\bar{Q}^2\mathcal{O} + #PQ\bar{Q}\mathcal{O} + #PP\mathcal{O}$</td>
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<tr>
<td>$\Delta$</td>
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<td>$\Delta + 1$</td>
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<td>$\Delta + 2$</td>
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<table>
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<th>spin</th>
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<tr>
<td>$l$</td>
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<tr>
<td>$l + 1, l - 1$</td>
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<tr>
<td>$l$</td>
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</tbody>
</table>

- Superconformal symmetry fixes coefficients of $\langle \Phi \Phi^\dagger J \rangle, \langle \Phi \Phi^\dagger N \rangle, \langle \Phi \Phi^\dagger D \rangle$ in terms of $\langle \Phi \Phi^\dagger \mathcal{O} \rangle$
- Must also normalize $J, N, D$ to have canonical 2pt functions
- Superconformal block is then a sum of $g_{\Delta, \ell}$’s for $\mathcal{O}, J, N, D$
Superconformal Blocks

We found, [DP, Simmons-Duffin '10]

\[ G_{\Delta, \ell} = g_{\Delta, \ell} + \frac{(\Delta + \ell)}{4(\Delta + \ell + 1)} g_{\Delta+1, \ell+1} + \frac{(\Delta - \ell - 2)}{4(\Delta - \ell - 1)} g_{\Delta+1, \ell-1} \]
\[ + \frac{(\Delta + \ell)(\Delta - \ell - 2)}{16(\Delta + \ell + 1)(\Delta - \ell - 1)} g_{\Delta+2, \ell} \]

- Unitarity bound \( \Delta \geq \ell + 2 \) saturated \( \rightarrow \) multiplet shortened
- \( G_{\Delta, \ell} \) can also be determined from consistency with \( \mathcal{N} = 2 \) superconformal blocks computed by [Dolan, Osborn '01]
- Similar results for current 4pt functions recently derived by [Fortin, Intriligator, Stergiou '11]
Scalar OPE Coefficient Bounds

Upper bounds on scalar OPE coefficients, $\Delta \phi = 1.01 \ldots 1.66$

- Bound on size of scalar OPE coefficient $\phi \times \phi \sim \lambda_{\mathcal{O}_0} \mathcal{O}_0$
- As $\Delta \phi \to 1$ nicely converges to free value, $\lambda_{\mathcal{O}_0} = \sqrt{2}$ at $\Delta_0 = 2$
Upper and Lower Bounds on $\Phi^2$ OPE Coefficient in SCFTs

- Now we consider the OPE $\Phi \times \Phi \sim \Phi^2 + \ldots$, where $\Delta_{\Phi^2} = 2\Delta_{\Phi}$
- Scalar descendants of non-chiral operators $\overline{Q}^2 \mathcal{O}$ can appear, but unitarity forces $\Delta_{\overline{Q}^2 \mathcal{O}} \geq |2\Delta_{\Phi} - 3| + 3$
- *Lower bounds* possible due to gap in dimensions for $\Delta_{\Phi} < 3/2$
Higher-Spin Protected Operators in $\Phi \times \Phi$

- $\Phi \times \Phi$ OPE also has higher-spin protected operators $\lambda(\overline{QO})_\ell$
- Gap since $\Delta(\overline{QO})_\ell = 2d + \ell$ while $\Delta(\overline{Q^2O})_\ell \geq |2d - 3| + 3 + \ell$
- Dashed lines large-$N$ values...deviations tightly constrained!
The Stress Tensor

$T^{ab}$ is a $\Delta = 4, \ell = 2$ operator present in every CFT:

- Ward identity fixes $\langle \phi \phi T \rangle \propto \Delta \phi$
- Only unknown: $\langle TT \rangle \propto c$, the central charge
- In SCFT, $T$ part of $U(1)_R$ current multiplet ($\Delta = 3, \ell = 1$)

$$J^a = J^a_R + \theta \sigma_b \bar{\theta} T^{ab} + \ldots$$

- Conformal block contributions are

$$\langle \phi \phi \phi \phi \rangle \sim \frac{\Delta^2 \phi}{360c} g_{4,2} \quad \text{(general CFTs)}$$

$$\langle \Phi \Phi^\dagger \Phi \Phi^\dagger \rangle \sim \frac{\Delta^2 \phi}{72c} g_{3,1} \quad \text{(SCFTs)}$$
Lower Bounds on $c$

- Bound smoothly approaches free values as $\Delta\phi \to 1$
  - $c_{\text{free}} = \frac{1}{120}$ (real scalar)
  - $c_{\text{chiral}} = \frac{1}{24}$ (chiral superfield)
- If a CFT contains a $\Delta\phi = 1$ scalar, $c = c_{\text{free}} + c_{\text{int}} \geq c_{\text{free}}$
- In dual AdS$_5$ description, $c \sim R^3 M_P^3$
  - Bound $\to$ Fundamental limit to strength of quantum gravity!
Lower Bounds on $c$ for $SO(N)$ or $SU(N)$, $N = 2..15$

- All lower bounds approach the free values $Nc_{\text{free}}$ or $Nc_{\text{chiral}}$ as $\Delta \phi \to 1$, growing linearly with $N$ near $\Delta \phi \sim 1$

- Also similar bounds on current 2pt functions: $\langle J^I J^J \rangle \propto \kappa \delta^{IJ}$
  - Bound on strength of bulk gauge couplings in AdS$_5$!
Current 2pt Function Bounds in SCFTs

SUSY lower bound on $\kappa$ for SU($N$) adjoint currents, $N = 2..15$

- Lower bounds on coefficient $\langle J^I J^J \rangle \propto \kappa \delta^{IJ}$, if $J^I$ is the adjoint SU($N$) global symmetry current appearing in $\Phi^i \times \Phi^{j\dagger}$
• Bounds on coefficient $\langle J^I J^J \rangle \propto \kappa \delta^{IJ}$, assuming $J^I$ is a singlet under the $SU(N)$ global symmetry

• In SCFTs $\kappa \delta^{IJ} = -3 \text{Tr}(F^I F^J R)$ is calculable!
Bounds on Current 2pt Function and Comparison to SQCD

SUSY lower bounds on $\kappa_R$ using $\text{SU}(N_f)_L$, $N_f = 2..15$

Conformal $\text{SU}(N_c)$ SQCD: $\frac{3}{2}N_c < N_f < 3N_c$, Mesons: $M = \bar{Q}\bar{Q}$

- $\text{SU}(N_f)_L \times \text{SU}(N_f)_R$: $M \times M^\dagger \sim J_L + J_R + \ldots$
- Use $\text{SU}(N_f)_L$ crossing relations to bound $\langle J_R J_R \rangle \propto \kappa_R$

Realized values: $d_M = 3 - \frac{3N_c}{N_f}$ and $\kappa_R = \frac{9}{16} \frac{N_c^2}{N_f}$
Bounding Strong/Conformal Technicolor

Preferred Regions for Strong/Conformal Technicolor Models

- Red: Flavor generic (4-ferm op’s have $O(1)$ flavor violation)
- Green: Flavor optimistic (4-ferm op’s Yukawa suppressed)
- 3 lines: Stability against perturbation $cH^+H$ with $c \sim (1, 0.1, 0.01)$