Quasiconformal Approach to Higher Spin Algebras, Their Deformations and Supersymmetric Extensions

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based mainly on Fernando & MG , 0908.3624, 1005.3580, 1008.0702, 1409.2185 , 1015-09130 , and Govil & MG, 1312.2907, 1401.6930.
For a review see MG 1603.02359.

- Geometric quasiconformal realizations of noncompact groups versus their conformal realizations
- Geometric quasiconformal realization of the group $SO(d,2)$ as the invariance group of a "quartic light-cone".
- Quantization of the geometric quasiconformal realizations and the minimal unitary representations
- Deformations of the minimal unitary representation of $SO(d,2)$ and massless conformal fields in d-dimensions.
- Minreps of conformal superalgebras in $d=4,5,6$ dimensions and their deformations via the quasiconformal approach.
- $AdS_{(d+1)}/CFT_d$ higher spin algebras as universal enveloping algebras of the minreps obtained via the quasiconformal approach and their deformations.
- $AdS_{(d+1)}/CFT_d$ higher spin superalgebras in $d=4,5,6$ and their deformations
- Comments and open problems

Conformal versus quasiconformal realizations of symmetry groups

- Minkowski spacetime can be coordinatized by Hermitian 2×2 matrices: $x = \sigma_\mu x^\mu$ where $\sigma_\nu = (1_2, \vec{\sigma})$ can be considered as elements of the Jordan algebra $J_2^{\mathbb{C}}$ with the Jordan product taken as $1/2$ the anticommutator.

$$x \cdot y = \frac{1}{2}(xy + yx) \quad , \qquad (x \cdot y) \cdot x^2 = x \cdot (y \cdot x^2)$$

- Automorphism group of $J_2^{\mathbb{C}} = SU(2) \rightarrow$ rotation group
- Invariance group of the norm form of $J_2^{\mathbb{C}}$:

$$N(x) = Det(x) = \eta_{\mu\nu} x^\mu x^\nu$$

  is $SL(2, C) =$ Lorentz group $=$reduced structure group
- Linear fractional group of $J_2^{\mathbb{C}} = SU(2,2) \rightarrow$ Conformal group $Con(J_2^{\mathbb{C}})$ which leaves invariant light-like distances with respect to the quadratic norm:

$$N(x - y) = 0$$

- Twistor theory is based on such a coordinatization of Minkowskian space-time in $d = 4$.

The Lie algebra of $Con(J_2^{\mathbb{C}})$ has a 3-grading with respect to the Lorentz subalgebra

$$Conf(J_2^{\mathbb{C}}) = K_\mu \oplus J_{\mu\nu} + \mathcal{D} \oplus P_\mu$$

Generalized spacetimes coordinatized by Jordan algebras J of arbitrary degree .

MG 1976

Symmetry Groups of Spacetimes coordinatized by Simple Jordan algebras

$n \times n$ Hermitian matrices over the division algebra $\mathbb{A}$ form a Jordan algebra $J_n^{\mathbb{A}}$ under the symmetric product $A \cdot B \equiv 1/2(AB + BA)$.

J	$Rot(J)$	$Lor(J)$	$Conf(J)$
$J_2^{\mathbb{C}}$	$SU(2)$	$SL(2,\mathbb{C})$	$SU(2,2)$
$J_n^{\mathbb{R}}$	$SO(n)$	$SL(n,\mathbb{R})$	$Sp(2n,\mathbb{R})$
$J_n^{\mathbb{C}}$	$SU(n)$	$SL(n,\mathbb{C})$	$SU(n,n)$
$J_n^{\mathbb{H}}$	$USp(2n)$	$SU^*(2n)$	$SO^*(4n)$
$J_3^{\mathbb{O}}$	F_4	$E_{6(-26)}$	$E_{7(-25)}$
$\Gamma_{(1,d)}$	$SO(d)$	$SO(d,1)$	$SO(d+1,2)$

Table: The complete list of simple Euclidean Jordan algebras and their rotation (automorphism), "Lorentz" (reduced structure) and "Conformal" (linear fractional) groups. The symbols $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ represent the four division algebras. $J_n^{\mathbb{A}}$ denotes a Jordan algebra of $n \times n$ hermitian matrices over $\mathbb{A}$. $\Gamma_{(1,d)}$ denotes the Jordan algebra of Dirac gamma matrices. $J_2^{\mathbb{R}} = \Gamma(1,2)$, $J_2^{\mathbb{C}} = \Gamma(1,3)$, $J_2^{\mathbb{H}} = \Gamma(1,5)$ and $J_2^{\mathbb{O}} = \Gamma(1,9)$

- Not all groups have conformal realizations. The exceptional groups G_2, F_4 and E_8 do not admit a three-graded decomposition with respect to any subalgebra of maximal rank.
- However all simple Lie algebras $\mathfrak{g}$ admit a 5-graded decomposition with respect to a subalgebra $\mathfrak{g}^0$ of maximal rank of the form

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{1} \oplus \mathfrak{g}^{2}$$

  such that $\dim(\mathfrak{g}^{\pm 2}) = 1$.
- The Lie algebra $\mathfrak{g}$ admits a geometric quasiconformal realization on a $2n+1$ dimensional space, where $2n = dim(\mathfrak{g}^1)$, which leaves invariant light-like separations with respect to a quartic distance function ("quartic light-cone").
  MG, Koepsell, Nicolai, 2000
- The quasiconformal realization of a noncompact Lie algebra $\mathfrak{g}$ corresponds to an extension of conformal realizations of certain subgroups of $\mathfrak{g}$.
- Conformal $SO(d,2) \Longrightarrow$ quasiconformal $SO(d+2,4)$.
  Conformal $E_{7(7)} \Longrightarrow$ quasiconformal $E_{8(8)}$.

Geometric realization of $SO(d,2)$ as a quasiconformal group that leaves a quartic light-cone invariant

MG, Koepsell, Nicolai (GKN) (2000) & MG, Pavlyk (2005)

$SO(d,2)$ admits a 5-graded decomposition

$$\mathfrak{so}(d,2) = \mathbf{1}^{(-2)} \oplus (\mathbf{d-2,2})^{(-1)} \oplus [\,\Delta \oplus \mathfrak{so}(d-2) \oplus \mathfrak{su}(1,1)\,] \oplus (\mathbf{d-2,2})^{(+1)} \oplus \mathbf{1}^{(+2)}$$

where Δ is the $\mathfrak{so}(1,1)$ generator that determines the 5-grading.
The generators of the $\mathfrak{so}(d,2)$ quasiconformal action are realized as nonlinear differential operators acting on a $(2d-3)$-dimensional space $\mathcal{T}$ .We shall denote the coordinates of the space $\mathcal{T}$ as $\mathcal{X} = (X^{i,a}, x)$, where $X^{i,a}$ transform in the $(d-2,2)$ representation of $\mathfrak{so}(d-2) \oplus su(1,1)$ subalgebra and x is a singlet coordinate.
$SO(d-2) \times SU(1,1)$ invariant quartic polynomial of the coordinates $X^{i,a}$ is

$$\mathcal{I}_4(X) = \delta_{ij}\delta_{kl}\epsilon_{ac}\epsilon_{bd}X^{i,a}X^{j,b}X^{k,c}X^{l,d}$$

$\epsilon_{ab} = -\epsilon_{ba} \qquad i,j,\cdots = 1,\ldots,d-2;\;\; a,b,\cdots = 1,2$

Quasiconformal realization of the generators of $SO(d,2)$

$$\mathfrak{so}(d,2) = K_- \oplus U_{i,a} \oplus \left[\Delta \oplus \mathcal{L}_{ij} \oplus \mathcal{M}_{ab}\right] \oplus \widetilde{U}_{i,a} \oplus K_+$$

where $\mathcal{L}_{ij}$ and $\mathcal{M}_{ab}$ are the generators of $SO(d-2)$ and $SU(1,1)$ subgroups, respectively.

$$K_+ = \frac{1}{2}\left(2x^2 - \mathcal{I}_4\right)\frac{\partial}{\partial x} - \frac{1}{4}\frac{\partial \mathcal{I}_4}{\partial X^{i,a}}\eta^{ij}\epsilon^{ab}\frac{\partial}{\partial X^{j,b}} + x\,X^{i,a}\frac{\partial}{\partial X^{i,a}}$$

$$U_{i,a} = \frac{\partial}{\partial X^{i,a}} - \eta_{ij}\epsilon_{ab}\,X^{j,b}\frac{\partial}{\partial x}$$

$$\mathcal{L}_{ij} = \eta_{ik}X^{k,a}\frac{\partial}{\partial X^{j,a}} - \eta_{jk}X^{k,a}\frac{\partial}{\partial X^{i,a}}$$

$$\mathcal{M}_{ab} = \epsilon_{ac}X^{i,c}\frac{\partial}{\partial X^{i,b}} + \epsilon_{bc}X^{i,c}\frac{\partial}{\partial X^{i,a}}$$

$$K_- = \frac{\partial}{\partial x} \quad , \quad \Delta = 2\,x\frac{\partial}{\partial x} + X^{i,a}\frac{\partial}{\partial X^{i,a}} \quad , \quad \widetilde{U}_{i,a} = \left[U_{i,a}\,,\,K_+\right]$$

The quartic norm (length) of a vector $\mathcal{X} = \left(X^{i,a}, x\right) \in \mathcal{T}$ is defined as

$$\mathcal{Q}\left(\mathcal{X}\right) = \mathcal{I}_4\left(X\right) + 2\,x^2 .$$

To see the geometric picture behind the above nonlinear realization, one defines a quartic distance function between any two points $\mathcal{X}$ and $\mathcal{Y}$ in the $(2d-3)$ dimensional space $\mathcal{T}$ as

$$d\left(\mathcal{X},\mathcal{Y}\right) = \mathcal{Q}\left(\delta\left(\mathcal{X},\mathcal{Y}\right)\right)$$

where the "symplectic" difference $\delta\left(\mathcal{X},\mathcal{Y}\right)$ is defined as

$$\delta\left(\mathcal{X},\mathcal{Y}\right) = \left(X^{i,a} - Y^{i,a}\,,\, x - y - \eta_{ij}\epsilon_{ab}\,X^{i,a}Y^{j,b}\right) = -\delta\left(\mathcal{Y},\mathcal{X}\right) .$$

where $\eta_{ij}\epsilon_{ab}\,X^{i,a}Y^{j,b}$ a skew symmetric bilinear form.
The lightlike separations between any two points with respect to the quartic distance function are left invariant under the quasiconformal action of $SO(d,2)$. In other words, $SO(d,2)$ acts as the invariance group of a "light-cone" with respect to a quartic distance function in a $(2d-3)$-dimensional space.

$$\mathcal{I}_4(X) = \delta_{ij}\delta_{kl}\epsilon_{ac}\epsilon_{bd}X^{i,a}X^{j,b}X^{k,c}X^{l,d}$$

$\epsilon_{ab} = -\epsilon_{ba} \qquad i,j,\dots = 1,\dots,d-2;\;\; a,b,\dots = 1,2$

Minimal Unitary Representations and Quasiconformal Realizations of Groups: GKN (2001) & MG, Pavlyk (2005)

- Quantization of the quasiconformal realization of a non-compact Lie group leads directly to its minimal unitary representation $\Rightarrow$ Unitary representation over an Hilbert space of square integrable functions of smallest number of variables possible.
- Lie algebra $\mathfrak{g}$ of a quasiconformal realization of a group G can be decomposed as :

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus (\mathfrak{h} \oplus \Delta) \oplus \mathfrak{g}^{+1} \oplus \mathfrak{g}^{+2}$$

$$\mathfrak{g} = E \oplus E^{\alpha} \oplus (J^a + \Delta) \oplus F^{\alpha} \oplus F$$

$\Delta = -\frac{i}{2}(yp + py)$ ($[y, p] = i$) determines the 5-grading and $\Omega^{\alpha\beta}$ is the symplectic invariant tensor of $\mathfrak{h}$ generated by J^a and $\left[\xi^{\alpha}, \xi^{\beta}\right] = \Omega^{\alpha\beta}$ $(\alpha, \beta, .. = 1, 2, ..., 2n)$

$$E = \frac{1}{2}y^2 \qquad E^{\alpha} = y\,\xi^{\alpha}, \qquad J^a = -\frac{1}{2}\lambda^a{}_{\alpha\beta}\xi^{\alpha}\xi^{\beta}$$

$$F = \frac{1}{2}p^2 + \frac{\kappa I_4(\xi^{\alpha})}{y^2}, \qquad F^{\alpha} = [E^{\alpha}, F]$$

$I_4(\xi^{\alpha}) = S_{\alpha\beta\gamma\delta}\xi^{\alpha}\xi^{\beta}\xi^{\gamma}\xi^{\delta} \Leftrightarrow$ *quartic invariant of* $\mathfrak{h}$
Choosing a polarization $\xi^{\alpha} = (x^i, p_j)$ one has $[x^i, p_j] = i\delta^i_j$ $(i, j = 1, 2, .., n)$
Gelfand-Kirillov dimension for the minimal unitary representation is $n + 1$ $\leftrightarrow$ (x^i, y).

- $(E, F, \Delta) \Longrightarrow SL(2, \mathbb{R})$ of Calogero model or conformal quantum mechanics with the quartic invariant I_4 playing the role of coupling constant.
$E + F = \frac{1}{2}(y^2 + p^2) + \frac{\kappa I_4(\xi^{\alpha})}{y^2} \leftrightarrow$ Calogero Hamiltonian.

To obtain the minimal unitary realization of $SO(d,2)$ we "quantize" its quasiconformal realization. Split the $2(d-2)$ variables $X^{i,a}$ into $(d-2)$ coordinates X^i and $(d-2)$ conjugate momenta P_i

$$X^i = X^{i,1} \qquad\qquad P_i = \eta_{ij} X^{j,2}$$

and introduce a momentum p conjugate to the singlet coordinate x and impose CCRs:

$$\left[X^i \,,\, P_j\right] = i\delta^i_j \qquad\qquad [x \,,\, p] = i$$

Convenient to use bosonic annihilation a_i and creation operators $a_i^\dagger$

$$a_i = \frac{1}{\sqrt{2}}\left(X^i + i\,P_i\right) \quad , a_i^\dagger = \frac{1}{\sqrt{2}}\left(X^i - i\,P_i\right) \quad , \left[a_i \,,\, a_j^\dagger\right] = \delta_{ij} \quad , [a_i \,,\, a_j] = \left[a_i^\dagger \,,\, a_j^\dagger\right] = 0\,.$$

Normal ordering ambiguity of coordinates and momenta in the quartic invariant $\mathcal{I}_4$ is resolved by the Jacobi identities of $SO(d,2)$. Note that $SO(d-2)$ is the simple part of the little group of massless particles in d dimensions.

The minimal unitary realization of $\mathfrak{so}(d,2)$ from its quasiconformal realization:

$$\mathfrak{so}(d,2) = \mathfrak{g}^{(-2)} \oplus \mathfrak{g}^{(-1)} \oplus [\Delta \oplus \mathfrak{so}(d-2) \oplus \mathfrak{su}(1,1)] \oplus \mathfrak{g}^{(+1)} \oplus \mathfrak{g}^{+2)}$$
$$K_- \oplus \begin{pmatrix} U_i \\ U_i^\dagger \end{pmatrix} \oplus (\Delta + L_{ij} + M_{ab}) \oplus \begin{pmatrix} W_i \\ W_i^\dagger \end{pmatrix} \oplus K_+$$

$K_- = x^2/2$, $\Delta = \frac{1}{2}(xp + px)$, $K_+ = \frac{1}{2}p^2 + \frac{1}{x^2}\mathcal{G} \Longrightarrow$ Calogero $SL(2,\mathbb{R})$

$$\begin{pmatrix} U_i \\ U_i^\dagger \end{pmatrix} = \begin{pmatrix} x\, a_i \\ x\, a_i^\dagger \end{pmatrix} \quad , \quad \begin{pmatrix} W_i \\ W_i^\dagger \end{pmatrix} = -i \begin{pmatrix} [U_i, K_+] \\ [U_i^\dagger, K_+] \end{pmatrix}$$

$$L_{ij} = i\left(a_i^\dagger a_j - a_j^\dagger a_i\right) \quad , \qquad M_+ = \frac{1}{2} a_i^\dagger a_i^\dagger \quad M_- = \frac{1}{2} a_i a_i \quad M_0 = \frac{1}{4}\left(a_i^\dagger a_i + a_i a_i^\dagger\right)$$

$$SO(d-2) \qquad , \qquad SU(1,1)$$

$$\mathcal{L}^2 = L_{ij} L_{ij} \qquad \mathcal{M}^2 = {M_0}^2 - \frac{1}{2}(M_+ M_- + M_- M_+)$$

$$\mathcal{L}^2 = 8\,\mathcal{M}^2 - \frac{1}{2}(d-2)(d-6)$$

$$\mathcal{G} = \frac{1}{4}\mathcal{L}^2 + \frac{1}{8}(d-3)(d-5) = 2\mathcal{M}^2 + \frac{3}{8}$$

$\mathcal{G}$ plays the role analogous coupling constant in conformal quantum mechanics or Calogero model.
$i, j, \cdots = 1, 2, \ldots, (d-2)$

$$
\begin{array}{ccccccc}
 & & & K_- & & & \\
 & T_i^P & & | & & T_i^X & \\
L_- & --- & & (D^X \oplus L_{ij} \oplus D^P) & & --- & L_+ \\
 & K_i^X & & | & & K_i^P & \\
 & & & K_+ & & &
\end{array}
$$

Table: The 5×5 grading of the Lie algebra of $\mathfrak{so}(d,2)$ in an Hermitian basis. The vertical 5-grading is determined by $\Delta = 1/2(xp + px) = 1/2(D^X + D^P)$ and the horizontal 5-grading is determined by $L_0 = (X_i P_i + P_i X_i) = 1/2(D^X - D^P)$.

$$
\begin{aligned}
[T_i^X, K_j^X] &= -2i\delta_{ij} D^X + 2iL_{ij} \\
[T_i^P, K_j^P] &= 2i\delta_{ij} D^P - 2iL_{ij} \\
[T_i^X, K_j^P] &= iL_+ \\
[T_i^P, K_j^X] &= iL_-
\end{aligned}
$$

where $L_+ = P_i P_i$ and $L_- = X_i X_i$. Therefore the quasiconformal realization of $SO(d,2)$ can be interpreted as the minimal Lie algebra containing the Euclidean conformal Lie algebra acting on transverse coordinates and the dual Euclidean conformal Lie algebra acting on the corresponding transverse momenta. The common subgroup of these two Euclidean conformal groups is $SO(d-2)$.

Considered as a conformal group $SO(d,2)$ has a three-graded decomposition determined by the dilatation generator $\mathcal{D}$:

$$\mathfrak{so}(d,2) = K_\mu \oplus (M_{\mu\nu} + \mathcal{D}) \oplus P_\mu$$

For the minrep the Poincare mass operator vanishes identically : $P_\mu P_\nu \eta^{\mu\nu} = 0$
By going to the compact three graded decomposition determined by the conformal Hamiltonian $H = \frac{1}{2}(K_+ + K_-) + M_0$:

$$\mathfrak{so}(d,2) = \mathfrak{C}^- \oplus (\mathfrak{so}(d) + H) \oplus \mathfrak{C}^+ \,.$$

one finds that the minrep is a unitary lowest weight (positive energy) representation with the lowest weight vector

$$H\,\psi_0^{\alpha_g}(x)\,|0\rangle = \frac{1}{4}(d-2)\,\psi_0^{\alpha_g}(x)\,|0\rangle$$

$$\psi_0^{(\alpha_g)}(x) = C_0\,x^{\alpha_g}\,e^{-x^2/2} \quad , \quad \alpha_g = \frac{(d-3)}{2} \quad , \quad a_i|0\rangle = 0$$

The Hilbert space of the minrep is spanned by states that are in the tensor product of Fock space of $(d-2)$ ordinary bosonic oscillators and the states of the singular oscillator that form irrep of $SU(1,1)_K$ subgroup with the lowest weight vector $\psi_0^{(\alpha_g)}(x)$.
The minrep describes a massless conformal scalar field in d dimensional Minkowski space.

Deformations of the minimal unitary realization of $\mathfrak{so}(d,2)$:

MG, Fernando 2015

$$\mathfrak{so}(d,2) = \mathfrak{g}^{(-2)} \oplus \mathfrak{g}^{(-1)} \oplus [\Delta \oplus \mathfrak{so}(d-2) \oplus \mathfrak{su}(1,1)] \oplus \mathfrak{g}^{(+1)} \oplus \mathfrak{g}^{+2)}$$

$$K_- \oplus \begin{pmatrix} U_i \\ U_i^\dagger \end{pmatrix} \oplus (\Delta + L_{ij} + M_{ab}) \oplus \begin{pmatrix} W_i \\ W_i^\dagger \end{pmatrix} \oplus K_+$$

$K_- = x^2/2$, $\Delta = \frac{1}{2}(xp+px)$, $K_+ = \frac{1}{2}p^2 + \frac{1}{x^2}\mathcal{G} \Longrightarrow$ Calogero $SL(2,\mathbb{R})$

$$\begin{pmatrix} U_i \\ U_i^\dagger \end{pmatrix} = \begin{pmatrix} x\, a_i \\ x\, a_i^\dagger \end{pmatrix} \quad , \quad \begin{pmatrix} W_i \\ W_i^\dagger \end{pmatrix} = -i \begin{pmatrix} [U_i, K_+] \\ [U_i^\dagger, K_+] \end{pmatrix}$$

$$J_{ij} = L_{ij} + S_{ij} \quad , \quad M_+ = \frac{1}{2} a_i^\dagger a_i^\dagger \quad M_- = \frac{1}{2} a_i a_i \quad M_0 = \frac{1}{4}\left(a_i^\dagger a_i + a_i a_i^\dagger\right)$$

$S_{ij} =$ " Spin Generators of little group" , $\mathcal{S}^2 = S_{ij}S_{ij}$, $\mathcal{J}^2 = J_{ij}J_{ij}$, $\mathcal{L}^2 = L_{ij}L_{ij}$

Coupling "constant" $\mathcal{G} = \left(\frac{1}{2}\mathcal{J}^2 - \frac{1}{4}\mathcal{L}^2 - \frac{(d-6)}{2(d-2)}\mathcal{S}^2 + \frac{1}{8}(d-3)(d-5)\right)$

Jacobi identities require

$$\Delta_{ij} = S_{ik}S_{jk} + S_{jk}S_{ik} - \frac{2}{(d-2)}\mathcal{S}^2\,\delta_{ij} = 0$$

Remarkably these are precisely the identities satisfied by the massless representations of Poincaré group in d dimensions that extend to the unitary representations of conformal group! (Angelopoulos & Laoues 1997).

Therefore there exists a one-to-one correspondence between the minrep of $SO(d,2)$ and its deformations and massless conformal fields in d dimensions.

Deformations of the minrep of $SO(d,2)$ for odd d:

There is a unique deformation of the minrep given by realizing the spin generators S_{ij} of $SO(d-2)$ as:

$$S_{ij} = \frac{1}{4}[\gamma_i, \gamma_j]$$

where γ_i are Euclidean gamma matrices in $(d-2)$ dimensions.

$$\mathcal{G} = \frac{1}{4}\,\mathcal{L}^2 + \epsilon\,\mathcal{L}\cdot\mathcal{S} + \epsilon\,\frac{1}{2}\,(d-3) + \frac{1}{8}\,(d-3)\,(d-5)$$

$\epsilon = 0$ for the minrep $\Longrightarrow$ Scalar singleton (massless conformal field)
Lowest energy irrep (K-type) is a singlet of $SO(d)$.
$\epsilon = 1$ deformed minrep $\Longrightarrow$ spinor singleton (massless spinor field).
Lowest energy irrep (K-type) is a spinor of $SO(d)$.
They are the analogs of the remarkable representations of $SO(3,2)$ discovered by Dirac.

Deformations of the minrep of $SO(d,2)$ for even d:

There exist infinitely many deformations of the minrep of $SO(d,2)$ for even d. One can realize the generators S_{ij} of the little group $SO(d-2)$ in the Fock space of fermionic oscillators transforming irreducibly under the subgroup $U((d-2)/2)$.

Construction of the representations of the little group $SO(d-2)_S$ of massless particles, generated by S_{ij} over the Fock space of Fermionic oscillators using its 3-grading w.r.t $\mathfrak{u}((d-2)/2)$:

$$\mathfrak{so}(d-2)_S = \mathfrak{g}^{(-1)} \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(+1)} = Z_{rs} \oplus T_{rs} \oplus Z^{\dagger}_{rs}$$

$$Z_{rs} = \vec{\alpha}_r \cdot \vec{\beta}_s - \vec{\alpha}_s \cdot \vec{\beta}_r + \varepsilon\, \xi_r \xi_s$$
$$T_{rs} = \vec{\alpha}^{\dagger}_r \cdot \vec{\alpha}_s - \vec{\beta}_s \cdot \vec{\beta}^{\dagger}_r + \frac{\varepsilon}{2}\left(\xi^{\dagger}_r \xi_s - \xi_s \xi^{\dagger}_r\right)$$
$$Z^{\dagger}_{rs} = -\vec{\alpha}^{\dagger}_r \cdot \vec{\beta}^{\dagger}_s + \vec{\alpha}^{\dagger}_s \cdot \vec{\beta}^{\dagger}_r - \varepsilon\, \xi^{\dagger}_r \xi^{\dagger}_s$$

where $r, s, \cdots = 1, 2, \ldots, (d-2)/2$; $\epsilon = 0, 1$ and $\vec{\alpha}_r \cdot \vec{\beta}_s = \sum_{K=1}^{P} \alpha_r(K)\beta_s(K)$.
The representations of even orthogonal groups $SO(d-2)$ that satisfy the constraint $\Delta_{ij} = 0$ are

$$(0, \ldots, 0, 0, f)_D = \left(\frac{f}{2}, \ldots, \frac{f}{2}, \frac{f}{2}\right)_{GZ} \qquad (0, \ldots, 0, f, 0)_D = \left(\frac{f}{2}, \ldots, \frac{f}{2}, -\frac{f}{2}\right)_{GZ}$$

where $f = 2P + \epsilon$ is the number of colors of fermionic oscillators.
They have the following lowest weight vectors of $SO(d-2)$ in the fermionic Fock space:

$$(0, \ldots, 0, 0, f)_D \iff |0\rangle$$
$$(0, \ldots, 0, f, 0)_D \iff \alpha^{\dagger}_{(r_1}(1)\beta^{\dagger}_{s_1}(1)\alpha^{\dagger}_{r_2}(2)\beta^{\dagger}_{s_2}(2)\ldots\alpha^{\dagger}_{r_P}(P)\beta^{\dagger}_{s_P}(P)\xi^{\dagger}_{t)}\,|0\rangle \equiv |Symvac\rangle$$

and $(r_1, s_1, \ldots, t)$ denotes complete symmetrization of indices.

K-type decomposition of the minrep of $SO(d,2)$ and its deformations

Eigenvalues of the quadratic Casimir of $\mathcal{S}^2$ are

$$\mathcal{S}^2 \Longrightarrow \frac{1}{4}(d-2)f(d+f-4)$$

for the "massless irreps" of $SO(d-2)$ for even d. The "coupling constant" $\mathcal{G}$ that appears in the generator K_+:

$$\mathcal{G} = \frac{1}{4}\mathcal{L}^2 + \mathcal{L}\cdot\mathcal{S} + \frac{2}{(d-2)}\mathcal{S}^2 + \frac{1}{8}(d-3)(d-5)$$

The state $\left|\Omega_{(f)}(0)\right\rangle = x^{\alpha_{g(f)}} e^{-x^2/2} \left|0\right\rangle$ is an eigenstate of the conformal Hamiltonian H with eigenvalue $E_0(f) = \frac{1}{2}(d+f-2)$ for $\alpha_{g(f)} = \frac{1}{2}(d+2f-3)$ and can be chosen as the lowest weight vector of a unitary representation of $SO(d,2)$ whose K-type decomposition with respect to $SO(2)\times SO(d)$ is:

$$\sum_{n=0} [(E_0(f)+n),(n,0,\ldots,f)_D]$$

Similarly, the unitary representation with the lowest weight vector $\left|\Omega_{(f)}(Symvac)\right\rangle = x^{\alpha_{g(f)}} e^{-x^2/2} \left|Symvac\right\rangle$ has the K-type decomposition

$$\sum_{n=0} [(E_0(f)+n),(n,0,\ldots,f,0)_D]$$

Minimal unitary representation of $4d$ conformal group $SU(2,2)$ by quantization of its quasiconformal action

MG, Pavlyk (2006), MG, Fernando 2009

The Lie algebra $\mathfrak{su}(2,2)$ admits a 5-grading:

$$\mathfrak{su}(2,2) = \mathbf{1}^{(-2)} \oplus \mathbf{4}^{(-1)} \oplus [\mathfrak{su}(1,1) \oplus \mathfrak{u}(1) \oplus \mathfrak{so}(1,1)] \oplus \mathbf{4}^{(+1)} \oplus \mathbf{1}^{(+2)}$$

$$\mathfrak{su}(2,2) = E \oplus (E^1, E^2, E_1, E_2) \oplus [J^n_m, U, \Delta] \oplus (F^1, F^2, F_1, F_2) \oplus F \,.$$

Quantization leads to a minimal unitary realization over the Hilbert space of functions in three variables which correspond to the tensor product of Fock spaces of two oscillators(d,g) with the state space of a singular oscillator (conformal):

$$\mathfrak{su}(1,1) = (dg, \frac{1}{2}(N_d + N_g + 1), d^\dagger g^\dagger) \quad , \mathfrak{u}(1) = U = N_d - N_g$$

$$E = \frac{1}{2}x^2, \quad \Delta = \frac{1}{2}(xp + px) \quad , F = \frac{1}{2}p^2 + \frac{1}{2\,x^2}\left[U^2 - \frac{1}{4}\right] \Leftrightarrow \mathfrak{sl}(2,\mathbb{R})_{conformal}$$

$$E^1 = x\,d^\dagger \qquad E^2 = x\,g \qquad E_1 = x\,d \qquad E_2 = -x\,g^\dagger$$

$$F^1 = [E^1, F] \quad F^2 = [E^2, F] \quad F_1 = [E_1, F] \quad F_2 = [E_2, F]$$

where $[x,p] = i$.

By going to the compact $SU(2) \times SU(2) \times U(1)$ basis one can show that the minimal unitary representation of $SU(2,2)$ is simply the scalar doubleton representation corresponding to a massless conformal scalar field in $4d$.

The above minrep admits a one parameter family of deformations obtained simply by the shift

$$\mathcal{I}_4 \longrightarrow \mathcal{I}_4(\zeta) = (N_d - N_g + \zeta)^2 - 1 = (U + \zeta)^2 - 1$$

Then grade $+2$ generator becomes

$$F(\zeta) = \frac{1}{2}p^2 + \frac{1}{2x^2}\left[(U+\zeta)^2 - \frac{1}{4}\right]$$

while the negative grade generators E, E^m and E_m remain unchanged and helicity generator U in $\mathfrak{g}^{(0)}$ becomes

$$U(\zeta) = U + \frac{\zeta}{2}\,.$$

The quadratic Casimir of $SU(2,2)$ takes on the value $\mathcal{C}_2(\zeta) = -\frac{1}{2}\left(\frac{\zeta}{2} - 1\right)\left(\frac{\zeta}{2} + 1\right)$.

The Poincaré mass operator $P_\mu P^\mu$ vanishes identically for the minrep and its deformations. For integer values of ζ they are isomorphic to the doubleton irreps of $SU(2,2)$ corresponding to conformal fields in the representation $(\zeta/2, 0)$ or $(0, -\zeta/2)$ of the Lorentz group.

For integer ζ they remain irreducible under the restriction to the Poincaré subgroup and describe massless particles of helicity $\zeta/2$. Mack & Todorov

Minimal Unitary Supermultiplets of $SU(2,2|p+q)$ and their deformations

Fernando , MG (2009)

The Lie superalgebra $\mathfrak{su}\,(2,2\,|\,\mathfrak{p}+\mathfrak{q})$ has the following 5-graded decomposition with respect to its subalgebra $\mathfrak{su}\,(1,1\,|\,\mathfrak{p}+\mathfrak{q}) \oplus \mathfrak{u}\,(1) \oplus \mathfrak{so}\,(1,1)$:

$$1^{(-2)} \oplus 2\,(2,\mathfrak{p}+\mathfrak{q})^{(-1)} \oplus [\mathfrak{su}\,(1,1\,|\,\mathfrak{p}+\mathfrak{q}) \oplus \mathfrak{u}\,(1) \oplus \mathfrak{so}\,(1,1)] \quad \oplus 2\,(2,\mathfrak{p}+\mathfrak{q})^{(+1)} \oplus 1^{(+2)}$$

The grade+2 generator is $F = \frac{1}{2}p^2 + \frac{1}{2\,x^2}\left[\left(N_d - N_g + N_\alpha - N_\beta + \zeta\right)^2 - \frac{1}{4}\right]$

ζ is the deformation parameter. $N_\alpha = \vec{\alpha}^\dagger \cdot \vec{\alpha}$ and $N_\beta = \vec{\beta}^\dagger \cdot \vec{\beta}$ are the number operators of the Fermionic oscillators . For $\zeta = 0$ one obtains the minimal unitary supermultiplet.

For $PSU(2,2|4)$ the minimal unitary supermultiplet is simply the $N=4$ Yang-Mills supermultiplet. The deformations of the Yang-Mills supermultiplet are the non-CPT self-conjugate higher spin doubleton supermultiplets (of maximal spin range 2).

For $SU(2,2|8)$ the minimal unitary supermultiplet corresponds to the CPT self-conjugate massless supermultiplet of maximal $N=8$ supergravity in four dimensions. The deformations of the minimal unitary supermultiplet $SU(2,2|8)$ are higher spin supermultiplets of maximal spin range 4.

Minrep of the 6d conformal superalgebra $OSp(8^*|2N)$ and its deformations

Fernando , MG (2010)

The superalgebra $\mathfrak{osp}(8^*|2N)$ has the 5-grading such that $dim(\mathfrak{g}^{(\pm 2)}) = 1$:

$$\mathfrak{osp}(8^*|2N) = \mathfrak{g}^{(-2)} \oplus \mathfrak{g}^{(-1)} \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(+1)} \oplus \mathfrak{g}^{(+2)}$$

$$\mathfrak{g}^{(0)} = \mathfrak{osp}(4^*|2N) \oplus \mathfrak{su}(2)_T \oplus \mathfrak{so}(1,1)_\Delta . \qquad \mathfrak{g}^{(\pm 2)} \equiv K_\pm$$

$$K_+ = \frac{1}{2} p^2 + \frac{1}{4\, x^2} \left(8\, \mathcal{T}^2 + \frac{3}{2} \right)$$

To deform the minimal unitary realization of $\mathfrak{so}^*(8)$, one extends the subalgebra $\mathfrak{su}(2)_T$ by adding a " Spin " term which can be realized in terms of fermionic oscillators. The deformations of the 6d conformal group $SO^*(8) = SO(6,2)$ and $OSp(8^*|2N)$ are labelled by the eigenvalues $t(t+1)$ of the quadratic Casimir of $SU(2)_T$ subgroup of the little group $SO(4)$ of massless particles in 6d. The true minrep of $SO^*(8)$ corresponds to a conformal scalar field in 6d ($t = 0$). Its discrete set of deformations labelled by t are isomorphic to the doubletons of $SO^*(8)$.

$$T_+ \rightarrow \mathcal{T}_+ = a^m b_m + \rho^x \chi_x \quad T_- \rightarrow \mathcal{T}_- = b^m a_m + \chi^x \rho_x \quad T_0 \rightarrow \mathcal{T}_0 = \frac{1}{2} (N_a - N_b + N_\rho - N_\chi)$$

The minimal unitary supermultiplet of $OSp(8^*|2N)$ and its deformations are isomorphic to doubleton supermultiplets.

Singletons of $SO(3,2) = Sp(4,\mathbb{R})$:

Consider now a pair of bosonic oscillators $a_i, a_i^\dagger$ ($i = 1,2$) and define a twistorial (Majorana) spinor Ψ and its Dirac conjugate in terms of these oscillators $\overline{\Psi} = \Psi^\dagger \gamma_0$

$$\Psi = \begin{pmatrix} a_1 \\ a_2 \\ a_2^\dagger \\ -a_1^\dagger \end{pmatrix}, \qquad \overline{\Psi} = \begin{pmatrix} a_1^\dagger & a_2^\dagger & -a_2 & a_1 \end{pmatrix}$$

Let Σ_{AB} be the matrices of $SO(3,2)$ generators in the spinor representation. Then the bilinears $M_{AB} = 2\overline{\Psi}\Sigma_{AB}\Psi$ generate the Lie algebra of $SO(3,2)$.
The generator J_{ABCD} of Joseph ideal vanishes identically for the singletonic realization.

The Fock space of these oscillators decompose into two ireducible unitary representations of $Sp(4,\mathbb{R}) = Spin(3,2)$ that are simply the two remarkable representations of Dirac which were called *Di* and *Rac* . They correspond to massless scalar and spinor fields in three dimensions.
$OSp(2N|4,\mathbb{R})$ admits two super singleton supermultiplets in which the *Di* and *Rac* fields transform in the chiral spinor representations of $SO(2N)$.
Note that for symplectic groups the quartic invariant I_4 that appears in the quasiconformal realization vanishes and the realization of the generators involve bilinears only (free field construction).

Minimal unitary realization of $SO(5,2)$ Fernando, MG 2014

$$\begin{aligned} \mathfrak{so}(5,2) &= \bar{1} \oplus \overline{(3,2)} \oplus [\, \mathfrak{so}(1,1) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(1,1) \,] \oplus (3,2) \oplus 1 \\ &= K_- \oplus (U_i,\ U_i^\dagger) \oplus (L_i\,, \Delta\,, (M_+, M_-\,, M_0)) \oplus (W_i + W_i^\dagger) \oplus K_+ \\ K_- &= \frac{1}{2}x^2 \qquad U_i = x\, a_i \qquad U_i^\dagger = x\, a_i^\dagger\,. \quad i = 1,2,3 \end{aligned}$$

$$\mathfrak{su}(1,1) \Rightarrow \qquad M_+ = \frac{1}{2} a_i^\dagger a_i^\dagger \qquad M_- = \frac{1}{2} a_i a_i \qquad M_0 = \frac{1}{4}\left(a_i^\dagger a_i + a_i a_i^\dagger \right)$$

$$\mathfrak{su}(2) \Rightarrow L_i = \epsilon_{ijk} a_j^\dagger a_k \qquad \mathcal{L}^2 = L_i L_i$$

$$K_+ = \frac{1}{2}p^2 + \frac{1}{4\,x^2}\left(8\,\mathcal{M}^2 + \frac{3}{2} \right) = \frac{1}{2}p^2 + \frac{1}{2\,x^2}\mathcal{L}^2$$

The minrep of $SO(5,2)$ describes a massless conformal scalar field in $d=5$.

Deformation of the minimal unitary representation of $SO(5,2)$:

$$L_i \Longrightarrow J_i = L_i + S_i \qquad S_i = \frac{1}{2}\zeta^\dagger \sigma_i \zeta \qquad , \zeta = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

$$K_+ = \frac{1}{2}p^2 + \frac{1}{2\,x^2}\left(2\,\mathcal{J}^2 - \mathcal{L}^2 + \frac{2}{3}\,\mathcal{S}^2 \right)$$

Deformed minrep describes a massless conformal spinor field in $d=5$. No other deformations!. Scalar and spinor singletons in $d=5$ similar to the situation in $d=3$.

Minimal unitary representation of the 5d superconformal algebra $F(4)$ with the even subalgebra $SO(5,2) \oplus SU(2)$

Fernando, MG 2014

$$\mathfrak{f}(4) = \mathfrak{g}^{(-2)} \oplus \mathfrak{g}^{(-1)} \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(+1)} \oplus \mathfrak{g}^{(+2)}$$
$$= 1_B \oplus (6_B \oplus 4_F) \oplus [d(2,1;2) \oplus \Delta] \oplus (6_B \oplus 4_F) \oplus 1_B$$

The $d(2,1;2)$ has the even subalgebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(1,1)$ and admits a 10-dimensional linear representation.
The conformal superalgebra $\mathfrak{f}(4)$ has a noncompact 5-grading in a manifestly $SO(4,1) = USp(2,2)$ covariant form:

$$\mathfrak{f}(4) = \mathcal{K}_\mu \;\oplus\; \mathfrak{S}_{\alpha r} \;\oplus\; [\mathcal{D}\,,\, \mathcal{M}_{\mu\nu}\,,\, T_{\pm,0}] \;\oplus\; \mathfrak{Q}_{\alpha r} \;\oplus\; \mathcal{P}_\mu$$

where $\mu,\nu = 0,1,2,3,4$; $\alpha = 1,2,3,4$; and $r = 1,2$.
$\mathcal{M}_{\mu\nu}$ are the generators of $Spin(4,1) \approx USp(2,2)$. $\mathcal{K}_\mu$ are the special conformal generators and $\mathcal{P}_\mu$ are the translations. The $\mathfrak{S}_{\alpha r}$ and $\mathfrak{Q}_{\alpha r}$ are the special conformal and Poincare supersymmetry generators.
Supersymmetry generators Ξ^r_α in the $(8,2)$ representation of $SO(5,2) \times SU(2)_T$ satisfy:

$$\left\{\Xi^r_\alpha\,,\, \Xi^s_\beta\right\} = i\epsilon^{rs}\, M_{AB}\left(\Sigma^{AB}\mathcal{C}_7\right)_{\alpha\beta} + 3i\,(\mathcal{C}_7)_{\alpha\beta}\,(i\sigma_2\sigma^i)^{rs}\, T_i$$

where $r,s = 1,2$ are the $SU(2)_T$ spinor indices, M_{AB} are the $SO(5,2)$ generators
The minimal unitary supermultiplet of $F(4)$ consists of two scalar singletons in a doublet of R-symmetry group $SU(2)_T$ and a spinor singleton which is a singlet of $SU(2)_T$.

Minimal unitary representation and the Joseph ideal

Among all the unitary representations of a noncompact Lie group the minimal one is distinguished by the fact that it is annihilated by the Joseph ideal inside its universal enveloping algebra. Denoting the generators of the Lie algebra of $SO(n-2,2)$ as G_{ab} the Joseph ideal is generated by the following elements of the enveloping algebra:

Eastwood et.al.(2005)

$$J_{abcd} = G_{ab}G_{cd} - G_{ab} \circledcirc G_{cd} - \frac{1}{2}[G_{ab}, G_{cd}] + \frac{n-4}{4(n-1)(n-2)}.\langle G_{ab}, G_{cd}\rangle$$

where $\langle G_{ab}, G_{cd}\rangle$ is the Killing form,
$G_{ab} \circledcirc G_{cd}$ is the Cartan product which for orthogonal groups ($SO(n-2,2)$) can be written as:

$$\begin{aligned} G_{ab} \circledcirc G_{cd} \equiv\ & \frac{1}{3}G_{ab}G_{cd} + \frac{1}{3}G_{dc}G_{ba} + \frac{1}{6}G_{ac}G_{bd} - \frac{1}{6}G_{ad}G_{bc} + \frac{1}{6}G_{db}G_{ca} - \frac{1}{6}G_{cb}G_{da} \\ & - \frac{1}{2(n-2)}\left(G_{ae}G_c^e\eta_{bd} - G_{be}G_c^e\eta_{ad} + G_{be}G_d^e\eta_{ac} - G_{ae}G_d^e\eta_{bc}\right) \\ & - \frac{1}{2(n-2)}\left(G_{ce}G_a^e\eta_{bd} - G_{ce}G_b^e\eta_{ad} + G_{de}G_b^e\eta_{ac} - G_{de}G_a^e\delta_{bc}\right) \\ & + \frac{1}{(n-1)(n-2)}G_{ef}G^{ef}\left(\eta_{ac}\eta_{bd} - \eta_{bc}\eta_{ad}\right) \end{aligned}$$

where η_{ab} is the $SO(n-2,2)$ invariant metric.

Higher spin algebras and superalgebras

Early work on the connection between high spin (super)algebras and the universal enveloping algebras of singleton representations of AdS Lie(super)algebras:

Fradkin-Vasiliev higher spin algebra in AdS_4 as the enveloping algebra of the singletonic realization of $Sp(4,\mathbb{R})$ and proposal to extend it to AdS_5 and AdS_7 HS algebras using the doubletonic realizations of $SU(2,2)$ and $SO^*(8)$ MG (1989)

AdS_3 higher spin algebras and universal enveloping algebras. Konstein and Vasiliev (1989)

The quotient of the universal enveloping algebra (UEA) $\mathcal{U}(o(n-2,2))$ of $o(n-2,2)$ by its annihilator on the scalar "singleton module" is the AdS_{n-1}/CFT_{n-2} higher-spin algebra. Vasiliev (2003)

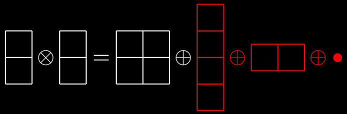
The ideal with which to quotient the UEA is the Joseph ideal that annihilates the minrep. Eastwood (2005)

I will adopt this definition and define the $AdS_{(d+1}/CFT_d$ higher spin algebras as the universal enveloping algebras $\mathcal{U}(SO(d,2))$ of $SO(d,2)$ quotiented by their Joseph ideals $\mathcal{J}(SO(d,2)$:

$$HS(SO(d,2)) \equiv \frac{\mathcal{U}(SO(d,2)}{\mathcal{J}(SO(d,2)}$$

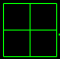
The generators J_{ABCD} of Joseph ideal vanish identically as operators for the singletonic realization of the Lie algebra $SO(3,2)$ as bilinears of covariant twistorial oscillators. Therefore the AdS_4/CFT_3 HS algebra is given simply by the enveloping algebra of the singletonic realization of $Sp(4,\mathbb{R})$.

Poincare-Birkhoff-Witt theorem: The enveloping algebra $\mathscr{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ decomposes into symmetric tensor products of the adjoint representation of $\mathfrak{g}$. For $\mathfrak{so}(d,2)$ the symmetric product of the adjoint representation decomposes as:



where the singlet $\bullet$ is the quadratic Casimir of $SO(d,2)$.

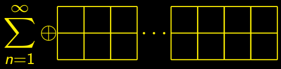
Vasiliev: the higher spin algebra $HS(\mathfrak{g})$ must be a quotient of $\mathscr{U}(\mathfrak{g})$ since the higher spin gauge fields are described by traceless two row Young tableaux. Hence the relevant ideal should quotient out all the diagrams except the first one in the above decomposition.

The Joseph ideal generators include all the diagrams indicated in red on the RHS and does not include the "window" diagram . Therefore by quotienting $\mathscr{U}(\mathfrak{g})$ by the Joseph ideal generated by J_{ABCD} we get rid of all the "unwanted" diagrams and obtain the standard higher spin algebra $HS(d,2)$ a la Eastwood & Vasiliev .

Quasiconformal construction of the minrep and the Joseph ideal:

For the minrep obtained by quantization of the quasiconformal realization of $SO(d,2)$ the generators of Joseph ideal vanish identically as operators ($J_{ABCD} \equiv 0$)

The universal enveloping algebras of the minreps of $SO(d,2)$ obtained by quasiconformal methods yield directly the higher spin algebras in the respective dimensions: The resulting enveloping algebra decomposes into operators whose $SO(d,2)$ Young tableaux have only two rows corresponding to higher spin gauge fields described by traceless two row Y-Ts. The operators in the symmetric product of the generators with four rows and one column and one row and two columns vanish identically. K. Govil & MG (2013/14) for $d = 3, 4, 6$ and Fernando & MG for $d = 5$ and $d > 6$.



Deformations of the minimal unitary representations and their associated ideals :

$4d$ covariant identities satisfied by the generators of the minrep of $SU(2,2)$and its deformations labelled by ζ:

$$P^2 = P^\mu P_\mu = 0 \quad , \quad K^2 = K^\mu K_\mu = 0$$

$$4\mathcal{D}\cdot\mathcal{D} + M^{\mu\nu}\cdot M_{\mu\nu} + P^\mu \cdot K_\mu = 0$$

$$P^\mu \cdot (M_{\mu\nu} + \eta_{\mu\nu}\mathcal{D}) = 0 \quad , \quad K^\mu \cdot (M_{\nu\mu} + \eta_{\nu\mu}\mathcal{D}) = 0$$

$$\eta^{\mu\nu} M_{\mu\rho}\cdot M_{\nu\sigma} - P_{(\rho}\cdot K_{\sigma)} + 2\eta_{\rho\sigma} = \frac{\zeta^2}{2}\eta_{\rho\sigma}$$

$$M_{\mu\nu}\cdot M_{\rho\sigma} + M_{\mu\sigma}\cdot M_{\nu\rho} + M_{\mu\rho}\cdot M_{\sigma\nu} = \zeta\epsilon_{\mu\nu\rho\sigma}\mathcal{D}$$

$$\mathcal{D}\cdot M_{\mu\nu} + P_{[\mu}\cdot K_{\nu]} = -\frac{\zeta}{2}\epsilon_{\mu\nu\rho\sigma}M^{\rho\sigma}$$

$\zeta = 0$ corresponds to the vanishing of the generators J_{ABCD} of the Joseph ideal.

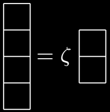
The Pauli-Lubanski vector, W^μ, and its conformal analogue, V^μ:

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} M_{\nu\rho} \cdot P_\sigma = \zeta P^\mu$$
$$V^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} M_{\nu\rho} \cdot K_\sigma = -\zeta K^\mu$$

Therefore the enveloping algebras of the minrep and its deformations provide a one-parameter family of AdS_5/CFT_4 higher spin algebras defined as

$$HS(4,2;\zeta) = \frac{\mathcal{U}(SO(4,2))}{\mathcal{J}_\zeta(SO(4,2))}$$

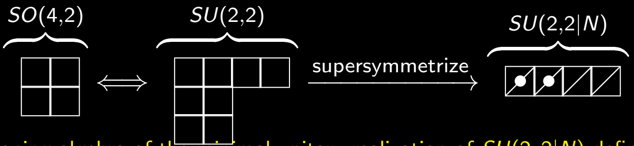
$\mathcal{J}_\zeta(SO(4,2)) \equiv$ deformed Joseph ideal. For the deformed minrep of $SO(4,2)$ one finds



Compare with the work of Boulanger, Ponomarev, Skvortsov and Taronna (2013) on the existence of one-parameter family of bosonic AdS_5/CFT_4 higher spin algebras using Young-Tableaux analysis. The physical meaning of this parameter is not clear within Y-T approach. Furthermore they do not find any deformations in higher dimensions.

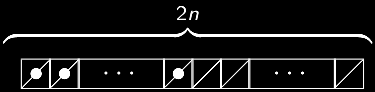
In the quasiconformal approach the deformation parameter is helicity. Furthermore QCG approach leads to unitary realizations for the entire one-parameter family of higher spin algebras as well as superalgebras.

For supersymmetric extensions obeying the usual spin and statistics connection one has to take the covering group $SU(2,2)$ of $SO(4,2)$ and use $SU(2,2)$ Young tableaux.



The enveloping algebra of the minimal unitary realization of $SU(2,2|N)$ defines a supersymmetric extension of the AdS_5/CFT_4 bosonic higher spin algebra obeying usual spin and statistics connection. the generators of the supersymmetric extension of the undeformed ($\zeta = 0$) bosonic higher spin algebra decompose with respect to $SU(2,2|N)$ as:

$$HS[SU(2,2|N);0] = \sum_{\oplus n}^{\infty}$$



Similarly, the enveloping algebras of the deformed minreps of $SU(2,2|N)$ define one parameter family of higher spin superalgebras:

$$HS(SU(2,2|N)_\zeta) = \frac{\mathcal{U}(SU(2,2|N))}{\mathcal{J}_\zeta(SU(2,2|N))}$$

Joseph ideal of the minrep of $SO(6,2)$ and its deformations

K. Govil & MG (2013)

$6d$ covariant identities satisfied by the generators of the minrep of $SO^*(8)$ are

$$P^\mu P_\mu = K^\mu K_\mu = 0 \quad \text{or,} \quad P^2 = K^2 = 0$$
$$6\mathcal{D}\cdot\mathcal{D} + M^{\mu\nu}\cdot M_{\mu\nu} + 2P^\mu\cdot K_\mu = 0$$
$$P^\mu\cdot(M_{\mu\nu} + \eta_{\mu\nu}\mathcal{D}) = 0 \quad , \quad K^\mu\cdot(M_{\nu\mu} + \eta_{\nu\mu}\mathcal{D}) = 0$$
$$\eta^{\mu\nu} M_{\mu\rho}\cdot M_{\nu\sigma} - P_{(\rho}\cdot K_{\sigma)} + 4\eta_{\rho\sigma} = 0$$
$$M_{\mu\nu}\cdot M_{\rho\sigma} + M_{\mu\sigma}\cdot M_{\nu\rho} + M_{\mu\rho}\cdot M_{\sigma\nu} = 0$$
$$\mathcal{D}\cdot M_{\mu\nu} + P_{[\mu}\cdot K_{\nu]} = 0$$
$$A_{\mu\nu\rho} = \frac{1}{3}M_{[\mu\nu}\cdot P_{\rho]} = 0 \qquad B_{\mu\nu\rho} = \frac{1}{3}M_{[\mu\nu}\cdot K_{\rho]} = 0$$

Deformations of the minrep of $SO^*(8)$ is obtained by extending the generators of the $SU(2)_T$ subgroup of the little group $SO(4)$ of massless representations of the Lorentz group $SU^*(4)$ to include bilinears of fermionic oscillators $\xi^x = (\xi_x)^\dagger$ and $\chi^x = (\chi_x)^\dagger$, $(x = 1, 2, \ldots, P)$ The modified $SU(2)_{\mathcal{T}}$ generators as then follows:

$$\mathcal{T}_+ = a^m b_m + \xi^x \chi_x$$
$$\mathcal{T}_- = b^m a_m + \chi^x \xi_x$$
$$\mathcal{T}_0 = \frac{1}{2}(N_a - N_b + N_\xi - N_\chi)$$

where $N_\xi = \xi^x\xi_x$ $N_\chi = \chi^x\chi_x$ are the fermionic number operators.

The deformations of the minrep of $SO^*(8)$ are labeled by the spin t of the $SU(2)_{\mathcal{T}}$ subgroup of the little group $SO(4)$ of light like vectors.
Under deformation one finds

$$\eta^{\mu\nu} M_{\mu\rho} \cdot M_{\nu\sigma} - P_{(\rho} \cdot K_{\sigma)} + 4\eta_{\rho\sigma} = \frac{3}{2}(N_\xi - N_\chi)^2 \eta_{\rho\sigma}$$

Two of the identities in the undeformed case do no longer hold separately but they combine into the following identity involving deformed generators

$$M_{\mu\nu} \cdot M_{\rho\sigma} + M_{\mu\sigma} \cdot M_{\nu\rho} + M_{\mu\rho} \cdot M_{\sigma\nu} = \epsilon_{\mu\nu\rho\sigma}{}^{\delta\tau}(P_{[\delta} \cdot K_{\tau]} + M_{\delta\tau} \cdot \mathcal{D})$$

The totally antisymmetric tensorial operators $A_{\mu\nu\rho}$ and $B_{\mu\nu\rho}$ do not vanish identically for the deformed generators. They become self-dual and anti-self-dual respectively:

$$A_{\mu\nu\rho} = \widetilde{A}_{\mu\nu\rho}$$
$$B_{\mu\nu\rho} = -\widetilde{B}_{\mu\nu\rho}$$

where the dual rank three tensors, which are the analogs of Pauli-Lubansky vector in $4d$, are defined as follows:

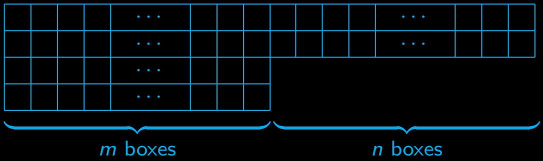
$$\widetilde{A}_{\mu\nu\rho} = \frac{1}{3!}\epsilon_{\mu\nu\rho\sigma\delta\tau} A^{\sigma\delta\tau}, \qquad \widetilde{B}_{\mu\nu\rho} = \frac{1}{3!}\epsilon_{\mu\nu\rho\sigma\delta\tau} B^{\sigma\delta\tau}$$

The rest of the identities remain unchanged by the deformations.

We have an infinite family of AdS_7/CFT_6 higher spin algebras obtained by quotienting the enveloping algebra of $SO(6,2)$ by the deformed Joseph ideal labelled by the spin t of an $SU(2)$ subgroup of the little group $SO(4)$ of massless particles:

$$HS(6,2;t) = \frac{\mathcal{U}(SO(6,2))}{\mathcal{J}_t(SO(6,2))}$$

The deformed AdS_7/CFT_6 higher spin algebra contain generators whose Y-T decomposition are of the form:



This suggests that the theories based on discrete deformations of the minrep describe higher spin theories of Fradkin-Vasiliev type in AdS_7 coupled to tensor fields that satisfy self-duality conditions and their higher extensions.
The corresponding higher spin superalgebras are given by the enveloping algebras of the deformed minimal unitary realizations of $OSp(8^*|2N)_t$ obtained via quasiconformal methods:

$$HS(OSp(8^*|2N)_t) = \frac{\mathcal{U}(OSp(8^*|2N))}{\mathcal{J}_t(OSp(8^*|2N))}$$

AdS_6/CFT_5 higher spin algebras and superalgebra

Fernando & MG (2014)

The Joseph ideal generators vanish identically as operators for the minrep of $SO(5,2)$ and a certain deformation of the Joseph ideal vanishes for the deformed minrep.

The enveloping algebra of the minrep (scalar singleton) of $SO(5,2)$ defines the AdS_6/CFT_5 bosonic higher spin algebra

$HS(5,2;t=0) = \frac{\mathcal{U}(SO(5,2))}{\mathcal{J}(SO(5,2))}$

and admits a single deformation

$HS(5,2;t=1/2) = \frac{\mathcal{U}(SO(5,2))}{\mathcal{J}_{(t=1/2)}(SO(5,2))}$

The enveloping algebra of the minimal unitary realization of $F(4)$ defines the unique AdS_6/CFT_5 higher spin superalgebra.

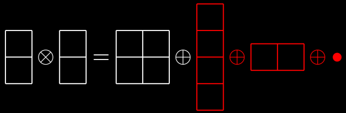
Existence of a scalar and a spinor singleton for $SO(5,2)$ is similar to $SO(3,2)=Sp(4,\mathbb{R})$. However only a unique R-symmetry group exists for supersymmetric extension, i.e $F(4)$ with even subalgebra $SO(5,2)\oplus SU(2)$.

$F(4)$ is the superconformal algebra of the unique exceptional superspace coordinatized by the exceptional Jordan superalgebra with 6 bosonic and 4 fermionic coordinates. It has no realization in terms of associative super matrices.

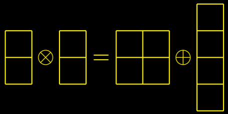
Minkowski spacetime is coordinatized by the Jordan algebra $J_2^{\mathbb{C}}$ of 2×2 Hermitian matrices.

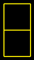
$Rot(J_2^{\mathbb{C}} = SU(2) \qquad Lor(J_2^{\mathbb{C}}) = SL(2,C) \qquad Con(J_2^{\mathbb{C}}) = SU(2,2)$

Poincare-Birkhoff-Witt theorem: The enveloping algebra $\mathscr{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ decomposes into symmetric tensor products of the adjoint representation of $\mathfrak{g}$. For $\mathfrak{so}(d,2)$ the symmetric product of the adjoint representation decomposes as:



where the singlet $\bullet$ is the quadratic Casimir of $SO(d,2)$.
Vanishing of the generators J_{ABCD} of the Joseph ideal implies that all the diagrams indicated in red on the RHS vanish. Therefore by quotienting $\mathscr{U}(\mathfrak{g})$ by the Joseph ideal generated by J_{ABCD} we get rid of all the "unwanted" diagrams and obtain the standard higher spin algebra $HS(d,2)$ a la Eastwood & Vasiliev .
The vanishing of the deformed Joseph ideal of $SO(d,2)$ implies



Therefore the gauge fields of deformed higher spin algebras in general dimensions will transform in representations of $SO(d,2)$ obtained by symmetric tensor products of  modulo the above generalized constraint.

THE EXCEPTIONAL SUPERSPACE:

- Rotation Lie superalgebra the exceptional superspace coordinatized by $JF(6/4)$ is
  $OSp(1/2) \times OSp(1/2) \supset SO(4) = SU(2) \times SU(2)$.
- Lorentz Lie superalgebra of $JF(6/4)$ is
  $OSp(2/4) \supset SO(2) \times Sp(4)$
- Superconformal Lie algebra of $JF(6/4)$ is
  $F(4) \supset SO(5,2) \times SU(2)$
  Non-linear action of $F(4)$ on the exceptional superspace can be obtained using the quadratic Jordan formulation. MG (1990).
- The exceptional $N = 2$ superconformal algebra $F(4)$ in five dimensions can not be embedded in any six dimensional super conformal algebra
  $OSp(8^*|2N) \supset SO(6,2) \times USp(2N)$ as expected from the exceptionality of the superspace defined by $JF(6/4)$.
- Minimal unitary realization of $F(4)$ was obtained via quasiconformal techniques recently. The enveloping algebra of the minimal unitary representation of $F(4)$ is the unique higher spin superconformal algebra in five dimensions.
  (Fernando and MG , 2014).
- According to Nahm's classification $d = 6$ is the maximal dimensions for the existence of superconformal field theories based on simple superconformal algebras! However,

Super Magic Rectangle that extends Tits Construction

	$J_3^{\mathbb{F}}$	$J_3^{\mathbb{F}\times\mathbb{F}}$	$J_3^{\mathbb{M}(\mathbb{F})_2}$	$J_3^{\mathbb{O}(\mathbb{F})}$	$J^{0\|2}$	D_t	$JF(6\|4)$
$\mathbb{F}$	$SO(3)$	$SU(3)$	$USp(6)$	$F(4)$	$Sp(2)$	$OSp(1\|2)$	$OSp(1\|2)^2$
$\mathbb{F}\times\mathbb{F}$	$SU(3)$	$SU(3)\times SU(3)$	$SU(6)$	E_6	$OSp(1\|2)$	$SU(2\|1)$	$OSp(2\|4)$
$\mathbb{M}(\mathbb{F})_2$	$USp(6)$	$SU(6)$	$SO(12)$	E_7	$SU(2\|2)$	$D(2,1;\alpha)$	$F(4)$
$\mathbb{O}(\mathbb{F})$	$F(4)$	E_6	E_7	E_8	$G(3)$	$F(4)_{t=2}$	$\mathcal{T}(55\|32)_5$

$\mathcal{T}(55|32)_5$ stands for a simple Lie superalgebra whose even subalgebra is $SO(11)$ and whose odd elements are in the spinor 32 representation of $SO(11)$ in characteristic five. Elduque 2007

Remarkable fact: Simple $AdS/Conformal$ superalgebra in $d = 10/9$ dimensions whose even subalgebra is $SO(9,2)$ and its odd generators are in the spinor representation 32 in characteristic five.

It corresponds to the quasiconformal algebra associated with the exceptional Jordan superalgebra $JF(6|4)$ just as E_8 is the quasiconformal algebra associated with the exceptional Jordan algebra $J_3^{\mathbb{O}}$.

Benkart, Cunha, Elduque, Shestakov, Zelmanov,....on super-extensions of Tits' construction.

Super Magic Square that extends Kantor Construction in characteristic 3

	$\mathbb{F}$	$\mathbb{F}\times\mathbb{F}$	$\mathbb{M}(\mathbb{F})_2$	$\mathbb{O}(\mathbb{F})$	$B(1,2)$	$B(4,2)$
$\mathbb{F}$	$SO(3)$	$SU(3)$	$USp(6)$	$F(4)$	$psl_{2,2}$	$sp_6 \oplus (14)$
$\mathbb{F}\times\mathbb{F}$	$SU(3)$	$SU(3)\otimes SU(3)$	$SU(6)$	E_6	$(pgl_3 \oplus sl_2) \oplus (psl_3 \otimes (2))$	$pgl_6 \oplus (20)$
$\mathbb{M}(\mathbb{F})_2$	$USp(6)$	$SU(6)$	$SO(12)$	E_7	$(sp_6 \oplus sl_2) \oplus ((13)\otimes(2))$	$so(12) \oplus (spin_{12})$
$\mathbb{O}(\mathbb{F})$	$F(4)$	E_6	E_7	E_8	$(f_4 \oplus sl_2) \oplus ((25)\otimes(2))$	$e_7 \oplus (56)$
$B(1,2)$					$so(7) \oplus 2(spin_7)$	$sp_8 \oplus (40)$
$B(4,2)$						$so(13) \oplus spin_{13}$

Elduque, Okubo, Shestakov,Cunha, Leites, ...

- $B(1,2)$ and $B(4,2)$ are composition superalgebras in characteristic three.
- Simple AdS/Conformal superalgebras 11/10 dimensions : $SO(10,2) \oplus (32)$ and 12/11 dimensions $SO(11,2) \oplus (64)$ and an exceptional simple conformal superalgebra : $E_7 \oplus (56)$
- $SO(10,2) \oplus (32) \subset E_{7(-25)} \oplus 56$
  $SO(6,6) \oplus (32) \subset E_{7(7)} \oplus 56$
- Novel higher spin AdS_{10}/CFT_9 higher spin algebra in characteristic 5 obtained as enveloping algebra quotiented by the Joseph ideal.
- Novel higher spin super algebras in AdS_{11}/CFT_{10}, AdS_{12}/CFT_{11} and an exceptional higher spin superconformal algebra in 27 dimensions !!

Further comments and open problems

- The above results show that there is a one-to-one correspondence between massless conformal fields and massless conformal supermultiplets and higher spin algebras and their deformations and supersymmetric extensions in all dimensions ($d > 2$). As such they are consistent with the results of Maldacena and Zhiboedov about the duality between free conformal field theories in $d = 3$ and higher spin theories in AdS_4 with unbroken higher spin symmetry. Therefore we expect this duality to extend to all the deformations of the $AdS_{(d+1)}/CFT_d$ higher spin theories and their susy extensions.
- The quasiconformal construction of the minrep and its deformations are non-linear , except for $d = 3$. This raises the question whether there exist interacting ,but integrable conformal field theories that are dual to unbroken higher spin gauge theories in higher dimensions.
- The quasiconformal construction of the minrep of $D(2, 1 : \alpha)$ and its deformations describe the spectra of $N = 8$ supersymmetric interacting quantum mechanical models obtained using harmonic superspace techniques by Fedoruk et.al Govil & MG
- Question: could these interacting yet integrable theories correspond to some dimensionally reduced CFTs ?
- Reformulation of nonlinear quasiconformal realizations of higher spin algebras and their supersymmetric extensions in terms of covariant fields.
- Construction of Vasiliev type interacting higher spin theories corresponding to deformed higher spin algebras and super algebras
- Relevance of AdS_7/CFT_6, AdS_6/CFT_5 and AdS_5/CFT_4 higher spin superalgebras and their deformations to M-theory and IIB superstring? What is special about 32 supercharges from the point of view of higher spin theories?

THANK YOU