Conformal vs Gauge Symmetry of (Partially) Massless Fields

Xavier BEKAERT

Laboratoire de Mathématiques et Physique Théorique (Tours, France)

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From joint work with G. Barnich and M. Grigoriev

- "Notes on conformal invariance of gauge fields," arXiv:1506.00595

Main question

The question which will be adressed in this talk is:

Which elementary fields
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Terminological remarks: In this talk,

- the adjective *elementary* will stand for an *irreducible* representation of the isometry algebra.
- the *fields* will always be *free*. Nevertheless, interesting issues already arise at this level.
- the adjective *conformal* will always be associated with *rigid* conformal transformations, and be distinguished from "Weyl", i.e. gauge conformal transformations.

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Classify the most general elementary fields that can appear in higher-dimensional CFTs (possibly even non-unitary ones).

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The aim is to provide a mathematically rigorous and exhaustive answer to the simple question

Which elementary fields
on maximally symmetric spacetimes
are conformal?

Although this question looks very simple, to provide a precise answer requires some care because several subtleties arise.

Outline

1. Fields on flat spacetime

Main question

The physical question to be adressed is:

Which elementary fields on Minkowski spacetime are conformal?

Main question

Even in flat spacetime, this issue is subtler than it may seem at first sight.

Some subtleties

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Some (very well known) examples of subtleties that can (nevertheless) be source of confusion:

1. **Masslessness $\not\Rightarrow$ Scale symmetry**

   Not all massless UIRs of the Poincaré algebra are scale invariant.

   The "infinite-component" (also called "continuous spin") massless UIRs are not scale invariant because (like massive UIRs) they are parametrised by a dimensionful continuous parameter.
   Technically, both the quadratic and quartic Casimir operators (the square of, respectively, the momentum and Pauli-Lubanski vectors) of the Poincaré subalgebra do *not* commute with the dilatation generator of the conformal algebra.

   Only the "finite-component" massless (also called "helicity") UIRs are scale invariant.

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Of course, symmetry under special conformal transformations (or, equivalently, inversion) is a logically independent requirement from symmetry under scale and Poincaré transformations.

This subtlety is of high importance for the study of interacting RG fixed points via CFT techniques.

But the subtlety even arises for free fields:

Not all scale-invariant massless fields are conformally-invariant for $d > 4$.

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The physical question to be adressed

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admit several (in principle, inequivalent) precise formulations depending on the mathematical settings in which they are formulated, e.g.

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Remarks: Heuristically, the reason why (1) and (2) are logically independent is that, generically,

$(2) \not\Rightarrow (1)$ because specifying rigorously the space of solutions requires some care and may depend on the desired structure (Hilbert space, Verma module, etc) the representation space should be endowed with.

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$(1) \not\Rightarrow (2)$ because nothing guarantees that symmetry generators are realised as local differential operators acting on fields, while locality is a foundational assumption of the above mentioned theory of PDEs.

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(2) PDE (local equations of motion) in terms of

- curvatures (no gauge symmetry)
- potentials (gauge fields)

Remarks: In particular, eqs in terms of curvature and potential may have distinct symmetries is that these two formulations are related by a non-local "integration" (when passing from curvature to potential).

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Remarks: In particular, eqs in terms of curvature and potential may have distinct symmetries is that these two formulations are related by a non-local "integration" (when passing from curvature to potential). Therefore, although these formulations may be equivalent from the group-theoretical point of view (1), i.e. possess formally isomorphic space of solutions, they are not necessarily equivalent from the PDE point of view (2), i.e. as *local* equations of motion.

Group theory

1. Fields on flat spacetime

(1) Group theory

Group theory

From a group-theoretic point of view, the physical question to be adressed

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can be reformulated as

Among all UIRs of the Poincaré algebra, which ones can be lifted to representations of the conformal algebra?

Group theory

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Notational remark: dimension d

$\mathbb{R}^{d-1,1}$ isometries		Conformal symmetries
$\mathfrak{iso}(d-1,1)$	$\subset$	$\mathfrak{so}(d,2)$

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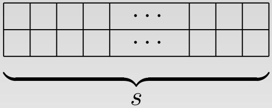
  $d = 4$: *only "helicity" representations*
- [Angelopoulos-Laoues (1998)]
  any $d \geqslant 3$: *all and only "singleton" representations*

Group theory

Definition *(as $\mathfrak{iso}(d-1,1)$-module)*

A singleton *is a massless unitary irreducible module of $\mathfrak{iso}(d-1,1)$ induced (à la Wigner) by a finite-dimensional irreducible representation of the stabilizer $\mathfrak{iso}(d-2)$ labeled by a partition in $\frac{d}{2}-1$ equal parts, i.e. a rectangular Young diagram made of $\frac{d}{2}-1$ rows of length $[s]$.*

e.g. for $d=6$ and integer s, the $\mathfrak{iso}(5,1)$-module is induced from the finite-dimensional $\mathfrak{so}(4)$-module labelled by



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Examples:

- $s=0,\frac{1}{2}$: $\forall d$ since the integer part $[s]=0$
  - Conformal scalar and spinor fields
- $s\geqslant 1$: even d only
  - For $d=4$, they are the "helicity" representations of the Poincaré algebra $\mathfrak{iso}(3,1)$.
  - For $d=2 \bmod 4$, they are the "chiral" representations (e.g. the chiral 2-form in 6 dimensions) in the sense that their fieldstrength is a self-dual (spinor-)tensor field carrying an irrep of $\mathfrak{so}(d-1,1)$ described by a rectangular Young diagram made of $d/2$ rows.

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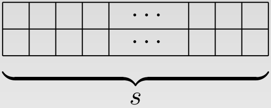
Definition *(as $\mathfrak{so}(d,2)$-module)*

A *singleton* *$\mathcal{D}(s+\frac{d}{2}-1\,;\,s,\ldots,s,\pm s)$ is a unitary lowest-weight irreducible module of $\mathfrak{so}(d,2)$ whose lowest-energy submodule is a finite-dimensional irreducible $\mathfrak{so}(d)$-module labeled by a partition in $\frac{d}{2}$ equal parts, i.e. a rectangular Young diagram made of $\frac{d}{2}$ rows of length $[s]$.*

e.g. for $d=4$ and integer s, the $\mathfrak{so}(4,2)$-module is built from the finite-dimensional $\mathfrak{so}(4)$-module labelled by



EOMs

1. Fields on flat spacetime

(2) Equations of motion

EOMs without gauge symmetry

Algebraic approaches to classifying symmetries of systems of PDEs are by now well-developed.

- They make use of techniques based on jet bundles and the variational bicomplex [initiated by Vinogradov and his school].
- In the absence of gauge symmetries, the particular problem of classifying linear PDEs preserved by a rigid symmetry group essentially reduces to the classification of (quasi-)invariant differential operators [for the conformal case, e.g. Penrose, Eastwood, Dobrev ...].
- In the presence of gauge symmetries, this problem admits a convenient reformulation in terms of local BRST cohomology in the Batalin-Vilkoviski formalism.

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In this setting, the physical question to be adressed

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- [Bracken-Jessup (1982)] $d=4$: *only Bargmann-Wigner eqs for helicity irreps*

  They are written in terms of "curvatures" (or "fiedlstrengths") e.g. for $s=2$

  $$\partial_{[\mu}C_{\nu\rho]\,\sigma\tau}=0\,,\qquad \partial^{\mu}C_{\mu\nu\,\rho\sigma}=0\,,$$

  where the tensor field $C_{\mu\nu\,\rho\sigma}$ possesses the symmetries of the linearized Weyl tensor, such as $C_{[\mu\nu\,\rho]\sigma}=0$ and $\eta^{\mu\rho}C_{\mu\nu\,\rho\sigma}=0$.

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  - $s = 0, \frac{1}{2}$: d'Alembert $\Box\phi = 0$ and Dirac(-Weyl) $\partial\!\!\!/\psi = 0$
  - $s \geqslant 1$: even d only, generalized Bargmann-Wigner eqs for "spinning singletons"

They are described in terms of "curvatures" which are tensor fields carrying an irreducible representation of the Lorentz algebra $\mathfrak{so}(d-1,1)$ labelled by a rectangular Young diagram whose columns are of lenght $d/2$. In particular, for $d = 2 \mod 4$, these curvatures are moreover (anti)self-dual.

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- [Shaynkman, Tipunin, Vasiliev (2004)] *general classification of linear PDEs for which Poincaré lifts to conformal symmetry*

  Investigating the catalogue of EOMs, one may check that indeed only these generalized BW eqs describe a single irrep of the Poincaré algebra.

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Open issue: In the presence of gauge symmetries, what about conformal symmetry up to a gauge transformation?

This could leave room for conformally-invariant EOMs in terms of potential.

EOMs without gauge symmetry

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Open issue: In the presence of gauge symmetries, what about conformal symmetry up to a gauge transformation?

This could leave room for conformally-invariant EOMs in terms of potential.

EOMs with gauge symmetry

In this setting, the physical question to be adressed

Which elementary fields on Minkowski spacetime are conformal?

can be reformulated as

Among all linear PDEs equivalent to Fronsdal-Labastida equations for massless fields on Minkowski spacetime, which ones are conformally-invariant?

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Remark: Two such eqs are considered equivalent if they are related by the introduction/elimination of "generalized auxilliary fields", i.e. fields that can be eliminated via their own algebraic EOMs ("auxilliary fields") or that are pure gauge ("Stuckelberg fields").

EOMs with gauge symmetry

Among all linear PDEs equivalent to Fronsdal-Labastida equations for massless fields on Minkowski spacetime, which ones are conformally-invariant?

Answer: For $d \geqslant 3$ and

- $s = 0, \frac{1}{2}$:
  - d'Alembert $\Box\phi = 0$ and Dirac(-Weyl) $\partial\!\!\!/\psi = 0$
- $s \geqslant 1$:
  - for symmetric tensor fields: only Maxwell $s = 1$ in $d = 4$
  - for tensor fields described by at least one row: only p-form in $d = 2p + 2$

EOMs with gauge symmetry

Idea of the proof:

- Show that "global reducibility parameters" (such as Killing tensor fields) of a gauge theory must carry a representation of any subalgebra of global symmetries of the theory.
- Identify the space of global reducibility parameters as a local BRST cohomology group.
  $\Longrightarrow$ They do not depend on the specific formulation of the theory (i.e. are invariant under the introduction/elimination of generalised auxiliary fields).
- Determine the Poincaré-module of *highest-order* global reducibility parameters.
- Check that the latter finite-dimensional Poincaré-module cannot lift to a conformal-module for $s > 1$.

2. Fields on constant curvature spacetime

Main question: constant curvature spacetime

Let us consider the case $\Lambda \neq 0$. The physical question to be adressed is:

Which elementary fields on (anti) de Sitter spacetime are conformal?

Some extra subtleties

Some further (well known) examples of subtleties that can (nevertheless) be source of confusion:

1. **Masslessness?**

   In curved spacetime, the quadratic Casimir operator of the isometry group does not identify with the Laplace-Beltrami operator.

   $\Longrightarrow$ The terminology "masslessness" is ambiguous.

   Usually, one retains the presence of gauge symmetries to be a manifestation of "masslessness".

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2. **Partial masslessness?**

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   However, on non-vanishing constant-curvature spacetimes, for each spin there exists a finite family of fields with distinct "critical" masses and gauge symmetries ("partial masslessness" phenomenon).

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Main question: constant curvature spacetime

For definiteness, let us consider the case $\Lambda < 0$.

The physical question to be adressed is:

Which elementary fields on anti de Sitter spacetime are conformal?

Main question: constant curvature spacetime

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The physical question to be adressed is:

Which elementary fields on anti de Sitter spacetime are conformal?

Remark: All final results equally hold for de Sitter spacetime since unitarity will not be assumed.

Three mathematical formulations of the question

The physical question to be adressed

Which elementary fields on anti de Sitter spacetime are conformal?

admits three precise formulations which will be considered

(1) Group theory (generalized Verma module)

(2) PDE (local equations of motion) in terms of

- curvatures (no gauge symmetry)
- potentials (gauge fields)

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- curvatures (no gauge symmetry)
- potentials (gauge fields)

Accordingly, partially massless symmetric tensor fields admit several definitions (e.g. group-theoretic vs field-theoretic).

Partially massless symmetric tensor fields

Definition *(Group theoretic)*
Non unitary lowest-weight irreducible module $\mathcal{D}(d+s-t-2; s, 0, \ldots, 0)$ *of* $\mathfrak{so}(d-1,2)$ *whose lowest-energy* ($E_0 = d+s-t-2$) *submodule is a finite-dimensional irreducible* $\mathfrak{so}(d-1)$*-module labeled by a single row of length* $s \geqslant t \geqslant 1$.

Definition *(Field theoretic)*
On-shell gauge field which is a symmetric tensor field $\varphi_{\mu_1 \ldots \mu_s}$ *on* AdS_d*, with tuned mass*

$$[\nabla^2 + (s-t-1)(s+t+d-4) - s]\varphi_{\mu_1 \ldots \mu_s} = 0$$

(in the TT gauge) and with (residual) depth-t Fronsdal-like gauge transformations

$$\delta\varphi_{\mu_1\mu_2\ldots\mu_s} = \nabla_{(\mu_1} \ldots \nabla_{\mu_t} \epsilon_{\mu_{t+1}\ldots\mu_s)} + \ldots$$

Some extra subtleties

Some further (well known) examples of subtleties that can (nevertheless) be source of confusion:

1. **Masslessness?**
2. **Partial masslessness?**
3. **Taxonomy of partially massless fields?**

   There is a very large collection of such exotic partially massless fields and of conformal multiplet they may form.

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   Examples: Weyl [Fradkin-Tseytlin (1985)] $\supset$ Partially massless [Deser-Waldron (2001)]
   Spin-s (depth-1) Fradkin-Tseytlin fields (generalizing $d=4$ Maxwell photon $s=1$ and Weyl graviton $s=2$) are irreducible $\mathfrak{so}(4,2)$-modules $\mathcal{D}(2;s,s)$ decomposing into the sum of irreducible $\mathfrak{so}(3,2)$-modules $\mathcal{D}(2+s-t;s)$ corresponding to spin-s partially massless fields on AdS_4 with depths $t=1,2,...,s$.

$$\mathcal{D}_{\mathfrak{so}(4,2)}(2;s,s) \;=\; \bigoplus_{t=1}^{s}\mathcal{D}_{\mathfrak{so}(3,2)}(2+s-t;s)$$

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1. **Masslessness?**
2. **Partial masslessness?**
3. **Taxonomy of partially massless fields?**

   There is a very large collection of such exotic partially massless fields and of conformal multiplet they may form.

   $\Longrightarrow$ It may sometimes be difficult to distinguish them properly because the terminology is not yet perfectly settled.

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1. **Masslessness?**
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   Particularly confusing is the fact that the critical mass and conformal weight are not necessarily sufficient to easily distinguish them.

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   Examples:

   s=2 [Deser-Nepomechie (1983)], $s \in \mathbb{N}$ [Erdmenger-Osborn (1998)]

   Spin-s maximal-depth ($t = s$) Fradkin-Tseytlin fields are described by second-order field eqs and are irreducible $\mathfrak{so}(4,2)$-modules $\mathcal{D}(2; s, 1)$ decomposing into the sum of irreducible $\mathfrak{so}(3,2)$-modules $\mathcal{D}(2; s')$ corresponding to maximal-depth partially massless fields on AdS_4 with spins $s' = 1, 2, ..., s$.

$$\mathcal{D}_{\mathfrak{so}(4,2)}(2; s, 1) = \bigoplus_{s'=1}^{s} \mathcal{D}_{\mathfrak{so}(3,2)}(2; s')$$

Group theory

2. Fields on constant curvature spacetime

(1) Group theory

Group theory

From a group-theoretic point of view, the physical question to be adressed

Which elementary fields on anti de Sitter spacetime are conformal?

can be reformulated as

Among all lowest-weight (not necessarily unitary) irreps of the anti de Sitter isometry algebra, which ones can be lifted to reps of the conformal algebra?

Group theory

Among all lowest-weight (not necessarily unitary) irreps of the anti de Sitter isometry algebra, which ones can be lifted to reps of the conformal algebra?

Notational remark: dimension d

AdS_d isometries		Conformal symmetries
$\mathfrak{so}(d-1,2)$	$\subset$	$\mathfrak{so}(d,2)$

Group theory

Among all lowest-weight (not necessarily unitary) irreps of the anti de Sitter isometry algebra, which ones can be lifted to reps of the conformal algebra?

Answers:

- [Metsaev (1995)] *among UIRs, only singletons*

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Definition *(as $\mathfrak{so}(d-1,2)$-module)*

A singleton $\mathcal{D}(s+\frac{d}{2}-1\,;\,s,\dots,s)$ is a unitary lowest-weight irreducible module of $\mathfrak{so}(d-1,2)$ whose lowest-energy submodule is a finite-dimensional irreducible $\mathfrak{so}(d-1)$-module labeled by a rectangular Young diagram made of $\frac{d}{2}-1$ rows of length $[s]$.

Group theory

Warning: distinguish AdS_d isometry vs conformal module definitions

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Among all lowest-weight (not necessarily unitary) irreducible representations of the anti de Sitter isometry algebra, which ones can be lifted to reps of the conformal algebra?

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- [Metsaev (1995)]
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- [Barnich, XB, Grigoriev (2015)]
  among all lowest-weight irreps, only singletons

Corollary *For any depth t (i.e. $2 \leqslant t \leqslant s$) and any symmetry,*

Partially massless fields are not conformal.

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Idea of the proof:

- Show that the question is equivalent to

Among all lowest-weight (not necessarily unitary) irreps of the conformal algebra $\mathfrak{so}(d,2)$, which ones remain irreducible upon restriction to the AdS_d isometry subalgebra $\mathfrak{so}(d-1,2)$?

- Notice that a necessary condition is that the finite-dimensional irrep of $\mathfrak{so}(d)$ carried by the lowest-energy submodule must remain irreducible upon restriction of $\mathfrak{so}(d)$ to $\mathfrak{so}(d-1)$.
  $\Longrightarrow$ If the lowest-energy submodule is nontrivial, then the dimension d must be even and the irrep must labelled by a rectangular Young diagram.
- Identify the corresponding list of candidates in the catalogue of [Shaynkman, Tipunin, Vasiliev (2004)]
- Apply the branching rules and check that only spinning singletons do not branch.

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EOMs

2. Fields on constant curvature spacetime

(2) Equations of motion

EOMs without gauge symmetry

In the setting of EOMs *without* gauge symmetry, the physical question to be adressed

Which elementary fields on anti de Sitter spacetime are conformal?

can be reformulated as

Among all linear PDEs associated with a single (maybe partially) massless field on (anti) de Sitter spacetime, which ones are conformally-invariant?

EOMs without gauge symmetry

Among all linear PDEs associated with a single (maybe partially) massless field on (anti) de Sitter spacetime, which ones are conformally-invariant?

Answer: analogue of flat case

For $d \geqslant 3$ and

- $s = 0, \frac{1}{2}$: *Yamabe and conformal Dirac(-Weyl) eqs*
- $s \geqslant 1$: *even d only, generalized Bargmann-Wigner eqs for spinning singletons*

EOMs without gauge symmetry

Idea of the proof:

- Minkowski case and (anti) de Sitter spacetimes are identical as conformal spaces
- Show that if an EOM on (A)dS is conformally invariant, then
  - up to a Weyl transformation it can be rewritten as an EOM on Minkowski spacetime,
  - the latter EOM is identical to its flat limit.
- This reduces the problem to the Minkowski case

  $\Longrightarrow$ analogous results.

EOMs with gauge symmetry

In the setting of EOMs *with* gauge symmetry, the physical question to be adressed

Which elementary fields on (anti) de Sitter spacetime are conformal?

can be reformulated as

Among all linear PDEs equivalent to Fronsdal-Labastida-type eqs for gauge fields on (anti) de Sitter spacetime, which ones are conformally-invariant?

EOMs with gauge symmetry

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Answer: For $d \geqslant 3$ and

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- $s \geqslant 1$:

  For $d = 0 \, mod \, 4$ *and a single real field*

  - for symmetric tensor fields: only Maxwell $s = 1$ in $d = 4$
  - for tensor fields described by more than one row: only p-form in $d = 2p + 2$

EOMs with gauge symmetry

Idea of the proof:

- Determine the AdS_d isometry $\mathfrak{so}(d-1,2)$-module of *highest-order* global reducibility parameters.
- Notice that the only finite-dimensional irreducible $\mathfrak{so}(d,2)$-modules that remain reducible as $\mathfrak{so}(d-1,2)$-modules are labelled by a rectangular Young diagram made of $\frac{d}{2}+1$ rows (thus d must be even).
  Moreover, if the module is over $\mathbb{R}$ then d must be equal to 2 modulo 4.
- Therefore a necessary condition for an $(A)dS_d$ gauge field to be conformal is that d is even and the space of highest-order global reducibility parameters span an irreducible $\mathfrak{so}(d-1,2)$-module labelled by a rectangular Young diagram made of $\frac{d}{2}$ rows.
- The latter case correspond to singletons.

EOMs with gauge symmetry

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Answer: For $d \geqslant 3$ and

- $s = 0, \frac{1}{2}$: *Yamabe and conformal Dirac(-Weyl)*
- $s \geqslant 1$:
  - $d = 0 \bmod 4$ *and a complex field, or*
  - $d = 2 \bmod 4$ *and a real field*

  The only possibility are singletons (but a priori no restriction on spin).

Remark: In fact, a manifestly conformal formulation of doubled real (or single complex) massless fields on AdS4 is known [Vasiliev (2002)].

Conclusion

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The answer to the question

Which elementary fields on maximally symmetric spacetimes are conformal?

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Which elementary fields on maximally symmetric spacetimes are conformal?

is

Only singletons

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Which elementary fields on maximally symmetric spacetimes are conformal?

is

Only singletons and only in terms of curvature

up to subtleties on (A)dS