Anomalous dimensions in CFT with weakly broken HS symmetry

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Based mainly on SG, Kirilin arXiv:1601.01310

Closely related work Skvortsov arXiv: 1512.05994

Introduction

- A classic result in CFT_d are the unitarity bounds. For a spin-*s* operator $J_{\mu_1 \mu_2 \cdots \mu_s}$ (in the symm. traceless of *SO(d)*):

$$\Delta_s \geq d - 2 + s \,, \qquad s \geq 1$$

- For a scalar operator $\mathcal{O}$:

$$\Delta_0 \geq \frac{d}{2} - 1$$

- Operators that saturate these inequalities belong to *short representations* of the conformal algebra.
- E.g., for a scalar operator, the shortening condition is just the free wave equation $\partial^2 \mathcal{O} = 0$, so $\mathcal{O}$ is a free conformal scalar field ($\Delta_0 = d/2 - 1$)

Introduction

- For a spin-*s* operator, saturation of the inequality bound implies that $J_{\mu_1\mu_2\cdots\mu_s}$ is a *conserved current*

$$\partial^\mu J_{\mu\mu_2\cdots\mu_s} = 0$$

- The cases *s=1* and *s=2* are familiar in any CFT. The case of exactly conserved currents of *s>2* is realized in free field theories. E.g., a free scalar field theory has the conserved currents of the form

$$J_{\mu_1\cdots\mu_s} = \sum_{k=0}^{s} c_{sk}\partial_{\{\mu_1}\cdots\partial_{\mu_k}\phi\partial_{\mu_{k+1}}\cdots\partial_{\mu_s\}}\phi$$

The HS currents in free scalar theory

- It is convenient to introduce a null "polarization vector" z^μ and construct the index-free object

$$\hat{J}_s(x,z) = J_{\mu_1\mu_2\cdots\mu_s} z^{\mu_1} z^{\mu_2} \ldots z^{\mu_s}$$

- Indices may be "freed up" using the operator (*Dobrev et al '76*)

$$D_z^\mu \equiv \left(\frac{d}{2}-1\right)\partial_{z_\mu} + z^\nu \partial_{z_\nu}\partial_{z_\mu} - \frac{1}{2} z^\mu \partial_{z_\nu}\partial_{z_\nu}$$

- E.g., we can recover the full symmetric traceless tensor by

$$J_{\mu_1\mu_2\cdots\mu_s} \propto D^z_{\mu_1} D^z_{\mu_2} \cdots D^z_{\mu_s} \hat{J}_s$$

The HS currents in free scalar theory

- Conservation can be expressed compactly as

$$\partial_\mu D_z^\mu \hat{J}_s = 0$$

- Writing (define $\hat{\partial} \equiv z^\mu \partial_\mu$):

$$\hat{J}_s(x,z) = \sum_{k=0}^{s} c_{sk} \hat{\partial}^k \phi \hat{\partial}^{s-k} \phi = f_s(\hat{\partial}_1, \hat{\partial}_2) \phi(x_1) \phi(x_2)|_{x_{1,2} \to x}$$

- Conservation then results in the differential equation

$$\left((d/2-1)(\partial_u + \partial_v) + u\partial_u^2 + v\partial_v^2\right) f_s(u,v) = 0$$

- This is solved by Gegenbauer polynomials, and one gets

$$\hat{J}_s(x,z) = \left(\hat{\partial}_1 + \hat{\partial}_2\right)^s C_s^{\frac{d-3}{2}} \left(\frac{\hat{\partial}_1 - \hat{\partial}_2}{\hat{\partial}_1 + \hat{\partial}_2}\right) \phi(x_1)\phi(x_2)|_{x_{1,2} \to x}$$

Free O(N) model

- If we consider *N* free scalars $\partial^2\phi^i = 0, \qquad i = 1,\ldots,N$
- The conserved currents

$$\hat{J}^{ij} = \sum_{k=0}^{s} c_{sk}\hat{\partial}^{s-k}\phi^i\hat{\partial}^k\phi^j$$

decompose in irreps of the global O(N) symmetry

$$J_s^{ij} = J_s^{(ij)} + J_s^{[ij]} + J_s$$

- In the higher spin/vector model duality, we restrict to the *O(N) singlet sector* . Then, the singlet conserved currents J_s and the scalar operatoı $J_0 = \phi^i\phi^i$ are the only "single trace" operators in the model

Higher spin/vector model duality

- By the usual AdS/CFT dictionary

$$\text{Single trace operators in CFT} \quad \Leftrightarrow \quad \text{Single particle states in AdS}$$

$$J_s, \qquad \partial \cdot J_s = 0 \quad \Leftrightarrow \quad \text{Massless HS gauge field}$$

$$J_0 \quad \Leftrightarrow \quad \text{Scalar field with} \quad m^2 = \Delta_0(\Delta_0 - d)$$

- Singlets built our of more than two scalars are "multi-trace" operators. They are dual to (massive) multi-particle states in AdS

$$\text{E.g.:} \;\; (\phi^i\phi^i)(\phi^j\partial\cdots\partial\phi^j) \quad \Leftrightarrow \quad \text{Two-particle state in AdS}$$

Exact HS symmetry

- Exactly conserved currents J_s --> symmetries generated by conserved charges Q_s. E.g. $[Q_s, \phi] \sim \partial^{s-1}\phi$

$$\text{HS algebra:} \qquad [Q_{s_1}, Q_{s_2}] = \sum Q_s$$

- Infinite dimensional. A charge of *s>2* requires whole tower.

$$\text{E.g.:} \quad [Q_4, Q_4] \sim Q_2 + Q_4 + Q_6$$

- Exact HS symmetry is very constraining. Assuming a CFT with a *s=4* exactly conserved current, one may show
  - Full HS tower is present in the theory
  - Correlation functions are fixed to be those of a free field theory

Maldacena, Zhiboedov '11

Weakly broken HS symmetry

- Consider a CFT which admits a small parameter *g* that controls the breaking of the currents

$$\partial \cdot J_s = g K_{s-1}$$

- The parameter *g* may be a power of 1/N, a power of ε in a Wilson-Fisher type fixed point, a marginal coupling constant, etc.
- When *g* is small, the weakly broken symmetry can still be used to constrain correlation functions (*Maldacena, Zhiboedov '12*: Three-point functions in large *N* CS theories with scalars/fermions)

Anomalous dimensions

$$\partial \cdot J_s = g K_{s-1}$$

- The non-conservation equation together with conformal invariance implies that the currents acquire small anomalous dimensions of the form

$$\Delta_s = d - 2 + s + \gamma_s$$

$$\gamma_s \propto g^2 \langle K_{s-1} K_{s-1} \rangle_{g=0}$$

- Leading order anomalous dimensions can be obtained by a *tree-level* calculation

Anomalous dimensions

- In the "index-free" method described before, we write

$$\partial_\mu D_z^\mu \hat{J}_s = g \hat{K}_{s-1}$$

- Conformal invariance implies

$$\langle \hat{J}_s(x_1, z_1) \hat{J}_{s'}(x_2, z_2) \rangle = \delta_{ss'} C_s(g) \frac{(I_{\mu\nu} z_1^\mu z_2^\nu)^s}{(x_{12}^2)^{\Delta_s}}$$

$$I_{\mu\nu} = \eta_{\mu\nu} - 2\frac{x_{12}^\mu x_{12}^\nu}{x_{12}^2}$$

- Acting with divergence on both currents and using non-conservation equation

$$\partial_{1\mu} D_{z_1}^\mu \partial_{2\mu} D_{z_2}^\mu \langle \hat{J}_s(x_1) \hat{J}_s(x_2) \rangle = g^2 \langle \hat{K}_{s-1}(x_1) \hat{K}_{s-1}(x_2) \rangle$$

Anomalous dimensions

- Directly differentiating the 2-point conformal structure and equating to the RHS one finds the relation

$$g^2\hat{x}^2\frac{\langle\hat{K}_{s-1}(x_1)\hat{K}_{s-1}(x_2)\rangle}{\langle\hat{J}_s(x_1)\hat{J}_s(x_2)\rangle} = -\gamma_s(g^2)s(s+d/2-2)\big[(s+d/2-1)(s+d-3) + \gamma_s(g^2)(s^2+sd/2-2s+d/2-1)\big]$$

- This allows to gain one order in perturbation theory. To leading order, we just need tree-level two-point functions

$$\gamma_s \sim g^2\frac{\langle K_{s-1}K_{s-1}\rangle_{g=0}}{\langle J_s J_s\rangle_{g=0}}$$

Anselmi '98; Henn, Jarczak, Sokatchev '05

Some general comments

$$\partial \cdot J_s = gK_{s-1}$$

- The operator K_{s-1} is a conformal primary in the representation $(\Delta = d-1+s, s-1)$ in the *g=0* theory

- The non-conservation equation is the statement that, when the coupling is switched on, the short representation $(\Delta = d-2+s, s)$ combines with the $(\Delta = d-1+s, s-1)$ to form a *long multiplet* (the non-conserved current)

$$(\Delta = d-2+s+\gamma_s, s) \simeq (\Delta = d-2+s, s) \oplus (\Delta = d-1+s, s-1)$$

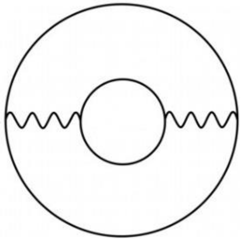
Bulk interpretation

- In the AdS bulk, this corresponds to a HS version of the Higgs mechanism: the massless gauge field combines with a spin *s-1* massive field to form a massive spin *s* field.
- In a large *N* theory, we can distinguish two cases, depending on whether K_{s-1} is a *single-trace* operator or *multi-trace*
- If K_{s-1} is single-trace:
  - It is dual to a single-particle state in the bulk
  - Higgsing occurs *classically*
  - Anomalous dimensions are non-zero at planar level
  - Typical example: adjoint theories (e.g. Yang-Mills)
  - Requires understanding coupling of Vasiliev's HS fields to matter

Quantum breaking

- If K_{s-1} is multi-trace, e.g. $K_{s-1} \sim \sum JJ$
  - Higgs is a multi-particle state (*Girardello, Porrati, Zaffaroni '02*)
  - Anomalous dimensions arise at non-planar level, $\gamma_s \sim O(1/N)$
  - Masses are generated via bulk loop effects

$$(\Delta_s + s - 2)(\Delta_s + 2 - d - s) = m_s^2 \ell_{AdS}^2$$
$$m_s^2 \ell_{AdS}^2 \approx (2s + d - 4)\gamma_s \sim O(1/N)$$



$G_N \sim 1/N$

  - Do not need additional fields on top of Vasiliev's spectrum

- Typical examples are Large *N* vector models:
  - Critical O(N) model
  - Gross-Neveu model
  - Conformal QED with N_f flavors, CP^N model
  - Large N Chern-Simons vector models

The critical O(N) model

- Take the interacting O(N) model

$$S = \int d^d x \Big(\frac{1}{2} \partial_\mu \phi^i \partial^\mu \phi^i + \frac{\lambda}{4} (\phi^i \phi^i)^2 \Big)$$

- In d<4, interaction is relevant and triggers RG flow to a non-trivial CFT in the IR: the critical O(N) model
- At large *N*, can be viewed as a double-trace perturbation of the free theory
- The dimension of the scalar operator $J_0 = \phi^i \phi^i$ goes from $\Delta_0 = d - 2$ in the UV to $\Delta_0 = 2 + O(1/N)$ in the IR
- Dual to the *same* Vasiliev HS theory, but with $\Delta = 2$ boundary conditions on the bulk scalar field (Klebanov, Polyakov)

The critical O(N) model

- The large *N* expansion can be developed with the aid of the Hubbard-Stratonovich auxiliary field

$$S = \int d^dx \Big(\frac{1}{2}\partial_\mu \phi^i \partial^\mu \phi^i + \frac{1}{2}\sigma\phi^i\phi^i - \frac{\sigma^2}{4\lambda}\Big)$$

- In the IR limit the last term can be dropped and one can develop the *1/N* expansion with the action

$$S = \int d^dx \Big(\frac{1}{2}\partial_\mu \phi_i \partial^\mu \phi^i + \frac{1}{2\sqrt{N}}\sigma\phi^i\phi^i \Big)$$

- At the critical point, σ plays the role of J_0, with induced 2-point function

$$\langle \sigma(x_1)\sigma(x_2)\rangle = \frac{C_{\sigma\sigma}}{(x_{12}^2)^2}\,, \qquad C_{\sigma\sigma} = \frac{2^{d+2}\Gamma(\frac{d}{2}-\frac{1}{2})\sin(\pi d/2)}{\pi^{\frac{3}{2}}\Gamma(\frac{d}{2}-2)}$$

The non-conservation equation

- Using the equation of motion

$$\partial^2 \phi^i = \frac{1}{\sqrt{N}} \sigma \phi^i$$

- One can derive the explicit form of the non-conservation equation for the HS currents

$$\partial_\mu D_z^\mu \hat{J}_s = \frac{1}{\sqrt{N}} \hat{K}_{s-1}$$

$$\hat{K}_{s-1} = \sum_{s'=2}^{s-2} \sum_{k=0}^{s-s'-1} c_{s'k} \hat{\partial}^{s-s'-k-1} \hat{J}_{s'} \hat{\partial}^k \sigma$$

$$c_{s'k} = (s-s')(2s'+d-3) \binom{s-s'-1}{k} \binom{-s-s'+k-d+3}{k+1}$$

The non-conservation equation

- As anticipated earlier, the operator on the right-hand side of non-conservation equation is “double-trace” and it is dual to a two-particle state in the bulk
- As an explicit example, the divergence of the spin 4 current is given by the spin 3 operator

$$\hat{K}_3 = (d+3)(d+1)\big((d+2)\hat{J}_2\hat{\partial}\sigma - 2(\hat{\partial}\hat{J}_2)\sigma\big)$$

- Using the method described above, we can extract the anomalous dimensions to order *1/N* from the tree-level two-point functions.
  No loops, no divergences or renormalization involved.

Anomalous dimensions

- The final result takes the form

$$\gamma_s = 2\gamma_\phi \frac{(s-1)(d+s-2) - \frac{\Gamma(d+1)\Gamma(s+1)}{2(d-1)\Gamma(d+s-3)}}{(d/2+s-2)(d/2+s-1)}$$

- Here γ_ϕ is the anomalous dimension of ϕ

$$\gamma_\phi = \frac{2\sin(\pi d/2)\Gamma(d-2)}{N\pi\Gamma(\frac{d}{2}-2)\Gamma(\frac{d}{2}+1)}$$

- HS anomalous dimensions agree with (and provides independent test of) the result obtained by Lang and Ruhl ('93) using 4-point functions and OPE

Anomalous dimensions

- In d=3, result takes the simple-looking form

$$\gamma_s = 2\gamma_\phi \frac{2(s-2)}{2s-1} = \frac{16(s-2)}{3\pi^2(2s-1)}\frac{1}{N}$$

corresponding to AdS_4 masses $m_s^2 \ell_{\mathrm{AdS}}^2 \approx \frac{16(s-2)}{3\pi^2 N}$

- The large spin behavior is

$$\gamma_s = 2\gamma_\phi - 2\gamma_\phi \frac{\Gamma(d+1)}{2(d-1)}\frac{1}{s^{d-2}} - 2\gamma_\phi \frac{d(d-2)}{4}\frac{1}{s^2} + \ldots$$

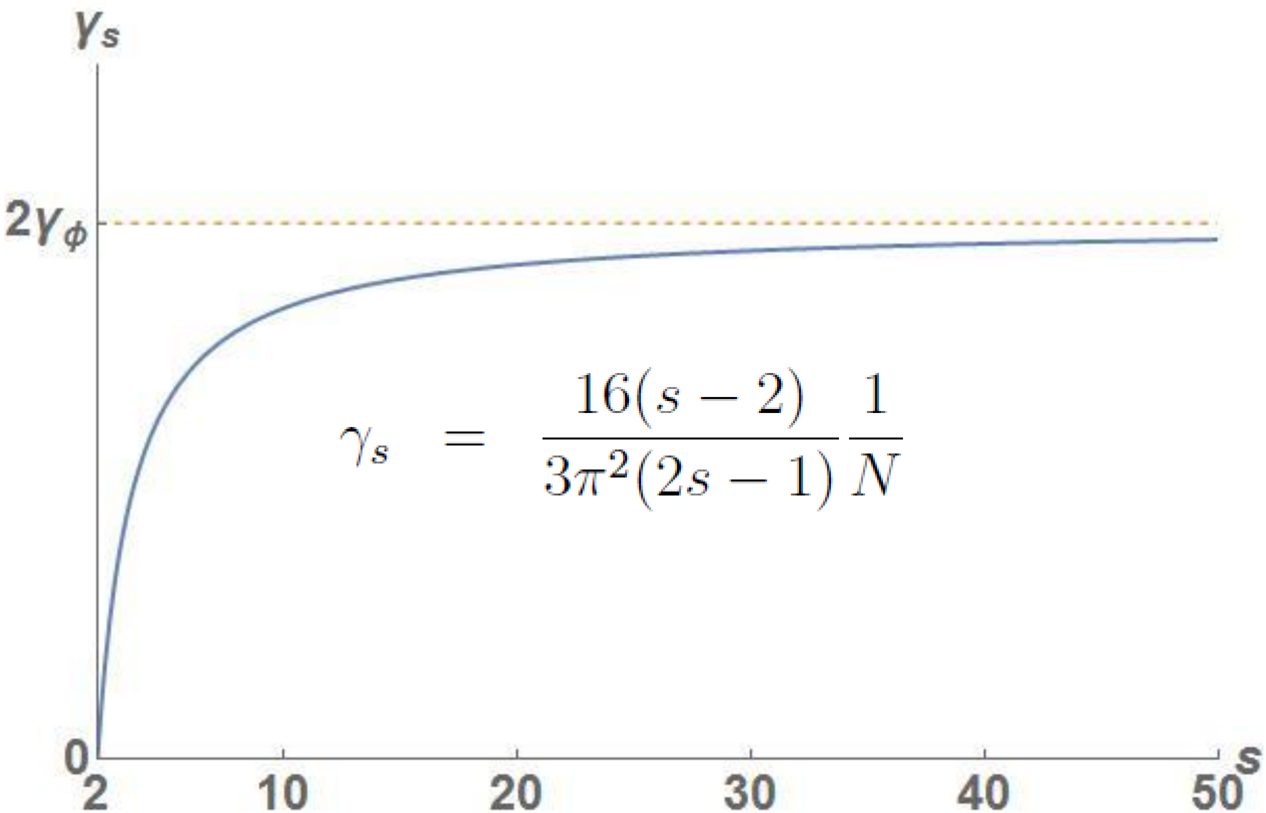
In agreement with general expectations

Callan, Gross '73. Nachtmann '73 $\quad \Delta_s \rightarrow s + 2\Delta_\phi$

Komargodski, Zhiboedov '12

Fitzpatrick et al '12

d=3 Higher Spin anomalous dimensions

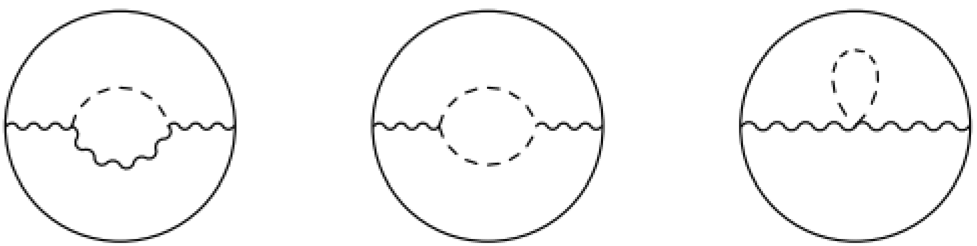


$$\gamma_s = \frac{16(s-2)}{3\pi^2(2s-1)}\frac{1}{N}$$

$$\gamma_\phi = \frac{4}{3\pi^2}\frac{1}{N}$$

One-loop in the bulk

- Consider the calculation of the one-loop correction to the current two-point function from the AdS theory
- Let us assume that for the $\Delta_0 = d-2$ boundary condition (dual to free CFT), the one-loop correction is trivial
- With $\Delta = 2$ boundary condition, the diagrams that differ are



where the dashed line is the $\Delta = 2$ scalar

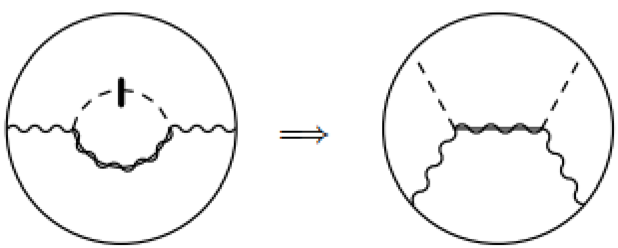
- One can then use the following identity for scalar Green's function (here AdS_4 for simplicity, and in momentum space)

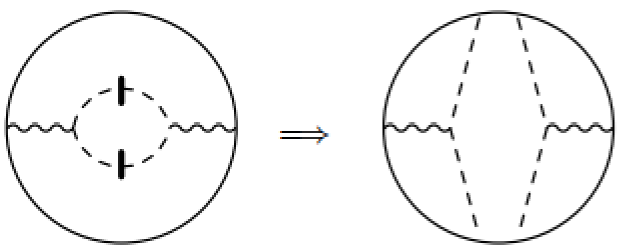
$$G_{\Delta=2}(p;z,z') - G_{\Delta=1}(p;z,z') = -|p| K_{\Delta=1}(p;z) K_{\Delta=1}(p;z')$$

Rastelli, Hartman '08. SG, Yin '11

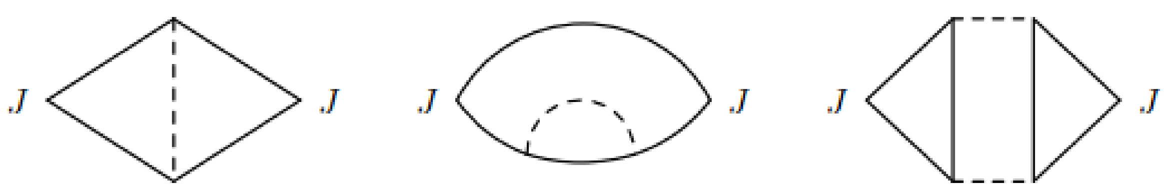
One-loop in the bulk

- This identity allows to replace each $\Delta = 2$ line by a $\Delta = 1$ line, plus a diagram where the scalar line is "cut" to two bulk-to-boundary propagators





- The difference of diagrams in $\Delta = 2$ and $\Delta = 1$ theories is then shown to match, diagram by diagram, the CFT computation (in the standard diagrammatic *1/N* approach)



One-loop in the bulk

- This implies that the 1/N anomalous dimensions in the critical O(N) model are correctly reproduced by bulk one-loop diagrams, provided the duality with free O(N) model holds
- What remains to be shown is that the bulk one loop two-point functions indeed vanish with the choice of boundary conditions corresponding to unbroken HS symmetry
- Perhaps it will be soon possible to check this directly, given the recent progress in constructing the explicit bulk vertices

Bekaert et al '15, '16

Skvortsov, Taronna '15

Sleight, Taronna '16

Finite N

- HS symmetry is weakly broken at large *N*. What happens at finite *N*?
- We can obtain some estimates for finite *N* by using the ε-expansion
- Recall that the interacting O(N) CFT has perturbative descriptions as
  - The IR fixed point of quartic scalar theory in $d=4-\varepsilon$
  - The UV fixed point of the non-linear sigma model in $d=2+\varepsilon$
- Both approaches work at finite *N*. We can combine information from the two expansions and obtain some estimate in the physical dimension d=3, for instance by using ``two-sided" Pade approximants

d=4-ε

$$S = \int d^d x \Big(\frac{1}{2} \partial_\mu \phi^i \partial^\mu \phi^i + \frac{\lambda}{4} (\phi^i \phi^i)^2 \Big)$$

$$\beta(\lambda) = -\epsilon\lambda + \frac{(N+8)\lambda^2}{8\pi^2} \qquad \lambda_* = \tfrac{8\pi^2}{N+8}\epsilon + O(\epsilon^2)$$

- We can use classical equation of motion to deduce the form of non-conservation $\partial \cdot J_s = \lambda_* K_{s-1}$, and we get the anomalous dimensions

$$\gamma_s = \frac{\epsilon^2 (N+2)}{2(N+8)^2} \Big(1 - \frac{6}{s(s+1)} \Big)$$

In agreement with Wilson, Kogut '74

d=2+ε

$$S = \int d^dx \left(\frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i + \sigma(\phi^i\phi^i - \frac{1}{g^2})\right)$$

$$\beta = \frac{\epsilon}{2}g - (N-2)\frac{g^3}{4\pi} \qquad g_*^2 = \frac{2\pi\epsilon}{N-2}$$

- From the HS non-conservation $\partial \cdot J_s = g_*^2 K_{s-1}$ we get

$$\gamma_s = \frac{\epsilon^2}{N-2}\left(\frac{1}{s} - \frac{1}{2} + \sum_{k=1}^{s-2}\frac{1}{k}\right)$$

- Note: large spin behavior is logarithmic in this case

$$\gamma_s \overset{s\gg 1}{\simeq} \frac{\epsilon^2}{N-2}\log(s)$$

d=3 estimates

- Combining d=4-ε and d=2+ε, as well as some input from large *N* expansion, we obtain the following estimates

N	3	4	5	6	10	20
$\gamma_{s=4}$ (Padé$_{[3,2]}$)	0.0261	0.0257	0.0208	0.0195	0.0158	0.0082
$\gamma_{s=6}$ (Padé$_{[3,2]}$)	0.0318	0.0310	0.0258	0.0240	0.0191	0.0100
$\gamma_{s=8}$ (Padé$_{[3,2]}$)	0.0342	0.0332	0.0278	0.0259	0.0206	0.0110
$\gamma_{s=10}$ (Padé$_{[3,2]}$)	0.0353	0.0343	0.0289	0.0269	0.0214	0.0115

- *N=1* and *N=2* are treated separately (NLσM cannot be used in this case). Ordinary Pade's give

$$\gamma_{s=4}^{N=1} = 0.0240\,, \qquad \gamma_{s=6}^{N=1} = 0.0300$$

$$\gamma_{s=8}^{N=1} = 0.0324\,, \qquad \gamma_{s=10}^{N=1} = 0.0336$$

$$\gamma_{s=4}^{N=2} = 0.0252\,, \qquad \gamma_{s=6}^{N=2} = 0.0315$$

$$\gamma_{s=8}^{N=2} = 0.0340\,, \qquad \gamma_{s=10}^{N=2} = 0.0353$$

Approximate HS symmetry?

- Results for *N=1* are compatible with the ones obtained by analytic bootstrap in *Alday, Zhiboedov '15*
- In all cases, anomalous dimensions appear to be very small. Approximate HS symmetry at finite *N* ?
- The smallness of the HS anomalous dimension is closely related to the fact that Δ_ϕ is very near free field. E.g. for *N=1* (3d Ising), $\Delta_\phi \approx 0.518$
- Other quantities apparently very close to free field value

$$c_T^{\text{3d Ising}} / c_T^{\text{3d free scalar}} \approx 0.9466 \qquad F_{3d \text{ Ising}} / F_{3d \text{ free sc}} \approx 0.976$$

El-Showk et al '13 — *Fei, SG, Klebanov, Tarnopolsky '15*

- Are all these facts related to the approximate HS symmetry? Can we use it to constrain further CFT data?