X, Y, Z
with Nonrelativistic Effective Field Theories

Nora Brambilla

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Non relativistic Effective Field Theories (plus lattice) give a QCD description of quarkonia below the strong decay threshold.
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• the hierarchy of NREFT is based on the hierarchy of scales in quarkonium
Non relativistic Effective Field Theories (plus lattice) give a QCD description of quarkonia below the strong decay threshold.

- The hierarchy of NREFT is based on the hierarchy of scales in quarkonium.

- In this framework, quarkonium becomes a golden system for the extraction of SM parameters (quark masses, alphas) and the study of confinement.
Can we use Non relativistic Effective Field Theories (plus lattice) to give a QCD description of exotic quarkonia ($X,Y,Z$) at or above the strong decay threshold?
• Can we use Non relativistic Effective Field Theories (plus lattice) to give a QCD description of exotic quarkonia (X,Y,Z) at or above the strong decay threshold?

• In this talk QQbar and glue: Hybrids multiplets with Lambda doubling effect and spin structure
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• This EFT picture may be extended to a comprehensive description of X, Y, Z
• Can we use Non relativistic Effective Field Theories (plus lattice) to give a QCD description of exotic quarkonia (X,Y,Z) at or above the strong decay threshold?

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Tetra quarks
• Can we use Non relativistic Effective Field Theories (plus lattice) to give a QCD description of exotic quarkonia (X,Y,Z) at or above the strong decay threshold?

• In this talk QQbar and glue: Hybrids multiplets with Lambda doubling effect and spin structure

• This EFT picture may be extended to a comprehensive description of X, Y, Z Tetra quarks

van der Waals bottomonia interaction : bound states?
Material for discussion/references

- **Heavy quarkonium: progress, puzzles, and opportunities**  

- **QCD and Strongly Coupled Gauge Theories: Challenges and Perspectives**  

- **Effective field theories for heavy quarkonium**  
  Nora Brambilla, Antonio Pineda, Joan Soto, Antonio Vairo  
  *Rev.Mod.Phys.* 77 (2005) 1423  

- **Quarkonium Hybrids with Nonrelativistic Effective Field Theories**  
  Matthias Berwein, Nora Brambilla, Jaume Tarrús Castellà, Antonio Vairo  
  e-Print: [arXiv:1510.04299](https://arxiv.org/abs/1510.04299)

- **Born-Oppenheimer approximation in an effective field theory language**  
  Nora Brambilla, Gastão Krein, Jaume Tarrús Castellà, Antonio Vairo  

- **Spin structure of heavy-quark hybrids**  
Quarkonium scales

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<td>Xb(2P)</td>
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<td>Xb(1P)</td>
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<td>Xc(1P)</td>
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<td>h(1P)</td>
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Normalized with respect to \( \chi_b(1P) \) and \( \chi_c(1P) \)

The mass scale is perturbative

\[ m_Q \gg \Lambda_{QCD} \]

\[ m_b \approx 5 \text{ GeV}; m_c \approx 1.5 \text{ GeV} \]
### Quarkonium scales

#### $S$ states

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#### Normalized with respect to $\chi_b(1P)$ and $\chi_c(1P)$

#### The system is nonrelativistic (NR)

\[
\Delta E \sim m v^2, \quad \Delta_{fs} E \sim m v^4
\]

\[
v_b^2 \sim 0.1, \quad v_c^2 \sim 0.3
\]

#### The mass scale is perturbative

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Quarkonium scales

NR bound states have at least 3 scales

\[ m \gg m v \gg m v^2 \quad v \ll 1 \]

\[ \Delta E \sim m v^2, \Delta_{fs} E \sim m v^4 \]

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The mass scale is perturbative

\[ 2S+1 L_J \]

Normalized with respect to \( \chi_b(1P) \) and \( \chi_c(1P) \)
**Quarkonium scales**

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**NR bound states have at least 3 scales**

\[
m \gg mv \gg mv^2 \quad v \ll 1
\]

\[
mv \sim r^{-1}
\]
Quarkonium scales

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NR bound states have at least 3 scales

\[ m \gg mv \gg mv^2 \quad v \ll 1 \]

\[ mv \sim r^{-1} \]

and \( \Lambda_{QCD} \)

The system is nonrelativistic (NR)

\[ \Delta E \sim mv^2, \Delta_f s E \sim mv^4 \]

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The mass scale is perturbative

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Normalized with respect to \( \chi_b(1P) \) and \( \chi_c(1P) \)
QCD theory of Quarkonium: a very hard problem

Close to the bound state $\alpha_s \sim \nu$

$p \sim m\alpha_s$

$\frac{g^2}{p^2} \left( 1 + \frac{m\alpha_s}{p} \right)$

$\sim \frac{1}{E - \left( \frac{p^2}{m} + V \right)}$
QCD theory of Quarkonium: a very hard problem

Close to the bound state $\alpha_s \sim \nu$

\[
p \sim m\alpha_s + \ldots \approx \frac{1}{E - (\frac{p^2}{m} + V)}
\]

\[
\frac{g^2}{p^2} \left(1 + \frac{m\alpha_s}{p}\right)
\]

\[
\frac{v^2}{m} + V = E \phi \rightarrow p \sim mv \text{ and } E = \frac{p^2}{m} + V \sim mv^2.
\]
QCD theory of Quarkonium: a very hard problem

Close to the bound state \( \alpha_s \sim \nu \)

\[ p \sim m\alpha_s \]

\[ \frac{g^2}{p^2} \left( 1 + \frac{m\alpha_s}{p} \right) \]

\[ g^2 \left( \frac{E}{m} - \left( \frac{p^2}{m} + V \right) \right) \]

- From \( \frac{p^2}{m} + V \phi = E\phi \rightarrow p \sim mv \) and \( E = \frac{p^2}{m} + V \sim mv^2 \).

multiscale diagrams have a complicate power counting and contribute to all orders in the coupling

\[ E \sim mv^2 \]

\[ p \sim mv \]

Difficult also for the lattice!

\[ L^{-1} \ll \lambda \ll \Lambda \ll a^{-1} \]
Quarkonium with Non-relativistic Effective Field Theories

\[ \mathcal{L}_{\text{EFT}} = \sum_{n} c_n \left( \frac{E_{\Lambda}}{\mu} \right) \frac{O_n(\mu, \lambda)}{E_{\Lambda}} \]

\[ \langle O_n \rangle \sim E_{\lambda}^n \]

Color degrees of freedom
3X3=1+8
singlet and octet QQbar

Hard

Soft
(relative momentum)

Ultrasoft
(binding energy)

\( m \)

perturbative matching

\( \mu \)

perturbative matching

\( \mu' \)

perturbative matching

\( \mu'' \)

perturbative matching

nonperturbative matching
(long-range quarkonium)

pNRQCD

QCD

NRQCD

m\nu

m\nu^2

singlet and octet QQbar

Color degrees of freedom
3X3=1+8
singlet and octet QQbar

Hard

Soft
(relative momentum)

Ultrasoft
(binding energy)
Quarkonium with Non relativistic Effective Field Theories

\[ \mathcal{L}_{\text{EFT}} = \sum_n c_n (E_\Lambda / \mu) \frac{O_n (\mu, \lambda)}{E_\Lambda} \]

\[ \langle O_n \rangle \sim E_\Lambda^n \]

Color degrees of freedom
3X3=1+8
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Hard

Soft
(relative momentum)

Ultrasoft
(binding energy)
Quarkonium with Non-relativistic Effective Field Theories

\[ \mathcal{L}_{EFT} = \sum_n c_n \left( \frac{E_\Lambda}{\mu} \right) \frac{O_n(\mu, \lambda)}{E_\Lambda} \]

\[ \langle O_n \rangle \sim E_\lambda^n \]
Quarkonium with NR EFT: Non Relativistic QCD (NRQCD)

\[ E \sim m v^2 \]

\[ p \sim m v \]

**QCD**

\[ m \]

*perturbative matching*

\[ m v \]

*nonperturbative matching (long-range quarkonium)*

\[ m v^2 \]

*perturbative matching (short-range quarkonium)*

**NRQCD**

\[ \mu \]

\[ \mu' \]

**pNRQCD**
Quarkonium with NR EFT: Non Relativistic QCD (NRQCD)

QCD

\[ m \]

perturbative matching

\[ m v \]

perturbative matching

\[ m v^2 \]

nonperturbative matching (long-range quarkonium)

perturbative matching (short-range quarkonium)

pNRQCD

E \sim mv^2

p \sim mv

\sim m
Quarkonium with NR EFT: Non Relativistic QCD (NRQCD)

\[ \mathcal{L}_{\text{NRQCD}} = \sum_n c(\alpha_s(m/\mu)) \times \frac{O_n(\mu, \lambda)}{m^n} \]
Quarkonium with NR EFT: potential NonRelativistic QCD (pNRQCD)
Quarkonium with NR EFT: potential Non-Relativistic QCD (pNRQCD)

\[ E \sim m v^2 \]

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perturbative matching (short-range quarkonium)
Quarkonium with NR EFT: potential NonRelativistic QCD (pNRQCD)

\[ \mathcal{L}_{\text{pNRQCD}} = \sum_k \sum_n \frac{1}{m_k^k} c_k(\alpha_s(m/\mu)) \times V(r\mu', r\mu) \times O_n(\mu', \lambda) r^n \]
Quarkonium with NR EFT: potential NonRelativistic QCD (pNRQCD)

$$\mathcal{L}_{pNRQCD} = \sum_k \sum_n \frac{1}{m_k} c_k(\alpha_s(m/\mu)) \times V(r\mu', r\mu) \times O_n(\mu', \lambda) r^n$$
In QCD another scale is relevant

\[ \Lambda_{\text{QCD}} \]

Quarkonium with NR EFT: pNRQCD

Pineda, Soto 97, N.B., Pineda, Soto, Vairo 99
N.B. Vairo, Pineda, Soto 00--014
Quarkonium systems with small radius \( r \ll \Lambda_{\text{QCD}}^{-1} \)
pNRQCD for quarkonia with small radius $r \ll \Lambda_{\text{QCD}}^{-1}$.

Degrees of freedom that \textit{scale} like $m v$ are integrated out:

- NRQCD
- $\times V(\mu r)$

$\Rightarrow$

- pNRQCD
pNRQCD for quarkonia with small radius $\Rightarrow r \ll \Lambda_{QCD}^{-1}$

Degrees of freedom that scale like $mv$ are integrated out:

\[ \text{NRQCD} \quad \Rightarrow \quad \text{pNRQCD} \times V(\mu r) \]

- If $mv \gg \Lambda_{QCD}$, the matching is perturbative

- Degrees of freedom: quarks and gluons
  - $Q$-$\bar{Q}$ states, with energy $\sim \Lambda_{QCD}$, $mv^2$ and momentum $\sim mv$
  - $\Rightarrow$ (i) singlet $S$  (ii) octet $O$

  - Gluons with energy and momentum $\sim \Lambda_{QCD}$, $mv^2$

- Definite power counting: $r \sim \frac{1}{mv}$ and $t, R \sim \frac{1}{mv^2}, \frac{1}{\Lambda_{QCD}}$

The gauge fields are multipole expanded:

\[ A(R, r, t) = A(R, t) + r \cdot \nabla A(R, t) + \ldots \]

Non-analytic behaviour in $r \rightarrow$ matching coefficients $V$
weak pNRQCD \( r \ll \Lambda_{\text{QCD}}^{-1} \)

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + \text{Tr} \left\{ S^\dagger \left( i\partial_0 - \frac{p^2}{m} - V_s \right) S + O^\dagger \left( iD_0 - \frac{p^2}{m} - V_o \right) O \right\} 
\]

LO in \( r \)

S singlet field
O octet field

Pineda, Soto 97; Brambilla, Pineda, Soto, Vairo 99-
weak pNRQCD \quad r \ll \Lambda_{QCD}^{-1}

\[ L = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu \, a} + \text{Tr} \left\{ S^\dagger \left( i\partial_0 - \frac{p^2}{m} - V_s \right) S \right\} S
\]
\[ + \ O^\dagger \left( iD_0 - \frac{p^2}{m} - V_o \right) O \}

Singlet static potential

Octet static potential

S singlet field

O octet field

Pineda, Soto 97; Brambilla, Pineda, Soto, Vairo 99-
weak pNRQCD \( r \ll \Lambda_{QCD}^{-1} \)

\( \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + \text{Tr} \left\{ S^\dagger \left( i\partial_0 - \frac{p^2}{m} - V_s \right) S \right\} \)

\( + O^\dagger \left( iD_0 - \frac{p^2}{m} - V_o \right) O \)

Singlet static potential

Octet static potential

- At leading order in \( r \), the singlet \( S \) satisfies the QCD Schrödinger equation.

Pineda, Soto 97; Brambilla, Pineda, Soto, Vairo 99-
At leading order in $r$, the singlet $S$ satisfies the QCD Schrödinger equation.

- The (weak coupling) static potential is the Coulomb potential:

$$V_s(r) = -C_F \frac{\alpha_s}{r} + \ldots,$$

$$V_o(r) = \frac{1}{2N} \frac{\alpha_s}{r} + \ldots, \quad N = 3, \quad C_F = \frac{4}{3}$$
weak pNRQCD $r \ll \Lambda_{QCD}^{-1}$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu}{}^a + \text{Tr} \left\{ S^\dagger \left( i\partial_0 - \frac{p^2}{m} - V_s \right) S \right. \\
+ \left. O^\dagger \left( iD_0 - \frac{p^2}{m} - V_o \right) O \right\}$$

LO in $r$

Pineda, Soto 97; Brambilla, Pineda, Soto, Vairo 99-
weak pNRQCD \quad r \ll \Lambda_{QCD}^{-1}

\begin{align*}
\mathcal{L} &= -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu} a + \text{Tr} \left\{ S^\dagger \left( i\partial_0 - \frac{p^2}{m} - V_s \right) S \right. \\
&\quad + \left. O^\dagger \left( iD_0 - \frac{p^2}{m} - V_o \right) O \right\} \\
&\quad + V_A \text{Tr} \left\{ O^\dagger r \cdot gE S + S^\dagger r \cdot gE O \right\} \\
&\quad + \frac{V_B}{2} \text{Tr} \left\{ O^\dagger r \cdot gE O + O^\dagger O r \cdot gE \right\} \\
&\quad + \cdots
\end{align*}

Pineda, Soto 97; Brambilla, Pineda, Soto, Vairo 99-
weak pNRQCD \( r \ll \Lambda_{QCD}^{-1} \)

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + \text{Tr} \left\{ S^\dagger \left( i\partial_0 - \frac{p^2}{m} - V_s \right) S \right. \\
\left. + O^\dagger \left( iD_0 - \frac{p^2}{m} - V_o \right) O \right\}
\]

\( \text{LO in } r \)

\[
+ V_A \text{Tr} \left\{ O^\dagger r \cdot gE S + S^\dagger r \cdot gE O \right\} \\
+ \frac{V_B}{2} \text{Tr} \left\{ O^\dagger r \cdot gE O + O^\dagger O r \cdot gE \right\} \\
+ \cdots
\]

\( \text{NLO in } r \)

- Feynman rules:

\[
\begin{array}{l}
\hline
\text{---} = \theta(t) e^{-itH_s} \\
\hline
\text{---} = \theta(t) e^{-itH_o} (e^{-i \int dt A^{\text{adj}}}) \\
\hline
\text{---} = O^\dagger r \cdot gE S \\
\hline
\text{---} = O^\dagger \{ r \cdot gE, O \}
\end{array}
\]
The QQbar potential is a matching coefficient of pNRQCD and can be calculated in perturbation theory:

\[
V = V_0 + \frac{1}{m} V_1 + \frac{1}{m^2} (V_{SD} + V_{VD})
\]

Quarkonium singlet potential

Small systems: \textbf{QQ} energies at \( m\alpha_s^5 \)

\[
E_n = 2m + \langle n | \frac{p^2}{m} + V_s | n \rangle + \langle n | \langle n | \mu \rangle
\]

\[
E_n = \langle n | H_s(\mu) | n \rangle - i \frac{g^2}{3N_c} \int_0^\infty dt \langle n | r e^{it(E_n^{(0)} - H_0)} r | n \rangle \langle E(t) E(0) \rangle (\mu)
\]
Applications to Quarkonium physics: systems with small radius

- $c$ and $b$ masses at NNLO, $N^3LO^*$, NNLL$^*$;
- $B_c$ mass at NNLO; Penin et al 04
- $B_c^*$, $\eta_c$, $\eta_b$ masses at NLL; Kniehl et al 04
- Quarkonium $1P$ fine splittings at NLO;
- $\Upsilon(1S)$, $\eta_b$ electromagnetic decays at NNLL;
- $\Upsilon(1S)$ and $J/\psi$ radiative decays at NLO;
- $\Upsilon(1S) \to \gamma \eta_b$, $J/\psi \to \gamma \eta_c$ at NNLO;
- $t\bar{t}$ cross section at NNLL;
- $QQq$ and $QQQ$ baryons: potentials at NNLO, masses, hyperfine splitting, ...; N. B. et al 010
- Thermal effects on quarkonium in medium: potential, masses (at $m\alpha_s^5$), widths, ...;

\begin{align*}
\mathcal{B}(J/\psi & \to \gamma \eta_c(1S)) = (1.6 \pm 1.1)\% \quad \text{N. B. Yu Jia A. Vairo 2005} \\
\mathcal{B}(\Upsilon(1S) & \to \gamma \eta_b(1S)) = (2.85 \pm 0.30) \times 10^{-4} \\
\Gamma(\eta_b(1S) & \to \gamma \gamma) = 0.54 \pm 0.15 \text{ keV} \quad \text{Y. Kiyo, A. Pineda, A. Signer 2010} \\
\Gamma(\eta_b(1S) & \to \text{LH}) = 7\text{-}16 \text{ MeV}
\end{align*}
Quarkonium systems with large radius $r \sim \Lambda_{QCD}^{-1}$.
Hitting the scale $\Lambda_{QCD}$

$(Q\bar{Q})_1 + \text{Glueball}$

$r \sim \Lambda_{QCD}^{-1}$

$(Q\bar{Q})_8 G$

hybrid
Hitting the scale $\Lambda_{QCD}$

$(Q\bar{Q})_1$

$(Q\bar{Q})_1 + \text{Glueball}$

$r \sim \Lambda_{QCD}^{-1}$

$(Q\bar{Q})_8G$

hybrid

Static qcd spectrum

Gluon excitations

$\beta=2.5$

$a_s=0.2\,\text{fm}$

$z=0.976(21)$

$a_s/\Lambda_s = z^5$

$|L_z|$

$c+=/-$

Symmetries of a diatomic molecule + C.C.

a) $|L_z| = 0, 1, 2, \ldots$

$= \Sigma, \Pi, \Delta \ldots$

b) CP (u/g)

c) Reflection (+/- )

(for $\Sigma$ only)

Jüge Kuti Morningstar 98-06 Static NRQCD
Hitting the scale $\Lambda_{QCD}$

$$(Q\bar{Q})_1 + \text{Glueball}$$

$$(Q\bar{Q})_8 G$$

**Static qcd spectrum**

- $a_t E_\Gamma$
- $\beta = 2.5$
- $a_s \sim 0.2 \text{ fm}$
- $z = 0.976(21)$

Symmetries of a diatomic molecule + C.C.

a) $|L_z| = 0, 1, 2, \ldots = \Sigma, \Pi, \Delta \ldots$

b) CP $(u/g)$

c) Reflection $(+/-)$ (for $\Sigma$ only)

\[ \mathcal{H}^{(0)} |n; x_1, x_2\rangle^{(0)} = E_n^{(0)}(x_1, x_2) |n; x_1, x_2\rangle^{(0)} \]

\[ |n; x_1, x_2\rangle^{(0)} = \psi^\dagger(x_1) \chi(x_2) |n; x_1, x_2\rangle^{(0)} \]
Hitting the scale $\Lambda_{QCD}$

$(Q\bar{Q})_1$

$(Q\bar{Q})_1 + \text{Glueball}$

$(Q\bar{Q})_8G$

Hybrid

Static QCD spectrum

Gluon excitations

\[ a_t E_\Gamma \]

Lat \, t \, c

Gluon excitations

\[ \beta = 2.5 \]

\[ a_s = 0.2 \, \text{fm} \]

\[ N = 4 \]

\[ N = 3 \]

\[ N = 2 \]

\[ N = 1 \]

\[ N = 0 \]

Gluon excitations

Crossover

String ordering

\[ a_s / a_t = z^* 5 \]

\[ z = 0.976(21) \]

Short distance degeneracies

\[ R / a_s \]

\[ |L_z| \]

\[ Q \]

\[ \bar{Q} \]

Symmetries of a diatomic molecule + C.C.

a) $|L_z| = 0, 1, 2, \ldots$

\[ = \Sigma, \Pi, \Delta \ldots \]

b) CP (u/g)

c) Reflection (+/-) (for $\Sigma$ only)

\[ |0\rangle^{(0)} = |(Q\bar{Q})_1\rangle \rightarrow \text{Quarkonium Singlet} \]

\[ |n > 0\rangle^{(0)} = |(Q\bar{Q})_g^{(n)}\rangle \rightarrow \text{Higher Gluonic Excitations} \]

\[ \mathcal{H}^{(0)} |n; x_1, x_2\rangle^{(0)} = E^{(0)}_n (x_1, x_2) |n; x_1, x_2\rangle^{(0)} \]

\[ \psi^\dagger (x_1) \chi (x_2) |n; x_1, x_2\rangle^{(0)} \]
$r \sim \Lambda_{QCD}^{-1}$  \[ mv \sim \Lambda_{QCD} \]

- Integrate out all scales above $mv$.
- Gluonic excitations develop a gap $\Lambda_{QCD}$ and are integrated out.

\[ mv^2 \]

$\Rightarrow$ The singlet quarkonium field $S$ of energy $mv^2$ is the only degree of freedom of pNRQCD (up to ultrasoft light quarks, e.g. pions).

\[ \mathcal{L} = \text{Tr} \left\{ S^\dagger \left( i\partial_0 - \frac{P^2}{m} - V_s \right) S \right\} \]
strongly coupled pNRQCD \[ r \sim \Lambda_{QCD}^{-1} \quad mv \sim \Lambda_{QCD} \]

- integrate out all scales above \[ mv^2 \]
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⇒ The singlet quarkonium field \( S \) of energy \( mv^2 \) is the only the degree of freedom of pNRQCD (up to ultrasoft light quarks, e.g. pions).

\[ \mathcal{L} = \text{Tr} \left\{ S^\dagger \left( i\partial_0 - \frac{p^2}{m} - V_s \right) S \right\} \]

- A potential description emerges from the EFT
- The potentials \( V = \text{Re}V + \text{Im}V \) from QCD in the matching: get spectra and decays
- \( V \) to be calculated on the lattice or in QCD vacuum models

Brambilla Pineda Soto Vairo 00

\begin{align*}
\kappa = 0.1575
\end{align*}
The matching condition is:

\[ \langle H | \mathcal{H} | H \rangle = \langle nljs | \frac{p^2}{m} + \sum_n \frac{V_s^{(n)}}{m^n} | nljs \rangle \]
The matching condition is:

\[
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\]

\[
H_{\text{NRQCD}} = H^{(0)} + \frac{1}{m_Q} H^{(1,0)} + \frac{1}{m_{\bar{Q}}} H^{(0,1)} + \ldots ,
\]

\[
H^{(0)} = \int d^3 x \frac{1}{2} (E^a \cdot E^a + B^a \cdot B^a) - \sum_{j=1}^{n_f} \int d^3 x \bar{q}_j iD \cdot \gamma q_j ,
\]

\[
H^{(1,0)} = -\frac{1}{2} \int d^3 x \psi^\dagger (D^2 + gc_F \sigma \cdot B) \psi ,
\]

\[
H^{(0,1)} = \frac{1}{2} \int d^3 x \chi^\dagger (D^2 + gc_F \sigma \cdot B) \chi ,
\]

\[
\mathcal{H}^{(0)} | \overline{n}; x_1, x_2 \rangle^{(0)} = E_n^{(0)}(x_1, x_2) | \overline{n}; x_1, x_2 \rangle^{(0)}
\]

\[
| \overline{n}; x_1, x_2 \rangle^{(0)} = \psi^\dagger(x_1) \chi(x_2) | n; x_1, x_2 \rangle^{(0)}
\]
The matching condition is:
\[ \langle H | \mathcal{H} | H \rangle = \langle nljs | \frac{p^2}{m} + \sum_n \frac{V_s^{(n)}}{m^n} | nljs \rangle \]

and from this we obtain the

Quarkonium singlet potential

\[ V = V_0 + \frac{1}{m} V_1 + \frac{1}{m^2} (V_{SD} + V_{VD}) \]
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**Quarkonium singlet potential**

\[ V = V_0 + \frac{1}{m} V_1 + \frac{1}{m^2} (V_{SD} + V_{VD}) \]

\[ V_s^{(0)} = \lim_{T \to \infty} \frac{i}{T} \ln \langle W(r \times T) \rangle = \lim_{T \to \infty} \frac{i}{T} \ln \langle \square \rangle \]

\[ W = \langle \exp\{ig \int A^\mu dx_\mu \} \rangle \]
QCD Spin dependent potentials

\[ V_{SD}^{(2)} = \frac{1}{r} \left( c_F \epsilon^{kij} \frac{2r^k}{r} i \int_{0}^{\infty} dt \left< \begin{array}{c} i \\ j \end{array} \right> - \frac{1}{2} V_s^{(0)r} \right) (S_1 + S_2) \cdot L \]

\[ -c_F^2 \hat{r}_i \hat{r}_j i \int_{0}^{\infty} dt \left( \left< \begin{array}{c} i \\ j \end{array} \right> - \frac{\delta_{ij}}{3} \left< \begin{array}{c} i \\ i \end{array} \right> \right) \times \left( S_1 \cdot S_2 - 3(S_1 \cdot \hat{r})(S_2 \cdot \hat{r}) \right) \]

\[ + \left( \frac{2}{3} c_F^2 i \int_{0}^{\infty} dt \left< \begin{array}{c} i \\ i \end{array} \right> - 4(d_2 + C_F d_4) \delta^{(3)}(r) \right) S_1 \cdot S_2 \]

- factorization: the NRQCD matching coefficients encode the physics at the large scale \( m \), the potentials are given in terms of low energy nonperturbative Wilson loops.
- the spin dependent potential has the usual structure with spin-orbit, tensor and spin-spin terms. The spin-orbit term has a confining contribution: they appear at order \( 1/m^2 \).
- the spin dependent potentials in the Schroedinger eq. give the multiplet spin structure.
EFTs (plus lattice) give a QCD description of quarkonium below threshold
EFTs (plus lattice) give a QCD description of quarkonium below threshold.

However, X Y Z states appear at or above the strong decay threshold.
For states close or above the strong decay threshold the situation is much more complicated.

there is no mass gap between quarkonium and the creation of a heavy-light mesons couple

\[ m_{Q\bar{q}} + m_{\bar{Q}q} = 2m + 2\Lambda_{QCD} \]
For states close or above the strong decay threshold the situation is much more complicated.

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Near the threshold heavy-light mesons and gluons excitations have to be included and many additional states built using the light quark quantum numbers may appear.

No systematic treatment is yet available; also lattice calculations are challenging.
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No systematic treatment is yet available; also lattice calculations are challenging.

Many phenomenological models exist.
States made of two heavy and light quarks

- Pairs of heavy-light mesons: $D\bar{D}, B\bar{B}$, ...

- Molecular states, i.e. states built on the pair of heavy-light mesons.
  - Tornqvist PRL 67(91)556

- The usual quarkonium states, built on the static potential, may also give rise to molecular states through the interaction with light hadrons (hadro-quarkonium).
  - Dubynskiy Voloshin PLB 666 (2008) 344

- Tetraquark states.

Having the spectrum of tetraquark potentials, like we have for the gluonic excitations, would allow us to build a plethora of states on each of the tetraquark potentials, many of them developing a width due to decays through pion (or other light hadrons) emission. Diquarks have been recently investigated on the lattice.

- Alexandrou et al. PRL 97(06)222002
- Fodor et al. PoS LAT2005(06)310

Vijande, Valcarce, Richard

Maiani, Piccinini, Polosa et al. 2005--
choosing one of these degrees of freedom and an interaction (a dynamical assumption) originates a model for exotics.
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It is particularly difficult to insert spin in these models, and when it is done the spin interaction is taken from standard quarkonium.
Start considering the simplified case of heavy quark, heavy antiquark plus glue.

**Characteristic Scales**

- Heavy-quarks are non-relativistic $m_Q \gg \Lambda_{QCD}$.
- Two components with very different dynamical time scales $\Lambda_{QCD} \gg m_Q v^2$.
  - Light d.o.f state $E_{\text{light}} \sim \Lambda_{QCD}$.
  - Heavy-quark binding $E_Q \sim m_Q v^2$ ($v \ll 1$ relative velocity).
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Quarkonium hybrids are a similar system to diatomic molecules

- Heavy d.o.f: Nuclei $\rightarrow$ Heavy Quark
- Light d.o.f: Electrons $\rightarrow$ Gluons&Light-quarks
Start considering the simplified case of heavy quark, heavy antiquark plus glue.

EFT approach: Exploit the hierarchy of scales at the Lagrangian level


Heavy-quark heavy antiquark plus glue
Heavy-quark heavy antiquark plus glue

Define the symmetries of the system and the system static energies in NRQCD

Static Lattice energies  Juge Kuti Morningstar 2003

Symmetries

Static states classified by symmetry group $D_{\infty h}$

Representations labeled $\Lambda_{\eta}$

Representations of $D_{\infty h}$

- $\Lambda = |\lambda|$ rotational quantum number
  $\lambda = \hat{r} \cdot \hat{K} = 0, \pm 1, \pm 2 \ldots$ corresponds to $\Lambda = \Sigma, \Pi, \Delta \ldots$
- $\eta$ eigenvalue of CP:
  $g \hat{=} +1, u \hat{=} -1$
- $\sigma$ eigenvalue of reflections (only $\Sigma$ states)

- The static energies correspond to the irreducible representations of $D_{\infty h}$.
- In general it can be more than one state for each irreducible representation of $D_{\infty h}$, usually denoted by primes, e.g. $\Pi_u, \Pi'_u, \Pi''_u \ldots$
Heavy-quark heavy antiquark plus glue

Lattice energies

- $\Sigma^+_g$ is the ground state potential that generates the standard quarkonium states.
- The rest of the static energies correspond to excited gluonic states that generate hybrids.
- The two lowest hybrid static energies are $\Pi_u$ and $\Sigma_u^-$, they are nearly degenerate at short distances.
- The static energies have been computed in quenched lattice QCD, the most recent data by Juge, Kuti, Morningstar, 2002 and Bali and Pineda 2003.
- Quenched and unquenched calculations for $\Sigma^+_g$ and $\Pi_u$ were compared in Bali et al 2000 and good agreement was found below string breaking distance.

Juge Kuti Morningstar PRL 90 (2003) 161601
Heavy-quark heavy antiquark plus glue

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Juge Kuti Morningstar PRL 90 (2003) 161601
In the short-range hybrids become *gluelumps*, i.e., quark-antiquark octets, \( O^a \), in the presence of a gluonic field, \( H^a: H(R, r, t) = H^a(R, t)O^a(R, r, t) \).

In the limit \( r \to 0 \) more symmetry: \( D_\infty \to O(3) \times C \)

- Several \( \Lambda_\eta^\sigma \) representations contained in one \( J^{PC} \) representation:
- Static energies in these multiplets have same \( r \to 0 \) limit.

**Gluonic excitation operators up to dim 3**

<table>
<thead>
<tr>
<th>( \Lambda_\eta^\sigma )</th>
<th>( K^{PC} )</th>
<th>( H^a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Sigma_u )</td>
<td>1++</td>
<td>( r \cdot B, r \cdot (D \times E) )</td>
</tr>
<tr>
<td>( \Pi_u )</td>
<td>1++</td>
<td>( r \times B, r \times (D \times E) )</td>
</tr>
<tr>
<td>( \Sigma^+_g )</td>
<td>1++</td>
<td>( r \times E, r \times (D \times B) )</td>
</tr>
<tr>
<td>( \Pi^+_g )</td>
<td>1--</td>
<td>( (r \cdot D)(r \cdot B) )</td>
</tr>
<tr>
<td>( \Sigma^+_g )</td>
<td>2++</td>
<td>( r \times ((r \cdot D)B + D(r \cdot B)) )</td>
</tr>
<tr>
<td>( \Pi^+_g )</td>
<td>2--</td>
<td>( (r \times D)^i(r \times B)^i + (r \times D)^i(r \times B)^i )</td>
</tr>
<tr>
<td>( \Delta_g )</td>
<td>2--</td>
<td>( (r \cdot D)(r \cdot E) )</td>
</tr>
<tr>
<td>( \Sigma^- + )</td>
<td>2--</td>
<td>( r \times ((r \cdot D)E + D(r \cdot E)) )</td>
</tr>
<tr>
<td>( \Delta_u )</td>
<td>2--</td>
<td>( (r \times D)^i(r \times E)^i + (r \times D)^i(r \times E)^i )</td>
</tr>
</tbody>
</table>

The gluelump multiplets \( \Sigma_u^-, \Pi_u, \Sigma^+_g, \Pi'_g, \Sigma_g^-, \Pi'_g, \Delta_g, \Sigma^+_u, \Pi'_u, \Delta_u \) are degenerate.

pNRQCD gives the multiplets at short distance: *gluelumps*.
In the short-range hybrids become gluelumps, i.e., quark-antiquark octets, $O^a$, in the presence of a gluonic field, $H^a: H(R, r, t) = H^a(R, t)O^a(R, r, t)$.

Match to pNRQCD: one can determine the form of the potential

$$E_H = V_o + \frac{i}{T} \ln \langle H^a \left( \frac{T}{2} \right) \phi_{ab} \phi^{\text{adj}} \phi^b \left( -\frac{T}{2} \right) \rangle$$

$$\langle H^a \left( \frac{T}{2} \right) \phi_{ab} \phi^{\text{adj}} \phi^b \left( -\frac{T}{2} \right) \rangle_{\text{np}} \sim h e^{-iT\Lambda_H}$$

$$E_H(r) = V_o(r) + \Lambda_H + b \Lambda_H r^2$$
Match to pNRQCD: one can determine the form of the potential

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Match to pNRQCD: one can determine the form of the potential

\[ E_H(r) = V_O(r) + \Lambda_H + b_H r^2 \]

- It is a non-perturbative quantity.
- It depends on the particular operator \( H^a \), however it is the same for operators corresponding to different projections of the same gluonic operators.
- The gluelump masses have been determined in the lattice. Foster et al 1999; Bali, Pineda 2004; Marsh Lewis 2014
- At the subtraction scale \( \nu_f = 1 \text{ GeV} \): \( \Lambda^{RS}_{1+-} = 0.87(15) \text{ GeV} \).
Match to pNRQCD: one can determine the form of the potential

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\[ b_H \]
- It is a non-perturbative quantity.
- Proportional to \( r^2 \) due to rotational invariance and the multipole expansion.
- We are going to fix it through a fit to the static energies lattice data.
- Breaks the degeneracy of the potentials.
Match to pNRQCD: one can determine the form of the potential

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- Breaks the degeneracy of the potentials.

Octet potential at two loops; renormalon subtraction realised among pole mass, octet potential and gluelump mass, use RS scheme
We consider hybrids that are excitations of the lowest lying static energies $\Pi_u$ and $\Sigma_u^-$. In the $r \to 0$ limit $\Pi_u$ and $\Sigma_u^-$ are degenerate and correspond to a gluonic operator with quantum numbers $1^{+-}$.
State multiplets

Hybrid spectrum for $\kappa = 1^{+-} \rightarrow \Lambda_\eta^\sigma = \Sigma_u^-, \Pi_u$

We consider hybrids that are excitations of the lowest lying static energies $\Pi_u$ and $\Sigma_u^-$. In the $r \to 0$ limit $\Pi_u$ and $\Sigma_u^-$ are degenerate and correspond to a gluonic operator with quantum numbers $1^{+-}$.

States are organized in spin multiplets.

<table>
<thead>
<tr>
<th>$H_1$</th>
<th>1</th>
<th>${1^{--}, (0, 1, 2)^{-+}}$</th>
<th>$\Sigma_u^-, \Pi_u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_2$</td>
<td>1</td>
<td>${1^{++}, (0, 1, 2)^{+-}}$</td>
<td>$\Pi_u$</td>
</tr>
<tr>
<td>$H_3$</td>
<td>0</td>
<td>${0^{++}, 1^{+-}}$</td>
<td>$\Sigma_u^-$</td>
</tr>
<tr>
<td>$H_4$</td>
<td>2</td>
<td>${2^{++}, (1, 2, 3)^{+-}}$</td>
<td>$\Sigma_u^-, \Pi_u$</td>
</tr>
<tr>
<td>$H_5$</td>
<td>2</td>
<td>${2^{--}, (1, 2, 3)^{-+}}$</td>
<td>$\Pi_u$</td>
</tr>
</tbody>
</table>

Braaten PRL 111 (2013) 162003
Braaten Langmack Smith PRD 90 (2014) 014044
Coupled radial Schrödinger equations

Projection vectors in matrix elements allow for two different solutions (coupled or uncoupled) for the \( \Sigma_u^- \) and \( \Pi_u \) radial wave functions:

1st solution

\[
\begin{bmatrix}
-\frac{1}{2\mu r^2} \partial_r r^2 \partial_r + \frac{1}{2\mu r^2} \\
\end{bmatrix} \begin{bmatrix}
\frac{l(l+1)+2}{2\sqrt{l(l+1)}} \\
\frac{2\sqrt{l(l+1)}}{l(l+1)}
\end{bmatrix} + \begin{bmatrix}
E^{(0)}_\Sigma \\
0
\end{bmatrix} \begin{bmatrix}
\psi_\Sigma \\
\psi_\Pi
\end{bmatrix} = \mathcal{E} \begin{bmatrix}
\psi_\Sigma \\
\psi_\Pi
\end{bmatrix}
\]

2nd solution

\[
\begin{bmatrix}
-\frac{1}{2\mu r^2} \partial_r r^2 \partial_r + \frac{l(l+1)}{2\mu r^2} + E^{(0)}_\Pi
\end{bmatrix} \psi_\Pi = \mathcal{E} \psi_\Pi
\]

- energy eigenvalue \( \mathcal{E} \) gives hybrid mass: \( m_H = m_Q + m_{\bar{Q}} + \mathcal{E} \)
- \( l(l+1) \) is the eigenvalue of angular momentum \( L^2 = (L_Q \bar{Q} + L_g)^2 \)
- the two solutions correspond to **opposite parity** states: \( (-1)^l \) and \( (-1)^{l+1} \)
- corresponding eigenvalues under charge conjugation: \( (-1)^{l+s} \) and \( (-1)^{l+s+1} \)
- Schrödinger equations can be solved numerically

For \( l = 0 \) the off-diagonal terms vanish, so the equations for \( \psi^{(N)}_\Sigma \) and \( \psi^{(N)}_\Pi \) decouple. There exists only one parity state, and its radial wave function is given by a Schrödinger equation with the \( E^{(0)}_\Sigma \) potential and an angular part \( 2/mr^2 \).
The Lambda-doubling effect breaks the degeneracy between opposite parity spin-symmetry multiplets and lowers the mass of the multiplets that get mixed contributions of different static energies.

1st solution
\[
\begin{pmatrix} \frac{1}{2\mu r^2} \frac{1}{2} r^2 \partial_r+\frac{l(l+1)+2}{2\sqrt{l(l+1)}} \frac{2\sqrt{l(l+1)}}{l(l+1)} \bigg[ \begin{pmatrix} E^{(0)}_{\Sigma} \\ 0 \end{pmatrix} \bigg] \end{pmatrix} = \varepsilon \begin{pmatrix} \psi_{\Sigma} \\ \psi_{\Pi} \end{pmatrix}
\]

2nd solution
\[
\begin{pmatrix} \frac{1}{2\mu r^2} \partial_r r^2 \partial_r + \frac{l(l+1)}{2\mu r^2} + E^{(0)}_{\Pi} \end{pmatrix} \psi_{\Pi} = \varepsilon \psi_{\Pi}
\]

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\[ E_H(r) = V_O(r) + \Lambda_H + b_H r^2 \]
Lattice data: Bali, Pineda 2004; Juge, Kuti, Morningstar 2003, dashed line $V^{(0.5)}$, solid line $V^{(0.25)}$

$V^{(0.25)}$

- $r \leq 0.25$ fm: pNRQCD potential.
  - Lattice data fitted for the $r = 0 - 0.25$ fm range with the same energy offsets as in $V^{(0.5)}$.
    
    $$b^{(0.25)}_{\Sigma} = 1.246 \text{ GeV/fm}^2, \quad b^{(0.25)}_{\Pi} = 0.000 \text{ GeV/fm}^2.$$

- $r > 0.25$ fm: phenomenological potential.
  - $V'(r) = \frac{a_1}{r} + \sqrt{a_2 r^2 + a_3 + a_4}$.
  - Same energy offsets as in $V^{(0.25)}$.
  - Constraint: Continuity up to first derivatives.
A doubling in quarkonium hybrid states

Charmonium hybrids

- no distinction between opposite parity states in BO
- mixed states lie lower than pure

Bottomonium hybrids

○ Berwein Brambilla Tarrus Vairo PRD 92 (2015) 114019
  data without mixing (dashed) from Braaten et al PRD 90 (2014) 114044
<table>
<thead>
<tr>
<th>State</th>
<th>$M$ (MeV)</th>
<th>$\Gamma$ (MeV)</th>
<th>$J^{PC}$</th>
<th>Decay modes</th>
<th>1st observation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X(3823)$</td>
<td>3823.1 ± 1.9</td>
<td>&lt; 24</td>
<td>?^{2-}</td>
<td>$\chi_c\gamma$</td>
<td>Belle 2013</td>
</tr>
<tr>
<td>$X(3872)$</td>
<td>3871.68 ± 0.17</td>
<td>&lt; 1.2</td>
<td>1^{++}</td>
<td>$J/\psi \pi^+\pi^-, J/\psi \pi^+\pi^-\pi^0$</td>
<td>Belle 2003</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$D^0\bar{D}^0\pi^0, D^0\bar{D}^0\gamma$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$J/\psi\gamma, \psi(2S)\gamma$</td>
<td></td>
</tr>
<tr>
<td>$X(3915)$</td>
<td>3917.5 ± 1.9</td>
<td>20 ± 5</td>
<td>0^{++}</td>
<td>$J/\psi\omega$,</td>
<td>Belle 2004</td>
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<tr>
<td>$\chi^c(2P)$</td>
<td>3927.2 ± 2.6</td>
<td>24 ± 6</td>
<td>2^{++}</td>
<td>$D\bar{D}$,</td>
<td>Belle 2005</td>
</tr>
<tr>
<td>$X(3940)$</td>
<td>3942^{+9}_{-8}</td>
<td>37^{+27}_{-17}</td>
<td>?^{2+}</td>
<td>$D^<em>\bar{D}, D\bar{D}^</em>$</td>
<td>Belle 2007</td>
</tr>
<tr>
<td>$G(3900)$</td>
<td>3943 ± 21</td>
<td>52 ± 11</td>
<td>1^{--}</td>
<td>$D\bar{D}$,</td>
<td>Babar 2007</td>
</tr>
<tr>
<td>$Y(4008)$</td>
<td>4008^{+121}_{-49}</td>
<td>226 ± 97</td>
<td>1^{--}</td>
<td>$J/\psi\pi^+\pi^-$,</td>
<td>Belle 2007</td>
</tr>
<tr>
<td>$Y(4140)$</td>
<td>4144.5 ± 2.6</td>
<td>15^{+11}_{-7}</td>
<td>?^{2+}</td>
<td>$J/\psi\phi$</td>
<td>CDF 2009</td>
</tr>
<tr>
<td>$X(4160)$</td>
<td>4156^{+29}_{-25}</td>
<td>139^{+113}_{-65}</td>
<td>?^{2+}</td>
<td>$D^<em>\bar{D}^</em>$</td>
<td>Belle 2007</td>
</tr>
<tr>
<td>$Y(4220)$</td>
<td>4216 ± 7</td>
<td>39 ± 17</td>
<td>1^{--}</td>
<td>$h_c(1P)\pi^+\pi^-$,</td>
<td>BESIII 2013</td>
</tr>
<tr>
<td>$Y(4230)$</td>
<td>4230 ± 14</td>
<td>38 ± 14</td>
<td>1^{--}</td>
<td>$\chi_{c0}\omega$,</td>
<td>BESIII 2014</td>
</tr>
<tr>
<td>$Y(4260)$</td>
<td>4263^{+8}_{-9}</td>
<td>95 ± 14</td>
<td>1^{--}</td>
<td>$J/\psi\pi^+\pi^-, J/\psi\pi^0\pi^0$</td>
<td>Babar 2005</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$Z_c(3900)\pi$,</td>
</tr>
<tr>
<td>$Y(4274)$</td>
<td>4293 ± 20</td>
<td>35 ± 16</td>
<td>?^{2+}</td>
<td>$J/\psi\phi$</td>
<td>CDF 2010</td>
</tr>
<tr>
<td>$X(4350)$</td>
<td>4350.6^{+4.6}_{-5.1}</td>
<td>13.3^{+18.4}_{-10.0}</td>
<td>0/2^{++}</td>
<td>$J/\psi\phi$,</td>
<td>Belle 2009</td>
</tr>
<tr>
<td>$Y(4360)$</td>
<td>4354 ± 11</td>
<td>78 ± 16</td>
<td>1^{--}</td>
<td>$\psi(2S)\pi^+\pi^-$,</td>
<td>Babar 2007</td>
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<tr>
<td>$X(4630)$</td>
<td>4634^{+9}_{-11}</td>
<td>92^{+41}_{-32}</td>
<td>1^{--}</td>
<td>$\Lambda_c^+\Lambda_c^-$,</td>
<td>Belle 2007</td>
</tr>
<tr>
<td>$Y(4660)$</td>
<td>4665 ± 10</td>
<td>53 ± 14</td>
<td>1^{--}</td>
<td>$\psi(2S)\pi^+\pi^-$,</td>
<td>Belle 2007</td>
</tr>
<tr>
<td>$Y_b(10890)$</td>
<td>10888.4 ± 3.0</td>
<td>30.7^{+8.9}_{-7.7}</td>
<td>1^{--}</td>
<td>$\Upsilon(nS)\pi^+\pi^-$</td>
<td>Belle 2010</td>
</tr>
</tbody>
</table>

TABLE V: Neutral mesons above open flavor threshold excluding isospin partners of charged states.
Charmonium Hybrid spectrum for $\kappa = 1^{+-}$


1. Solid blue bars: Neutral exotic charmonium states (Belle, CDF, BESIII, Babar, LHCb).

2. Bands: Predicted masses for hybrid spin-symmetry multiplets $\pm$ uncertainty of $\Lambda_{1^{+-}}$.

Spin-symmetry multiplets

<table>
<thead>
<tr>
<th></th>
<th>$l$</th>
<th>$J^{PC}$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_1$</td>
<td>1</td>
<td>${1^{--}, (0, 1, 2)^{-+}}$</td>
<td>$\Sigma_u^-, \Pi_u$</td>
</tr>
<tr>
<td>$H_2$</td>
<td>1</td>
<td>${1^{++}, (0, 1, 2)^{+-}}$</td>
<td>$\Pi_u$</td>
</tr>
<tr>
<td>$H_3$</td>
<td>0</td>
<td>${0^{++}, 1^{-+}}$</td>
<td>$\Sigma_u^-$</td>
</tr>
<tr>
<td>$H_4$</td>
<td>2</td>
<td>${2^{++}, (1, 2, 3)^{+-}}$</td>
<td>$\Sigma_u^-, \Pi_u$</td>
</tr>
</tbody>
</table>
Introducing the spin of the quark

Up to now we have worked at the leading order, now we want to include the correction coming from the quark spin.

We calculate the spin dependent potentials matching NRQCD and pNRQCD: we get a purely perturbative contribution in the form of spin dependent octet potential integrating out $m_v$ and then we get nonperturbative correlators depending only on glue when integrating out $\Lambda_{QCD}$.

The nonperturbative correlators should be calculated on the lattice or in QCD vacuum models.

We fix them on lattice data of charmonium—> we can then predict hybrids spin multiplet for bottomonium.

We obtain for the first time the spin dependent potentials for the hybrids.
In the short distance limit the static energies are characterized by $O(3) \times C$ instead of $D_\infty h$.

$G^i_{\kappa}^a$ create a basis of color-octet eigenstates of $h_0(R)$ in the presence of a static, local, color-octet source $O^a$.

$$h_0(R)G^i_{\kappa}^a(R)|0\rangle = \Lambda_\kappa G^i_{\kappa}^a(R)|0\rangle$$

The light d.o.f Hamiltonian density leading order in the multipole expansion.

$$h_0 = \frac{1}{2} \left(E^2 - B^2\right) - \sum_{j=1}^{n_f} \bar{q}_j iD \cdot \gamma q_j$$

States are constrained to satisfy the Gauss law.

At the pNRQCD level a basis of hybrid states is defined as

$$|\kappa, \lambda\rangle = P^i_{\kappa, \lambda} O^a \ (r, R) \ G_{\kappa}^i a(R)|0\rangle$$

The hybrid EFT is formulated for the subspace spanned by

$$\int d^3 r d^3 R \sum_{\kappa} |\kappa, \lambda\rangle \ \Psi_{\kappa, \lambda} (t, r, R)$$

$\Psi_{\kappa, \lambda}$ is the basic degree of freedom upon which we build the EFT.

$P^i_{\kappa, \lambda}$ projects $G_{\kappa}^i a$ into a representation of $D_\infty h$. 

Born-Oppenheimer EFT for hybrids

- After projecting and integrating out $\Lambda_{QCD}$:

$$\mathcal{L}_{BOEFT} = \int d^3r \sum_{\kappa} \sum_{\lambda' \lambda} \psi^\dagger_{\kappa \lambda}(t, r, R) \left\{ \delta_{\lambda' \lambda} i \partial_t - V_{\kappa \lambda \lambda'}(r) - P^i_{\kappa \lambda} \frac{\nabla_r^2}{m_Q} P^i_{\kappa \lambda'} \right\} \psi_{\kappa \lambda'}(t, r, R) + \ldots$$

The potential $V_{\kappa \lambda \lambda'}$ can be organized into an expansion in $1/m_Q$

$$V_{\kappa \lambda \lambda'}(r) = V^{(0)}_{\kappa \lambda}(r) \delta_{\lambda' \lambda} + \frac{V^{(1)}_{\kappa \lambda \lambda'}(r)}{m_Q} + \frac{V^{(2)}_{\kappa \lambda \lambda'}(r)}{m_Q^2} + \ldots$$

The static potential, $V^{(0)}_{\kappa \lambda}$, can be matched to the lattice NRQCD static energies and a short distance weak-coupling pNRQCD description:

$$E^{(0)}_{|\lambda|, \sigma}(r) = V_0(r) + \Lambda_{\kappa} + b_{\kappa \lambda} r^2 + \cdots = V^{(0)}_{\kappa \lambda}(r)$$

The nonadiabatic coupling mixes states which are different projections of the same light d.o.f operator.

$$P^i_{\kappa \lambda} \left[ \frac{\nabla_r^2}{m_Q}, P^i_{\kappa \lambda'} \right] = P^i_{\kappa \lambda} \frac{\nabla_r^2}{m_Q} P^i_{\kappa \lambda'} - \frac{\nabla_r^2}{m_Q}$$
Hybrids spin dependent potentials

\[ V_{1\lambda \lambda', SD}^{(1)}(r) = V_{1SK}(r) \left( P_{1\lambda}^{i\dagger} K_1^{ij} P_{1\lambda'}^j \right) \cdot S + \ldots, \]

\[ V_{1\lambda \lambda', SD}^{(2)}(r) = V_{1LSa}(r) \left( P_{1\lambda}^{i\dagger} L_{Q\bar{Q}}^i P_{1\lambda'}^j \right) \cdot S + V_{1LSb} P_{1\lambda}^{i\dagger} (r) \left( L_{Q\bar{Q}}^i S^j + S^i L_{Q\bar{Q}}^i \right) P_{1\lambda'}^j \]

\[ + V_{1S^2}(r) S^2 \delta_{\lambda \lambda'} + V_{1S_{12a}}(r) S_{12} \delta_{\lambda \lambda'} + V_{1S_{12b}}(r) P_{1\lambda}^{i\dagger} P_{1\lambda'}^j \left( S_1^i S_2^j + S_2^i S_1^j \right) + \ldots \]

where \( L_{Q\bar{Q}} \) is the orbital angular momentum of the heavy-quark-antiquark pair, \( S_1 \) and \( S_2 \) are the spin vectors of the heavy quark and heavy antiquark respectively, \( S = S_1 + S_2 \) and \( S_{12} = 12(S_1 \cdot \hat{r})(S_2 \cdot \hat{r}) - 4S_1 \cdot S_2 \).

\((K^{ij})^k = i \epsilon^{ijk}\) is the angular momentum operator for the spin-1 gluons.
Hybrids spin dependent potentials

\[ V_{1\lambda \lambda'}^{(1)}_{SD}(r) = V_{1SK}(r) \left( P_{1\lambda}^{i\dagger} K_1^{ij} P_{1\lambda'}^{j} \right) \cdot S + \ldots \, , \]

\[ V_{1\lambda \lambda'}^{(2)}_{SD}(r) = V_{1LSa}(r) \left( P_{1\lambda}^{i\dagger} L_{Q\bar{Q}} P_{1\lambda'}^{j} \right) \cdot S + V_{1LSb} P_{1\lambda}^{i\dagger}(r) \left( L_{Q\bar{Q}}^i S^j + S^i L_{Q\bar{Q}}^j \right) P_{1\lambda'}^{j} \]

\[ + V_{1S2}(r) S^2 \delta_{\lambda \lambda'} + V_{1S_{12a}}(r) S_{12} \delta_{\lambda \lambda'} + V_{1S_{12b}}(r) P_{1\lambda}^{i\dagger} P_{1\lambda'}^{j} \left( S_1^i S_2^j + S_2^i S_1^j \right) + \ldots \]

\[ \kappa = 1^{+-} \rightarrow \Lambda_{\eta}^{\sigma} = \Sigma_u^{-}, \Pi_u \]

\[ \rightarrow \text{Unlike standard quarkonium spin appear at } 1/m \]

where \( L_{Q\bar{Q}} \) is the orbital angular momentum of the heavy-quark-antiquark pair, \( S_1 \) and \( S_2 \) are the spin vectors of the heavy quark and heavy antiquark respectively, \( S = S_1 + S_2 \) and \( S_{12} = 12(S_1 \cdot \hat{r})(S_2 \cdot \hat{r}) - 4S_1 \cdot S_2 \).

\( (K^{ij})^k = i \epsilon^{ijk} \) is the angular momentum operator for the spin-1 gluons.
Hybrids spin dependent potentials

\[ V_{1\lambda\lambda'}^{(1)}_{SD}(r) = V_{1SK}(r) \left( P_{1\lambda}^i \mathbf{K}^{ij}_{1\lambda} P_{1\lambda'}^j \right) \cdot \mathbf{S} + \ldots , \]

\[ V_{1\lambda\lambda'}^{(2)}_{SD}(r) = V_{1LSa}(r) \left( P_{1\lambda}^i L_{Q\bar{Q}}^i P_{1\lambda'}^j \right) \cdot \mathbf{S} + V_{1LSb} P_{1\lambda}^i (r) \left( L_{Q\bar{Q}}^i S_j^i + S^i L_{Q\bar{Q}}^j \right) P_{1\lambda'}^j \]

\[ + V_{1S^2} (r) S^2 \delta_{\lambda\lambda'} + V_{1S_{12a}} (r) S_{12} \delta_{\lambda\lambda'} + V_{1S_{12b}} (r) P_{1\lambda}^i P_{1\lambda'}^j \left( S^i S^j_1 + S^j_2 S^i_1 \right) + \ldots \]

\[ \kappa = 1^{+-} \rightarrow \Lambda_{\eta}^\sigma = \sum_u \Pi_u \]

\[ \kappa = 1^{+-} \rightarrow \Lambda_{\eta}^\sigma = \sum_u \Pi_u \]

—>Unlike standard quarkonium spin appear at 1/m
—>These are new operators not present in standard quarkonium

where \( L_{Q\bar{Q}} \) is the orbital angular momentum of the heavy-quark-antiquark pair, \( S_1 \) and \( S_2 \) are the spin vectors of the heavy quark and heavy antiquark respectively, \( S = S_1 + S_2 \) and \( S_{12} = 12(S_1 \cdot \hat{r})(S_2 \cdot \hat{r}) - 4S_1 \cdot S_2 \).

\( (K^{ij})^k = i\epsilon^{ijk} \) is the angular momentum operator for the spin-1 gluons.
Hybrids spin dependent potentials

- In the short distance we can use weakly coupled pNRQCD to calculate $V_0$, it is given by the QQbar octet potential.

- The $V^{np}$ depend on non perturbative gluon correlators not yet calculated on the lattice: 6 unknown.

- The only flavour dependence is carried by the NRQCD matching coefficients.
Charmonium Hybrids Multiplets H\textsubscript{1}

lattice data from

with a pion of about 240 MeV

Power counting: we include terms up to order $\Lambda^3/m^2$ and $m v^4$ to the spin splittings.
height of the boxes is an estimate of the uncertainty: estimated by the parametric size of higher order corrections, $m \alpha_s^5$ or the perturbative part, powers of $\Lambda_{\text{QCD}}/m$ for the nonperturbative part, plus the statistical error on the fit
the perturbative part produces a pattern opposite to the lattice and to ordinary quarkonia \( \rightarrow \) discrepancy can be reconciled thanks to the nonperturbative parts, especially the one at order \( 1/m \) which goes like \( \Lambda^2/m \) and is parametrically larger than the perturbative contribution that i border \( m v^4 \)
Charmonium Hybrids  Multiplets $H_1$ and $H_2$

$H_1$ and $H_2$ corresponds to $l=1$ and are negative and positive parity resp. The mass splitting between $H_1$ and $H_2$ is a result of lambda-doubling.

$H_3$ and $H_4$ are also calculated.
Since the nonperturbative correlators are not depending on flavour we can use the result of the fits on charmonium to predict the hybrid bottomonium spin multiplets.
Prediction for the Bottomonium Hybrids Multiplets H₁, H₂

For bottomonium hybrids lattice data are scarce and not at the level of charmonium hybrids.

- **Spin structure of heavy-quark hybrids**
Tetraquarks

- Light-quark operators for tetraquarks \( H = H^a T^a (q = (u, d), \tau^i \) isospin Pauli matrices)

\[
\Lambda_{\eta}^{\sigma} | K^{PC} | H^a (l = 0, l = 1) \\
\begin{array}{c|c|c}
\Sigma_g^+ & 0^{++} & \bar{q} T^a (1, \tau) q \\
\Sigma_u^+ & 0^{+-} & \bar{q} \gamma^0 T^a (1, \tau) q \\
\Sigma_u^- & 0^{-+} & \bar{q} \gamma^5 T^a (1, \tau) q \\
\Sigma_g^- & 1^{--} & \bar{q} (\hat{r} \cdot \gamma) T^a (1, \tau) q \\
\Pi_g & 1^{--} & \bar{q} (\hat{r} \times \gamma) T^a (1, \tau) q \\
\Sigma_g^+ & 1^{++} & \bar{q} (\hat{r} \cdot \gamma) \gamma^5 T^a (1, \tau) q \\
\Pi_g & 1^{++} & \bar{q} (\hat{r} \times \gamma) \gamma^5 T^a (1, \tau) q \\
\end{array}
\]

\( \kappa = \{ J^{PC}, f \} \), isospin

- The lowest masses adjoint light-quark operators have been studied in quenched lattice QCD M. Foster and C. Michael Phys.Rev. D59 (1999)
- No lattice determination of the static energies.

Light-quark operators masses

<table>
<thead>
<tr>
<th>[MeV]</th>
<th>( K^{PC} )</th>
<th>( q\bar{q} (l = 0) )</th>
<th>( s\bar{s} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Lambda_{\kappa} - \Lambda_{1++} )</td>
<td>1^{--}</td>
<td>47 ± 90</td>
<td>120 ± 70</td>
</tr>
<tr>
<td></td>
<td>0^{-+}</td>
<td>91 ± 216</td>
<td>170 ± 99</td>
</tr>
</tbody>
</table>

Tetraquark static energies

- Short distance: \( V_0 + \Lambda_{\kappa} + \ldots \)
- Long distance: Unknown, String picture suggests \( \sigma r + \text{const.} \)


<table>
<thead>
<tr>
<th>( K^{PC} )</th>
<th>( \Lambda_{\eta}^{\sigma} )</th>
<th>( l )</th>
<th>( S = 0 )</th>
<th>( S = 1 )</th>
<th>isospin-1</th>
<th>isospin-0</th>
<th>( s\bar{s} )</th>
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<tbody>
<tr>
<td>1^{--} | ( \Sigma_g^+ - \Pi_g )</td>
<td>1 1^{--} (0, 1, 2)^{++} | ( Z_c^+(4023) )</td>
<td>( X(3918) )</td>
<td>( Y(4145) )</td>
<td>4100</td>
<td>4327</td>
<td>4205</td>
<td></td>
</tr>
<tr>
<td>2 2^{--} (1, 2, 3)^{++}</td>
<td>| | 4312</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>\Pi_g</td>
<td>1 1^{--} (0, 1, 2)^{--} | ( Z_c^+(3898) )</td>
<td>( 3793 )</td>
<td>( 4020 )</td>
<td>4080</td>
<td>3975</td>
<td>4201</td>
<td></td>
</tr>
<tr>
<td>2 2^{--} (1, 2, 3)^{++}</td>
<td>| | 4312</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Sigma_g^+ )</td>
<td>0 0^{++} 1^{-} (4368)</td>
<td>( Y(4263) )</td>
<td>( 4490 )</td>
<td>| | 4205</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Sigma_u^- )</td>
<td>0 0^{++} 1^{-} (4368)</td>
<td>( Y(4263) )</td>
<td>( 4490 )</td>
<td>| | 4205</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Sigma_u^- )</td>
<td>1 1^{--} (0, 1, 2)^{--}</td>
<td>| | 4205</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Conclusions

Quarkonium is a golden system to study strong interactions
For states below threshold non relativistic EFTs provide a systematic tool
to investigate a wide range of observables in the realm of QCD and
quarkonium becomes a

NREFT Allow us to make calculations with unprecented precision, where
high order perturbative calculations are possible
and to systematically factorize short from long range contributions where
observables are sentitive to the nonperturbative dynamics of QCD

For states close or above the strong decay threshold the situation is
much more complicated.

Many degrees of freedom show up and the absence of a clear
systematic is an obstacle to a universal picture

We have presented results obtained for the hybrid masses
in pNRQCD that show a very rich structure of multiplets.
Conclusions

- study of hybrids in EFT framework (NRQCD and pNRQCD)
- study of the spectrum of the static Hamiltonian with non-static corrections
- short distance degeneracy of static energies
- mixing of different static states
- breaking of degeneracy between opposite parity states

We have included spin in the hybrids multiplet structure:
—could interpret the lattice result
—make independent predictions for the bottomonium sector
—no spin symmetry suppression

Same approach can be used to include light quarks: “tetraquarks”
This approach holds the promise to be able to explain all exotics (including pentaquark) from QCD in the same framework

Input from the lattice is needed: more precise calculations of the gluelump masses, static energies for the hybrids and the tetra quarks, correlators of gluons fields..

Exotics may be generated also by QCD van der Waals forces: for example eta b-eta b bound states?