The electroweak effective field theory the on-shell way

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Introduction: EFT amplitudes without $\mathcal{L}$

$\mathcal{L}_{\text{EFT}}$ follows from **bottom-up** construction using

- SM **field** content
- Lorentz symmetry
- Global symmetry
- **Gauge**

An on-shell construction is natural:

$\mathcal{M}_{\text{EFT}}$ too is **bottom-up** starting from

- SM **particle** content
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- **Unitarity + locality**

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Use what we **know** to parametrize what we **don't know**!

Introduction: EFT amplitudes without $\mathcal{L}$

Amplitudes are great for simplifying computations!

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Amplitudes are great for simplifying computations!

$\mathcal{L}_{\text{EFT}}$ has many fields + derivatives $\Longrightarrow$ could be simplified by on-shell methods!

On-shell methods were used to:

- derive selection rules for (non) interference of EFT amplitudes and the SM (tree)

  Azatov, Contino, Machado and Riva

- explian non-trivial zeros in the anomalous dimension matrix in SMEFT (loop)

  Cheung, Kampf, Novotny, Shen and Trunka Bern, Sawyer and Parra-Martinez

Work only with physical particles $\Longrightarrow$ no redundancies, no field redefinitions

Shadmi, YW Durieux, Kitahara, Shadmi, YW

Ma, Shu and Xiao Aoude and Machado

Jiang, Shu, Xiao and Zheng Durieux and Machado

Falkowski Bachu and Yelleshpur

Herderschee, Koren and Trott Christensen, Field, Moore and Pinto

...

Introduction: EFT amplitudes without $\mathcal{L}$

Use on-shell methods to derive EFT amplitudes without $\mathcal{L}$!

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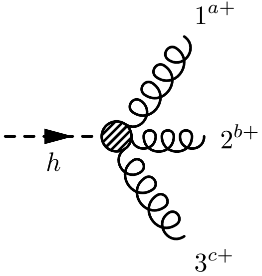
Use on-shell methods to derive EFT amplitudes without $\mathcal{L}$!

Example: $hggg$ (all incoming!):



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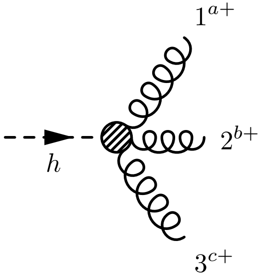

$$\frac{[12]\,[13]\,[23]}{\Lambda}\Bigg[f^{abc}\Bigg(-\,i\,\frac{m^4\,g_s\,c_5^{hgg}}{s_{12}s_{13}s_{23}}+\frac{c_7}{\Lambda^2}$$
$$+\frac{c_{11}}{\Lambda^6}(s_{12}s_{23}+s_{13}s_{23}+s_{12}s_{13})$$
$$+\frac{c_{13}}{\Lambda^8}s_{12}s_{13}s_{23}+\dots\Bigg)$$
$$+\,d^{abc}(\dots)\Bigg]$$

c's are analogs of Wilson coefficients!

Introduction: EFT amplitudes without $\mathcal{L}$

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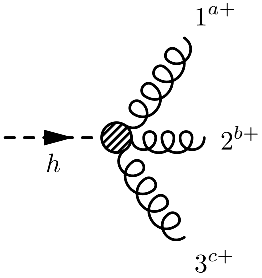

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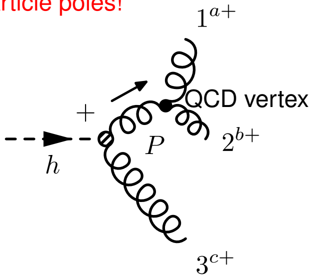
This very simple expression captures the EFT contributions up to dimension 13!

Introduction: EFT amplitudes without $\mathcal{L}$

Factorizable part $\Longrightarrow$ one-particle poles!



$$\frac{[12]\,[13]\,[23]}{\Lambda}\left[f^{abc}\left(-\,i\,\frac{m^4\,g_s\,c_5^{hgg}}{s_{12}s_{13}s_{23}}+\frac{c_7}{\Lambda^2}\right.\right.$$
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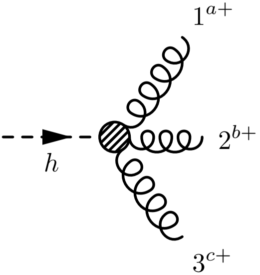
We determined it without a $\mathcal{L}$! $\Longrightarrow$ ***No redundancies associated with field redefinitions, EOM, IBP...***

Introduction: EFT amplitudes without $\mathcal{L}$

Contact terms $\Longrightarrow$ no poles!



$$\frac{[12]\,[13]\,[23]}{\Lambda}\Bigg[f^{abc}\Bigg(-\,i\,\frac{m^4\,g_s\,c_5^{hgg}}{s_{12}s_{13}s_{23}}+\frac{c_7}{\Lambda^2}$$
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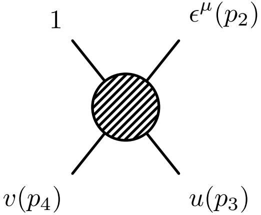
Little group scaling, factorization and Bose statistics (almost: c's) fix the amplitude

How to Bootstrap amplitudes

Backtrack to explain how to bootstrap amplitudes
Generically any amplitude can be written

$$\mathcal{M} = \underbrace{\text{External polarizations}}_{\text{massless/massive little group}} \times \underbrace{\text{Mandelstam variables, couplings}}_{\text{Functions of momenta, no little group!}}$$



The strategy is:

- Little group+spin statistics $\Longrightarrow$ 3-pt, contact terms
- Glue the 3-pt functions $\Longrightarrow$ higher point +contact

Massive amplitudes in the SM broken phase EFT

We assume

- a SM-like EW spectrum: $Z, W^{\pm}, \psi, \psi^c, h$
- electric charge conservation
- fermion number conservation

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We derive

- The necessary ingredients for SMEFT calculations (in particular all 3-pt's)
- How aspects of $\mathrm{SU}(2) \times \mathrm{U}(1)$/Higgs emerge from the massive amplitudes

perturbative unitarity/good massless limit

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perturbative unitarity/good massless limit

What we'll see in this talk

- the massive gauge boson 3-point function $\implies$ $\mathrm{SU}(2)$ structure
- the $\psi^c \psi Z$ 3-point function
- the $\psi^c \psi Z h$ 4-point function

Three **degenerate** gauge bosons

We can see $\mathrm{SU}(2)$ emerging already the three point function!

Three **degenerate** gauge bosons

- Consider the three-point function of massive **degenerate** gauge bosons, W^a with $a = 1, \ldots, n$

$$\mathcal{M}(\mathbf{1}_W^a, \mathbf{2}_W^b, \mathbf{3}_W^c) = \langle \mathbf{12} \rangle \langle \mathbf{13} \rangle \langle \mathbf{23} \rangle \; C_6^{abc}/\bar{\Lambda}^2$$

$$+ \Big(\langle \mathbf{12} \rangle \langle \mathbf{13} \rangle [\mathbf{23}] + \langle \mathbf{12} \rangle [\mathbf{13}] \langle \mathbf{23} \rangle + [\mathbf{12}] \langle \mathbf{13} \rangle \langle \mathbf{23} \rangle \Big) \, C^{abc}/m_W^2$$
$$+ \langle \ldots \rangle \to [\ldots]$$

- Each spinor structure is **totally** antisymmetric under any exchange!
- Bose symmetry $\implies$ C's are totally antisymmetric
- For $n = 3$: all the $C^{abc} = \epsilon^{abc} \implies \mathrm{SU}(2)$ structure constants emerge already at the level of the three-point function
- (In some ways this is simpler than the massless case)
- (Jacobi identity from 4-pt function)

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$$+ \Big(\langle \mathbf{12} \rangle \langle \mathbf{13} \rangle [\mathbf{23}] + \langle \mathbf{1} \ \ldots \ \langle \mathbf{23} \rangle + [\mathbf{12}] \langle \mathbf{13} \rangle \langle \mathbf{23} \rangle \Big) C^{abc} / m_W^2$$

$$+ \langle \ldots \rangle \to [\ldots]$$

SU(2) from little group + Bose!

- Each spinor structure … **ally** antisymmetric under any exchange!
- Bose symmetry … C's are totally antisymmetric
- For $n = 3$: all … $C^{abc} = \epsilon^{abc} \implies \mathrm{SU}(2)$ structure constants emerge already at the level of the three-point function
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Massive amplitudes in the broken phase EFT

Can we see relations between couplings (*e.g.* Higgs mechanism) from the on-shell approach?

Massive amplitudes in the broken phase EFT

$$\mathcal{M}(\mathbf{1}_{\psi^c}, \mathbf{2}_{\psi}, \mathbf{3}_Z) =$$
$$\frac{c^{RRR}_{\psi^c \psi Z}}{\bar{\Lambda}} [\mathbf{13}] [\mathbf{23}] + \frac{c^{LLL}_{\psi^c \psi Z}}{\bar{\Lambda}} \langle \mathbf{13} \rangle \langle \mathbf{23} \rangle + \frac{c^{LR0}_{\psi^c \psi Z}}{m_Z} \langle \mathbf{13} \rangle [\mathbf{23}] + \frac{c^{RL0}_{\psi^c \psi Z}}{m_Z} [\mathbf{13}] \langle \mathbf{23} \rangle$$

- Very simple expression includes *all* renormalizable and non-renormalizable contributions
- The c's are functions of the mass ratios, *e.g.*, m_ψ / m_Z , $m_\psi / \bar{\Lambda}$ etc
- Contains **all** spin information of the external particles

Massive amplitudes in the broken phase EFT

$$\mathcal{M}(\mathbf{1}_{\psi^c}^{I_1}, \mathbf{2}_{\psi}^{J_1}, \mathbf{3}_Z^{K_1,K_2}) \ni \frac{c_{\psi^c \psi Z}^{RL0}}{m_Z} [\mathbf{13}] \langle \mathbf{23} \rangle$$

Bolded spinors carry massive little group indices

- For each external massive spin s particle we have $2s$ bolded spinors

Massive amplitudes in the broken phase EFT

$$\mathcal{M}(\mathbf{1}_{\psi^c}^{I_1}, \mathbf{2}_{\psi}^{J_1}, \mathbf{3}_Z^{K_1,K_2}) \ni \frac{c^{RL0}_{\psi^c\psi Z}}{m_Z} [\mathbf{13}] \langle \mathbf{23} \rangle \quad \left[1^{I_1} 3^{\{K_1}\right] \left\langle 2^{J_1} 3^{K_2\}} \right\rangle .$$

Fermion $\implies$ one index $I_1 = 1, 2$ $\implies$ one “half bracket”

Bolded spinors carry massive little group indices

- For each external massive spin s particle we have $2s$ bolded spinors

Massive amplitudes in the broken phase EFT

$$\mathcal{M}(\mathbf{1}_{\psi^c}^{I_1}, \mathbf{2}_{\psi}^{J_1}, \mathbf{3}_{Z}^{K_1,K_2}) \ni \frac{c_{\psi^c\psi Z}^{RL0}}{m_Z} [\mathbf{13}] \langle \mathbf{13} \rangle \quad \left[1^{I_1} 3^{\{K_1}\right] \left\langle 2^{J_1} 3^{K_2\}} \right\rangle$$

Gauge boson $\implies$ two indices $K_1, K_2 = 1, 2$ $\implies$ two "half brackets"

Bolded spinors carry massive little group indices Arkani-Hamed, Huang and Huang

- For each external massive spin s particle we have $2s$ bolded spinors
- Repeated indices are totally symmetrized, *e.g.*, the K's
- This formalism can handle any massive spinning particle

Massive amplitudes in the broken phase EFT

Take massless limit:

$$\mathcal{M}(\mathbf{1}_{\psi^c}^{I_1=2}, \mathbf{2}_{\psi}^{J_1=1}, \mathbf{3}_{Z}^{K_1,K_2=1}) =$$

$$\frac{c^{RRR}_{\psi^c\psi Z}}{\bar{\Lambda}}[\mathbf{13}][\mathbf{23}] + \frac{c^{LLL}_{\psi^c\psi Z}}{\bar{\Lambda}}\langle\mathbf{13}\rangle\langle\mathbf{23}\rangle + \frac{c^{LR0}_{\psi^c\psi Z}}{m_Z}\langle\mathbf{13}\rangle[\mathbf{23}] + \frac{c^{RL0}_{\psi^c\psi Z}}{m_Z}[\mathbf{13}]\langle\mathbf{23}\rangle$$

$$\sim \mathcal{O}\left(\frac{m^2}{\bar{\Lambda}}\right) \qquad \sim \mathcal{O}\left(\frac{m^2}{\bar{\Lambda}}\right) \qquad \sim \mathcal{O}(1) \qquad \sim \mathcal{O}(m)$$

$$\mathcal{M}(1^-_{\psi^c}, 2^+_{\psi}, 3^+_Z) = +\frac{[23]^2}{[12]}g_L + \mathcal{O}\left(\frac{m}{\bar{\Lambda}}\right)$$

keep only constant term in $c^{LR0}_{\psi^c\psi Z} \to g_L + \mathcal{O}(m/\bar{\Lambda})$

Massive amplitudes in the broken phase EFT

Take massless limit:

$$\mathcal{M}(\mathbf{1}_{\psi^c}^{I_1=1}, \mathbf{2}_{\psi}^{J_1=2}, \mathbf{3}_{Z}^{K_1,K_2=1}) =$$

$$\frac{c_{\psi^c\psi Z}^{RRR}}{\bar{\Lambda}} [\mathbf{13}][\mathbf{23}] + \frac{c_{\psi^c\psi Z}^{LLL}}{\bar{\Lambda}} \langle\mathbf{13}\rangle\langle\mathbf{23}\rangle + \frac{c_{\psi^c\psi Z}^{LR0}}{m_Z} \langle\mathbf{13}\rangle[\mathbf{23}] + \frac{c_{\psi^c\psi Z}^{RL0}}{m_Z} [\mathbf{13}]\langle\mathbf{23}\rangle$$

$\sim \mathcal{O}\left(\frac{m^2}{\bar{\Lambda}}\right)$ $\sim \mathcal{O}\left(\frac{m^2}{\bar{\Lambda}}\right)$ $\sim \mathcal{O}(m)$ $\sim \mathcal{O}(1)$

$$\mathcal{M}(1_{\psi^c}^{+}, 2_{\psi}^{-}, 3_{Z}^{+}) = -\frac{[13]^2}{[12]} g_R + \mathcal{O}\left(\frac{m}{\bar{\Lambda}}\right)$$

keep only constant term in $c_{\psi^c\psi Z}^{RL0} \to g_R + \mathcal{O}(m/\bar{\Lambda})$

Massive amplitudes in the broken phase EFT

Take massless limit:

$$\mathcal{M}(\mathbf{1}_{\psi^c}^{I_1=1}, \mathbf{2}_{\psi}^{J_1=1}, \mathbf{3}_{Z}^{K_1,K_2=\{1,2\}}) =$$

$$\frac{c_{\psi^c\psi Z}^{RRR}}{\bar{\Lambda}} [\mathbf{13}][\mathbf{23}] + \frac{c_{\psi^c\psi Z}^{LLL}}{\bar{\Lambda}} \langle\mathbf{13}\rangle\langle\mathbf{23}\rangle + \frac{c_{\psi^c\psi Z}^{LR0}}{m_Z} \langle\mathbf{13}\rangle[\mathbf{23}] + \frac{c_{\psi^c\psi Z}^{RL0}}{m_Z} [\mathbf{13}]\langle\mathbf{23}\rangle$$

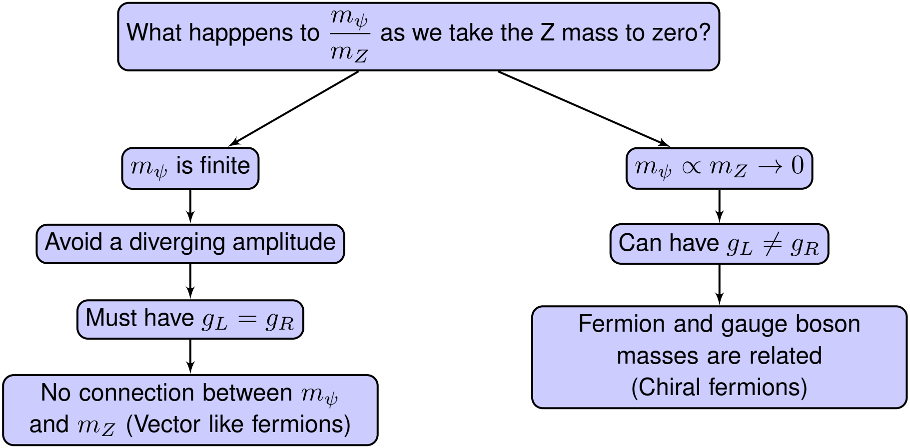
$$\sim \mathcal{O}\left(\frac{m^2}{\bar{\Lambda}}\right) \qquad \sim \mathcal{O}\left(\frac{m^2}{\bar{\Lambda}}\right) \qquad \sim \mathcal{O}\left(\frac{m_\psi}{m_Z}\right) \qquad \sim \mathcal{O}\left(\frac{m_\psi}{m_Z}\right)$$

$$\mathcal{M}(1_{\psi^c}^+, 2_{\psi}^+, 3_Z^0) = -[12]\ (g_L - g_R)\ m_\psi/\sqrt{2}m_Z + \mathcal{O}\left(\frac{m}{\bar{\Lambda}}\right)$$

Massive amplitudes in the broken phase EFT

From the longitudinal mode we obtain

$$\mathcal{M}(1^+_{\psi^c}, 2^+_{\psi}, 3^0_Z) = -\,[12]\ (g_L - g_R)\ \frac{m_\psi}{\sqrt{2}m_Z}$$



Massive amplitudes in the broken phase EFT

From the longitudinal mode we obtain

$$\mathcal{M}(1^+_{\psi^c}, 2^+_\psi, 3^0_Z) = -\,[12]\;(g_L - g_R)\;\frac{m \ldots}{\ldots Z}$$

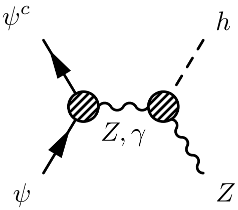
What happpens to $\frac{m_\psi}{m_Z}$ as we take the Z ... to zero?

- m_ψ is finite
  - Avoid a diverging amplitude
  - Must have $g_L = \ldots$
  - No connection bet...en m_ψ and m_Z (Vector like fermions)
- $m_\psi \propto m_Z \to 0$
  - Can have $g_L \neq g_R$
  - Fermion and gauge boson masses are related (Chiral fermions)

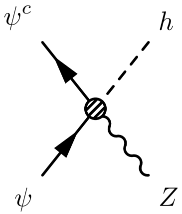
Relation between m_ψ and m_Z without $\mathcal{L}$!

The $\psi^c \psi Z h$ four point

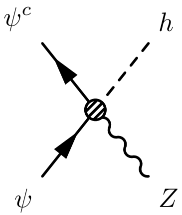
Factorizable piece with *single particle* poles



Contact terms *without* poles



The $\psi^c \psi Z h$ four point



Little group + no poles $\implies$ contact terms:

$$\mathcal{M}^{\text{nf}}(\mathbf{1}_{\psi^c}, \mathbf{2}_{\psi}, \mathbf{3}_Z, \mathbf{4}_h) =$$

$$\frac{c^{RRR}_{\psi^c\psi Zh}}{\bar{\Lambda}^2}\,[\mathbf{13}]\,[\mathbf{23}] + \frac{[\mathbf{12}]}{\bar{\Lambda}^3}\langle\mathbf{3}\ \{c^{RR0_A}_{\psi^c\psi Zh}(\mathbf{1}+\mathbf{2}) + c^{RR0_S}_{\psi^c\psi Zh}(\mathbf{1}-\mathbf{2})\}\ \mathbf{3}]$$

$$+\frac{c^{RL0}_{\psi^c\psi Zh}}{\bar{\Lambda}^2}\,[\mathbf{13}]\,\langle\mathbf{23}\rangle + \frac{c^{RLR}_{\psi^c\psi Zh}}{\bar{\Lambda}^3}[\mathbf{312}\rangle\,[\mathbf{13}] + \frac{c^{LRR}_{\psi^c\psi Zh}}{\bar{\Lambda}^3}[\mathbf{321}\rangle\,[\mathbf{23}] + ([..] \to \langle..\rangle)\,,$$

the c's: positive powers of s_{ij}'s.

Most general form of the amplitude!

The $\psi^c\psi Zh$ four point

Back to the factorizable part!

The $\psi^c \psi Z h$ four point

Perturbative unitarity $\implies \mathcal{O}(E/m) = 0 \implies$ relations between coefficients!

Avoiding energy growth in the $(\pm \pm 00)$ modes we obtain the renormalizable level conditions

$$(g_L - g_R) \; (c^{00}_{ZZh} \, m_\psi / 2 m_Z - c^{RR}_{\psi^c \psi h}) = 0 + \mathcal{O}(m/\bar{\Lambda}) \, .$$

The $\psi^c \psi Z h$ four point

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Gauge!

Yukawa!

$$(g_L - g_R)\;\left(c^{00}_{ZZh}\, m_\psi / 2m_Z - c^{RR}_{\psi^c \psi h}\right) = 0 + \mathcal{O}(m/\bar{\Lambda})\,.$$

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Vector-like fermions

No relation between the Yukawa and fermion mass

Chiral fermions $\implies g_L \neq g_R$

Fermion mass related to the Yukawa $c^{00}_{ZZh}\, m_\psi/2m_Z = c^{RR}_{\psi^c\psi h}$

The $\psi^c \psi Zh$ four point

Perturbative unitarity $\implies \mathcal{O}(E/m) = 0 \implies$ relation … een coefficients!

Avoiding energy growth in the $(\pm\pm 00)$ modes … in the renormalizable level conditions

Gauge!

…kawa!

$$(g_L - g_R) \quad (c^{00}_{ZZ\ldots} \ldots 2m_Z - c^{RR}_{\psi^c\psi h}) = 0 + \mathcal{O}(m/\bar{\Lambda})\,.$$

Vector-like ferm…

No relati… ween the … va and f… on mass

Chiral fermions $\implies g_L \neq g_R$

Fermion mass related to the Yukawa $c^{00}_{ZZh}\, m_\psi / 2m_Z = c^{RR}_{\psi^c\psi h}$

Relation between m_ψ and Yukawa without $\mathcal{L}$!

Conclusions

- We derive the necessary ingredients for on-shell SMEFT analysis:
  - all the three-point amplitudes: both massive and massless
  - one example of fully massive four point
  - matching to the SMEFT Lagrangian
- We adopted massive on-shell amplitude methods to examine a theory of the electroweak spectrum from the bottom up, including both renormalizable and non-renormalizable interactions
- We demonstrated the emergence of patterns due to electroweak symmetry breaking, starting from a purely bottom-up approach, **without ever using a Lagrangian**
- In the on-shell approach, we never encounter the operator or gauge redundancies that are associated with Lagrangians.

Thanks for listening!

Backup slides

Future outlook

- Consider higher point functions to find relations between different non-renormalizable couplings
- These methods can be extended to include loops, where the effects of operator mixing and RGE would arise.

The little group and amplitudes

Scattering amplitudes transform directly under the little group (LG) of each external leg.

- For a *massless* particle the LG is $\mathrm{SO}(2) = \mathrm{U}(1)$.
- For a *massive* particle the LG is $\mathrm{SO}(3) = \mathrm{SU}(2)$.

If we had variables that transform directly under the LG we could (begin to) bootstrap amplitudes!

These are the spinor helicity variables.

The little group and amplitudes

Scattering amplitudes transform directly under the little group (LG) of each external leg.

- For a *massless* particle the LG is $\mathrm{SO}(2) = \mathrm{U}(1)$.
- For a *massive* particle the LG is $\mathrm{SO}(3) = \mathrm{SU}(2)$.
- A lightlike momentum ($p^2 = 0$) can be decomposed into a pair of **helicity spinors**

$$\not{p} \equiv p_\mu \sigma^\mu_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$$
$$\text{LG: } \lambda \to \xi\lambda \,, \tilde{\lambda} \to \xi^{-1}\tilde{\lambda} \,, \xi = e^{i\varphi} \,,$$

while a timelike momentum ($p^2 = m^2$) can be decomposed into a *two* pairs of **helicity spinors**

$$\not{p} \equiv p_\mu \sigma^\mu_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} + \eta_\alpha \tilde{\eta}_{\dot{\alpha}} \equiv \lambda^I_\alpha \tilde{\lambda}_{I,\dot{\alpha}}$$
$$\text{LG: } \lambda^I \to W^I_J \lambda^J \,, \tilde{\lambda}_I \to \left(W^{-1}\right)^J_I \tilde{\lambda}_J \,,$$

where $I = 1, 2$ are $\mathrm{SU}(2)$ indices.

Invariants from spinors

- There are two types of invariants, for two lightlike momenta $p_i = \lambda\tilde{\lambda}\,, p_j = \chi\tilde{\chi}$

$$[\lambda\chi] = e^{-i\phi_{ij}}\sqrt{s_{ij}} = \bar{u}_-(p_i)u_+(p_j) = \tilde{\lambda}\tilde{\chi}\,,$$
$$\langle\lambda\chi\rangle = e^{i\phi_{ij}}\sqrt{s_{ij}} = \bar{u}_+(p_i)u_-(p_j) = \lambda\chi\,,$$

  where $s_{ij} = (p_i + p_j)^2 = [ij]\,\langle ji\rangle$.
- The brackets are **antisymmetric** under exchange.
- Each massless, helicity h particle, induces a phase ξ^{-2h} in the amplitude.
- Each massive spin s particle, introduces $2s$ totally symmetrized $\mathrm{SU}(2)$ indices.

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$$[\lambda\chi] = e^{-i\phi_{ij}}\sqrt{s_{ij}} = \bar{u}_-(p_i)u_+(p_j) = \tilde{\lambda}\tilde{\chi}\,,$$
$$\langle\lambda\chi\rangle = e^{i\phi_{ij}}\sqrt{s_{ij}} = \bar{u}_+(p_i)u_-(p_j) = \lambda\chi\,,$$

  where $s_{ij} = (p_i + p_j)^2 = [ij]\,\langle ji\rangle$.
- The brackets are **antisymmetric** under exchange.
- Each massless, helicity h particle, induces a phase ξ^{-2h} in the amplitude.
- Each massive spin s particle, introduces $2s$ totally symmetrized $\mathrm{SU}(2)$ indices.

Let's assume we a four point function of two massless fermions, a massive Z and h, we'd write

$$\mathcal{M}(1_{\psi^c}, 2_\psi, 3_Z, 4_h) \to \mathcal{M}\left(1_{\psi^c}^{h_1}, 2_\psi^{h_2}, 3_Z^{I_1,I_2}, 4_h\right),$$

where $h_{1,2} = \pm 1/2$, and I_1, I_2 are massive LG indices.

Polarization vectors

For a massless spin one particle we can write the two polarizations (with Lorentz indices suppressed)

$$\epsilon_p^-(r) = \sqrt{2}\frac{\lambda_p \tilde{\lambda}_r}{[pr]}\,, \quad \epsilon_p^+(r) = \sqrt{2}\frac{\lambda_r \tilde{\lambda}_p}{\langle rp\rangle}\,,$$

where r^μ is a lightlike reference momentum, which we are free to choose as long as $p \cdot r \neq 0$ (gauge invariance!). If we perform a little group transformation for $r\,,p$ we'll get

$$\epsilon_p^-(r) \xrightarrow{\text{LG}} \epsilon_p^-(r)\xi_p^2\,, \quad \epsilon_p^+(r) \xrightarrow{\text{LG}} \epsilon_p^+(r)\xi_p^{-2}\,,$$

which means that the polarizations are **not** invariant under the massless little group.