The Non-Conservative Maxwell's Equations and the Electrodynamics of Time Varying Voltage Sources using Impressed Electric Fields and Magnetic Currents

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Consideration is given to the electrodynamics of alternating current voltage sources with either free or bound charge. For the free charge ideal voltage source, it is necessary to included an impressed non-conservative electric field vector, which causes the force essential to separate and move the free charge in a lossless way. This is equivalent to the force per unit charge converted from an external energy source that drives the electromotive force. For the bound charge voltage source, we show that it is similar to the idealized bar electret, which is polarized permanently and uniformly parallel to its cylindrical z-axis. Usually a bar electret is charged with a permanent DC polarization, however in this work we consider the more general case of a time dependent polarization, which is shown to be equivalent to an impressed time dependent electric field. We also show that the bound charge voltage source has a capacitive source impedance, which can in principle reduce the output voltage depending on the source load. For both cases Faraday's law is modified through the addition of an effective impressed magnetic current boundary source. The impressed electric field can be shown to be related to the effective magnetic current via the left hand rule, and is also described by an electric vector potential. For both cases, in their no-load state, they are shown to be equivalent to an idealized Hertzian dipole.

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INTRODUCTION

The conservation of energy is a fundamental law of physics, but only applies to an isolated system. The law means that energy cannot be created or destroyed, but only transferred from one form to another. For example, the creation of photons or electricity can be achieved, however the energy must come from another source, such as a nuclear reaction where mass is converted into other forms of energy through $E = mc^2$. If the other form of energy is electricity, then we can call this device a nuclear battery, such devices use the energy from the decay of a radioactive isotope to generate electricity and electromagnetic fields and can produces large DC electric fields and voltages of up to 10-100 kV [1]. The modern form of the nuclear battery is a micro-electromechanical system or MEMS device [2, 3], which can also be configured as an AC battery capable of generating radio frequencies of $60 - 260 MHz$ [4]. Other types of batteries and voltage sources include chemical batteries and permanently polarized electrets. These too can be turned into AC voltage sources, for example a DC electret is typically coupled to an acoustic diaphragm and implemented as a microphone/speaker, while DC chemical driven batteries can be configured as AC sources by incorporating a built in inverter to provide AC power.

From the view point of the electromagnetic environment, the creation of electromagnetic energy is a non-conservative process. Thus, when we consider the electrodynamics in isolation to the whole system, the standard Maxwell's equations must be made more general to take into account the non-conservative processes. The standard Maxwell's equation in differential form and in vacuum are given by (in SI units),

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_f}{\epsilon_0}, \tag{1}$$

$$\vec{\nabla} \times \vec{B} - \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}_f, \tag{2}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{3}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0. \tag{4}$$

Here $\vec{E}$ is the electric field intensity, $\vec{B}$ is the magnetic flux density, $\vec{J}_f$ is the electric current density, ρ_f is the electric charge density and ϵ_0 and μ_0 are the permittivity and permeability of free space. This representation has no source terms for the fields and in a source free medium there are no currents or charges adding energy to the system. Thus, the free currents and charges in the above equations either propagate without loss due to the interaction with the electromagnetic fields, or can describe a dissipative (or resistive) system where electromagnetic energy is lost, usually by conversion to heat (in this case the electric and magnetic field phasors can become complex).

The non-electric energy sources (for example, nuclear energy as discussed previously) capable of transmitting energy and hence a force to electric charges are commonly referred to as 'impressed' sources of the field. Theoretically, they can be represented either as an ideal current or voltage generator of electronic network theory [5]. Thus in electrodynamics, an impressed electric current will be added as a current source and hence will modify Ampere's law, while an impressed electric voltage will be added as a magnetic current source to modify Faraday's law. The later does not mean that magnetic monopole particles exists, but is a consistent way to model boundary value problems when considering the electrodynamics of a non-conservative voltage source [6–8]. This technique is more generally known as the Compensation Theorem [5, 9]. We also note the impressed sources are not influenced by Maxwell's equations because they represent creation of electromagnetic energy from an external source.

MAXWELL'S EQUATIONS FOR NON-CONSERVATIVE ELECTRODYNAMICS

Differential Form

Assuming that all the electromagnetic field sources in a region are reduced to impressed sources in vacuum (compensation theorem), then using the Weber convention the differential form of the non-conservative Maxwell's equations in vacuum may be written as,

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_e}{\epsilon_0}, \tag{5}$$

$$\vec{\nabla} \times \vec{B} - \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 (\vec{J}_f + \vec{J}_e^i) = \mu_0 \vec{J}_e, \tag{6}$$

$$\vec{\nabla} \cdot \vec{B} = \rho_m, \tag{7}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = -\vec{J}_m^i. \tag{8}$$

Here, the total electric current, $\vec{J}_e$, is a sum of the impressed current source, $\vec{J}_e^i$, driven by the external energy process and the free current, $\vec{J}_f$, in the system. Because magnetic monopoles do not exist, the effective magnetic current, $\vec{J}_m^i$, and the effective magnetic charge terms, ρ_m, in equations (8) and (6) can only exist due to impressed external non-electromagnetic sources. Due to the conservation of charge, the source currents must satisfy the continuity equations,

$$\frac{\partial \rho_e}{\partial t} = -\vec{\nabla} \cdot \vec{J}_e, \tag{9}$$

$$\frac{\partial \rho_m}{\partial t} = -\vec{\nabla} \cdot \vec{J}_m^i. \tag{10}$$

Here we have represented equations (5) to (8) with only the local sources on the right hand side of the equations, so the set of Maxwell's equations are a set of differential equations representing a retarded-field theory [5, 10–12]. A dual formulation of Maxwell's equations exists if we define the electric displacement current, $\vec{J}_{ed} = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ and the magnetic displacement current $\vec{J}_{md} = \frac{\partial \vec{B}}{\partial t}$, as non-local sources. In this case Maxwell's equations become a set of integrodifferential equations representing an instantaneous action-at-a-distance theory [13].

Integral Form

One can also write down the integral form of the non-conservative Maxwell's equations in vacuum as,

$$\oiint_S \vec{E} \cdot d\vec{a} = \frac{Q_{e_{enc}}}{\epsilon_0}, \tag{11}$$

$$\oint_P \vec{B} \cdot d\vec{l} - \mu_0 \epsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot d\vec{a} = \mu_0 I_{e_{enc}} \tag{12}$$

$$\oiint_S \vec{B} \cdot d\vec{a} = Q_{m_{enc}}, \tag{13}$$

$$\oint_P \vec{E} \cdot d\vec{l} + \frac{d}{dt} \int_S \vec{B} \cdot d\vec{a} = -I^i_{m_{enc}} \tag{14}$$

Here, $Q_{e_{enc}}$ in eqn.(11) is the enclosed total electric charge by the surface, S. $I_{e_{enc}}$ in eqn.(12) is the enclosed total electric current by the line path, P. $Q_{m_{enc}}$ in eqn(13) is the effective enclosed magnetic charge by the surface, S. $I^i_{m_{enc}}$ in eqn(14) is the effective enclosed impressed magnetic current by the line path, P. From these integral equations the boundary conditions for the non-conservative electrodynamics can be determined. For example, a magnetic current boundary source is required to model an ideal electromotive force generator (or voltage source), which converts an external energy into electromagnetic energy (like a battery). This is a prevalent technique in antenna theory, circuit theory and in the application of finite element software, which utilises two-potential theory with active sources [6–8, 14].

In general from a circuit theory perspective, the structures under consideration are considered as electrically small (quasi-static limit), no time delay exists between sources and the rest of the circuit and the only loss occurs through dissipation. From an antenna theory perspective these assumptions in general can be relaxed as time delays may be important. In this work we just consider the quasi-static limit relevant for antenna and circuit theory in the limit that the structures are very small compared to the wavelength.

Two Potential Formulation

The general two potential formulation of impressed current and voltage sources has been discussed in detail in standard text books on Electrical Engineering [6–8, 14]. The two potential formulation is used in electrodynamics to model voltage sources (non conservative electric fields), when there is conversion of external energy into electromagnetic energy (such as from mains power or batteries as discussed in the introduction). It has also been used to describe duality in electrodynamics and axion electrodynamics[15–18].

Using superposition, we can consider the electric and magnetic current sources from equations (5) to (8) separately. So setting the magnetic sources to zero, the electric and magnetic fields may be written in terms of the magnetic vector potential, $\vec{A}$, and the electric scalar potential, ϕ,

$$\begin{aligned}\vec{E}_A &= -\nabla\phi - \tfrac{\partial \vec{A}}{\partial t}\\ \vec{B}_A &= \nabla \times \vec{A}.\end{aligned} \tag{15}$$

Then by setting the electric sources to zero the electric and magnetic fields may be written in terms of the electric vector potential, $\vec{C}$, and the magnetic scalar potential, ϕ_m,

$$\begin{aligned}\vec{E}_C &= -\tfrac{1}{\epsilon_0}\nabla \times \vec{C}\\ \vec{B}_C &= -\mu_0\nabla\phi_m - \mu_0\tfrac{\partial \vec{C}}{\partial t}\end{aligned} \tag{16}$$

The total electric and magnetic fields may be calculated using the principle of superposition and are given by [6, 7];

$$\vec{E} = \vec{E}_A + \vec{E}_C = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} - \frac{1}{\epsilon_0}\nabla \times \vec{C} \tag{17}$$

$$\vec{B} = \vec{B}_A + \vec{B}_C = -\mu_0\nabla\phi_m - \mu_0\frac{\partial \vec{C}}{\partial t} + \nabla \times \vec{A}. \tag{18}$$

Considering the electric field given by equation (17), in the quasi-static limit we can ignore the time dependent terms and the main source terms are due to the charge distributions defined by the electric charge and the effective magnetic current, with the electric vector potential given by [6, 7],

$$\vec{C}(\vec{r},t) = \frac{\epsilon_0}{4\pi}\int_\Omega \frac{\vec{J}^i_m(\vec{r}',t')}{|\vec{r}-\vec{r}'|}\mathrm{d}^3\vec{r}'. \tag{19}$$

and the electric scalar potential given by,

$$\phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int_\Omega \frac{\rho(\vec{r}',t')}{|\vec{r}-\vec{r}'|}\mathrm{d}^3\vec{r}'. \tag{20}$$

Here $\vec{C}$ and ϕ at point $\vec{r}$ and time t is calculated from magnetic current and charge distribution at distant position $\vec{r}'$ at an earlier time $t' = t - |\vec{r}-\vec{r}'|/c$ (known as the retarded time). The location $\vec{r}'$ is a source point within volume Ω that contains the magnetic current distribution. The integration variable, $d^3\vec{r}'$, is a volume element around position r'. In a similar way the magnetic potentials may be written in terms of the electric current density and magnetic charge density, this work focuses on voltage sources, so is not written down here [6, 7].

The two potential formulation also means we can separate Maxwell's equations into two parts given by,

$$\vec{\nabla}\cdot\vec{E}_A = \frac{\rho_e}{\epsilon_0}, \tag{21}$$

$$\vec{\nabla}\times\vec{B}_A - \epsilon_0\mu_0\frac{\partial \vec{E}_A}{\partial t} = \mu_0(\vec{J}_f + \vec{J}^i_e) = \mu_0\vec{J}_e, \tag{22}$$

$$\vec{\nabla}\cdot\vec{B}_A = 0, \tag{23}$$

$$\vec{\nabla}\times\vec{E}_A + \frac{\partial \vec{B}_A}{\partial t} = 0, \tag{24}$$

for the electric sources, and

$$\vec{\nabla}\cdot\vec{E}_C = 0, \tag{25}$$

$$\vec{\nabla}\times\vec{B}_C - \epsilon_0\mu_0\frac{\partial \vec{E}_C}{\partial t} = 0, \tag{26}$$

$$\vec{\nabla}\cdot\vec{B_C} = \rho_m, \tag{27}$$

$$\vec{\nabla}\times\vec{E}_C + \frac{\partial \vec{B}_C}{\partial t} = -\vec{J}^i_m, \tag{28}$$

for the magnetic sources. Equations (5) to (8) represents just the sum of these two, consistent with equations (17) and (18). Equations (25) to (28) are the dual representation of equations (21) to (24).

ELECTRODYNAMICS OF AN FREE CHARGE ALTERNATING CURRENT VOLTAGE SOURCE

To analyse a free charge voltage source in the quasi static limit we can start with the equations given in advanced electrodynamics text books such as Griffiths [19], where he shows that the total force per unit charge, $\vec{f}$ involved in a free charge voltage source is given by,

$$\vec{f} = \vec{E}_A + \vec{f}_S. \tag{29}$$

Here $\vec{f}_S$ is the force per unit charge, which supplies the work to seperate the charges and supply an electromotive force from an external energy source. Following this a resulting electric field, $\vec{E}_A$, is produced by the separated charges. Harrington [6] presents essentially the same equation as Griffiths [19] for a general AC source, but using different terminology. Harrington considers the electric field more generally such that $\vec{E} \equiv \vec{f}$, which is consistent with the $\vec{E}$ in equations (5)-(8) in this paper. Harrington also defines $\vec{E}^i_C \equiv \vec{f}_S$ as the source impressed electric field. Here to be consistent with Griffith [19] and the left hand rule for the relation between magnetic current and electric field, we have defined $\vec{E}^i_C$ in the opposite direction to Harrington, so that

$$\vec{E} = \vec{E}_A + \vec{E}^i_C, \tag{30}$$

which reveals a form consistent with equation (17).

It should be emphasized that the impressed field, $\vec{E}^i_C$, is ordinarily confined to the voltage source and does not

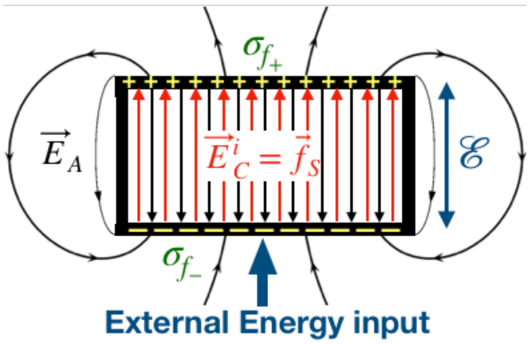

FIG. 1: Illustration of the electric field generated in a free-charge ideal voltage source from an external energy source. The associated delivered force per unit charge, $\vec{f}_S$, supplies the work to seperate the free charges, $\sigma_{f\pm}$. This is equivalent to an impressed non-conservative electric field, $\vec{E}^i_C$, as given by equations (29) and (30). The non-conservative nature means an electromotive force, $\mathcal{E}$, is generated, resulting in a voltage output that can drive an electric circuit. The separated free charges then generate a conservative electric field, $\vec{E}_A$.

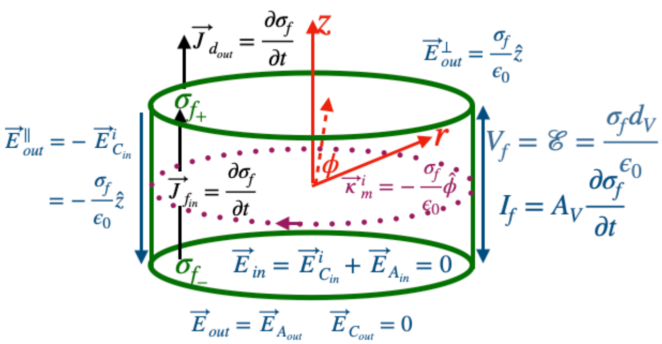

FIG. 2: Schematic of the cylindrical ideal free charge voltage source of axial length, d_V and cross sectional area A_V. For the ideal AC voltage source of terminal voltage, V_f, the free charge electric current density, $\vec{J}_f$, flows up and down the cylindrical axis as the terminal charges, $\sigma_{f\pm}$, oscillate in polarity

exist outside it. For example, in a battery where chemical energy is converted to electromagnetic energy [20–22], the impressed electric field allows the electrons to move in the opposite direction to the electric field created by the electrons themselves, even though the internal resistance inside an ideal free charge voltage source is near zero, with $\vec{E}_A \approx -\vec{E}^i_C$ and thus $\vec{E} \approx 0$, which is illustrated in fig.1.

Ideal Cylindrical Free Charge Voltage Source

In the following we will analyse the electrodynamics of an ideal zero resistance cylindrical AC voltage source, as shown in fig.2. We recognise that the net charge in the system is zero, and the charge will be present as a surface charge density, $\sigma_{f\pm}$, at the upper and lower boundaries of the cylinder. Furthermore, there will be a non-familiar boundary condition to be determined at the cylindrical boundary, since $\vec{E}^i_C$ must be contained within the cylinder. Also, we assume no impressed electrical currents as we are modelling a voltage source, so there can only be an impressed magnetic current with the magnetic charge density set to zero. In this case Maxwell's equations in differential form (5)-(8) become,

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_f}{\epsilon_0}, \tag{31}$$

$$\vec{\nabla} \times \vec{B} - \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}_f, \tag{32}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{33}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = -\vec{J}^i_m. \tag{34}$$

Following this, the integral form of Maxwell's equations may be written as,

$$\oiint_S \vec{E} \cdot d\vec{a} = \frac{Q_{f_{enc}}}{\epsilon_0}, \tag{35}$$

$$\oint_P \vec{B} \cdot d\vec{l} - \mu_0 \epsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot d\vec{a} = \mu_0 I_{f_{enc}} \tag{36}$$

$$\oiint_S \vec{B} \cdot d\vec{a} = 0, \tag{37}$$

$$\oint_P \vec{E} \cdot d\vec{l} + \frac{d}{dt} \int_S \vec{B} \cdot d\vec{a} = -I^i_{m_{enc}} \tag{38}$$

In the quasi static limit, assuming $\frac{\partial \vec{B}}{\partial t} \approx 0$, we can calculate the voltage source emf, $\mathcal{E}$, by,

$$\mathcal{E} = -I^i_{m_{enc}} = \oint_P \vec{E} \cdot d\vec{l} = \oint_P \vec{E}^i_C \cdot d\vec{l}, \tag{39}$$

and the surface charge on the end faces caused by the impressed source field, $\vec{E}^i_C$, can be calculated to be,

$$\sigma_f = \epsilon_0 \vec{E}^i_C \cdot \hat{n}, \text{ where } \vec{E}^i_C = \frac{\sigma_f}{\epsilon_0} \hat{z}. \tag{40}$$

Here $\hat{n}$ is the normal to the surface, which is equal to $\hat{z}$ on the top surface. Then, from eqn.(39) the emf generated in the quasi static limit is given by,

$$\mathcal{E} = E^i_C d_V = \frac{\sigma_f d_V}{\epsilon_0} = V_f, \tag{41}$$

which is similar to a voltage across a capacitor, we label this the free charge terminal voltage, V_f.

Next we consider the magnetic surface current per unit length, which will be apparent at the radial boundary of the voltage source. This surface magnetic current will determine the parallel boundary condition, and can be calculated from equation (39), using the left hand rule to be,

$$\vec{\kappa}_m^i = -\frac{\sigma_f}{\epsilon_0}\hat{\phi}. \tag{42}$$

From the integral equations (35)-(38) it is straightforward to derive the boundary conditions. Here subscript "in" refers to inside the ideal voltage source and subscript "out" refers to outside the voltage source, while the subscript "⊥" refers to the perpendicular components of the field with respect to a surface and the subscript "∥" refers to the parallel components of the fields with respect to the surface. We also note that $\vec{E}_{in} = 0$, and that there is no free surface electric current (only volume current). Thus, the boundary conditions can be determined from,

$$\vec{E}_{out}^{\perp} = \frac{\sigma_f}{\epsilon_0}, \tag{43}$$

$$\vec{B}_{out}^{\parallel} = \vec{B}_{in}^{\parallel}, \tag{44}$$

$$\vec{B}_{out}^{\perp} = \vec{B}_{in}^{\perp}. \tag{45}$$

$$\vec{E}_{out}^{\parallel} = -\vec{\kappa}_m^i \times \hat{n} = -\vec{E}_{Cin}^{i\parallel}, \tag{46}$$

To calculate the electromagnetic fields of the system, we need to consider the boundary conditions. Applying the radial boundary condition given by equation (46), gives at the boundary, $\vec{E}_{out}^{\parallel} = -\vec{E}_C^i = -\frac{\sigma_f}{\epsilon_0}\hat{z}$. Then by applying the axial boundary condition given by equation (43), gives at the boundary, $\vec{E}_{out}^{\perp} = \frac{\sigma_f}{\epsilon_0}\hat{z}$. This means the electric field just outside the source is maximum on the boundaries, while the electric field inside the source is zero. Thus, the solution of the fields outside the voltage source will be essentially that of a Hertzian dipole. Because the electric field inside is zero, there is no displacement current inside the voltage source, however, there will be an electric current density. This is opposite outside the voltage source, where there is a displacement current and no free current.

The electric current density inside the voltage source can be calculated directly by the time rate of change of the surface charge density, to be,

$$\vec{J}_{f_{in}} = \frac{\partial \sigma_f}{\partial t} \tag{47}$$

On the axial boundaries, we can show that the displacement current outside the voltage source matches the free current inside the voltage source, with both terminated by the same surface charges, $\sigma_{f\pm}$, so that,

$$\vec{J}_{d_{out}} = \epsilon_0 \frac{\partial \vec{E}_{out}^{\perp}}{\partial t} = \epsilon_0 \frac{\partial \vec{E}_C^i}{\partial t} = \frac{\partial \sigma_f}{\partial t} = \vec{J}_{f_{in}} \tag{48}$$

To calculate the magnetic field in the system we can start inside the voltage source, and since the total electric field, $\vec{E}_{in} = 0$, the magnetic field can be calculated from $\oint_P \vec{B} \cdot d\vec{l} = \mu_0 I_{f_{enc}} = \mu_0 \int_S \vec{J}_{f_{in}} \cdot d\vec{a}$ and is similar to that of the $\vec{B}$-field in a cylindrical conductor. Then by applying the boundary conditions and solving Maxwell's equations, with no circuit attached to the voltage source, the solution is that of an ideal Hertzian dipole.

Assuming a harmonic surface charge density of the form, $\sigma_f = \sigma_{f0} e^{j\omega_0 t}$, then the terminal voltage may be determined to be, $V_f = V_{f0} e^{j\omega_0 t} = \frac{\sigma_{f0} d_V}{\epsilon_0} e^{j\omega_0 t}$. Likewise, the current inside the voltage source may be determined to be, $I_f(t) = I_{f0} e^{j\omega_0 t} = j\omega_0 \sigma_{f0} A_V e^{j\omega_0 t}$. The ratio of the terminal voltage to internal current can be calculated to be $\frac{V_f}{I_f} = \frac{1}{j\omega_0 C_{eff}}$ where $C_{eff} = \frac{\epsilon_0 A_V}{d_V}$ is similar to a capacitance, where V_f lags I_f by $\frac{\pi}{2}$. Note, this is not a capacitance, but just the phase relationship between the current and voltage of the source. This is an active component where the impressed electric field maintains the charge separation, and in the ideal voltage source there is no power dissipation, so the internal current and terminal voltage must be out of phase to satisfy this condition. One must remember the resistance and hence impedance of the ideal source is zero, similar to a perfect conductor. This condition is exactly the same as the ideal Hertzian dipole antenna solution, where the dipole field outside the source resembles a charged capacitor.

As in the Hertzian dipole solution, in the quasi static limit, the near $\vec{B}$-field is suppressed compared to the $\vec{E}$-field due to the large wavelength of the source. In the DC limit as the frequency goes to zero, the $\vec{B}$-field is in fact zero. However, if we attach a circuit load to the voltage source, it will draw a current and then the dynamics will be mainly determined by the source load, depending on the source load properties properties.

TIME VARYING BOUND CHARGE VOLTAGE SOURCE

A bound charge voltage source is essentially a bar electret. The ideal bar electret exhibits a permanent electrical dipolar field and has overall charge neutrality[23]. Like the free charge Hertzian dipole voltage source discussed previously, the electric field of an electret is often compared to a parallel plate capacitor. The most common way to make an electret is to heat a polar dielectric material under the influence of a large electric field (thermo electret) [24]. Once cooled and removed from the electric field a net polarization will be maintained. The electret thus becomes a bound charge voltage source, and can supply a current and be discharged in a similar way to a battery [25].

In this work we generalise the concept of an electret to a time varying impressed permanent polarization, which is similar to the impressed electric field defined previously to describe a free charge voltage source. This system describes a putative AC voltage source based on oscillating bound charge. We find that the impressed polarization, $\vec{P}^i_{C_b}$ is equivalent to the force per unit charged required to drive the voltage source, which leads to a generalization of Faraday's Law.

Impressing a Source Term into Maxwell's Equations to Describe an Electret

The starting point to consider a bound charge voltage source, is to consider Maxwell's equations for a dielectric media, with an impressed magnetic current represented by the following equations,

$$\vec{\nabla} \cdot \vec{D} = 0, \tag{49}$$

$$\vec{\nabla} \times \vec{B} - \mu_0 \frac{\partial \vec{D}}{\partial t} = 0, \tag{50}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{51}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = -\vec{J}^i_m. \tag{52}$$

Here we assume a lossless dielectric media so ideally there is no free charge or current in the system and hence they are set to zero, which highlights one of the main difference between the Maxwell's equations for the bound charge (49)-(52) and the free charge voltage source equations (31)-(34). The last term in equation (52) is the impressed magnetic current source, which drives the system to create an AC voltage output. In the following we show how the values of the electromagnetic fields, output voltage and magnetic current may be calculated with the aid of the constitutive relations and boundary conditions that define an ideal electret.

In a similar way to the free charge voltage source, the forces in the bound charge system may be defined using equation (29), originally presented in Griffiths [19]. Thus, the total force per unit charge, $\vec{f}_b$ acting on the bound charges is given by,

$$\vec{f}_b = \vec{E}_A + \vec{f}_{S_b}. \tag{53}$$

Here, $\vec{f}_{S_b}$, is the force per unit charge, which supplies the work to seperate the bound charges from an external energy source and hence supply an electromotive force. Following this, an electric field, $\vec{E}_A$, is produced by the separated bound charges. The analogous notation used by Harrington, describes all terms that supply a force to a charge as an electric field, so that the total field $\vec{E} \equiv \vec{f}_b$ and $\vec{E}^i_{C_b} \equiv \vec{f}_{S_b}$, is defined as the equivalent impressed electric field, with the following relation.

$$\vec{E} = \vec{E}_A + \vec{E}^i_{C_b}. \tag{54}$$

Accordingly the impressed electric field, $\vec{E}^i_{C_b}$, maybe be identified to be related to the impressed permanent source polarization $\vec{P}^i_{C_b}$ by,

$$\vec{E}^i_{C_b} = \vec{P}^i_{C_b}/(\epsilon_0 \epsilon_r), \tag{55}$$

and then if we multiply equation (54) through by the permittivity, $\epsilon_0\epsilon_r$, we obtain

$$\vec{D} = \epsilon_0 \epsilon_r \vec{E}_A + \vec{P}^i_{C_b}. \tag{56}$$

where $\vec{D} = \epsilon_0\epsilon_r\vec{E}$ and ϵ_r is the relative permittivity of the dielectric. Equation (56) is the well known constitutive relation between the $\vec{D}$-field, $\vec{E}$-field and $\vec{P}$-field in the electret. Thus, equation (54), which balances the Lorentz forces in the voltage source is essentially on the same footing as a constitutive relationship between fields.

For a linear dielectric, we note that equation (56) may also be written as,

$$\vec{D} = \epsilon_0 \vec{E}_A + \vec{P}, \tag{57}$$

where

$$\vec{P} = \chi_e \epsilon_0 \vec{E}_A + \vec{P}^i_{C_b}. \tag{58}$$

Here $\chi_e = \epsilon_r - 1$ is the electronic susceptibility of the medium.

In general it is well known that the curl of both $\vec{D}$ and $\vec{P}^i_{C_b}$ are non-zero for an electret [19], so if we take the curl of equation (56) we obtain,

$$\vec{\nabla} \times \vec{D} = \epsilon_0 \epsilon_r \vec{\nabla} \times \vec{E}_A + \vec{\nabla} \times \vec{P}^i_{C_b}, \tag{59}$$

which, by dividing through by, $\epsilon_0\epsilon_r$, and combining equations (24) and (28), we can show (59) may also be written as

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = -\vec{J}^i_m, \tag{60}$$

which is the same as equation (52), justifying our initial starting point given by equations (49)-(52). Here the $\frac{\partial \vec{B}}{\partial t}$ term in eqn.(60) can be identified as the magnetic displacement current. In this system, the the magnetic current term, $\vec{J}^i_m$ drives the voltage source and also sets the boundary condition for the parallel components of the fields on the boundary of the voltage source.

Boundary Conditions

The boundary conditions of the fields on the normal and parallel surfaces of the electret can be calculated from the integral version of equations (49)-(52), and are given by,

$$\oiint_S \vec{D} \cdot d\vec{a} = 0, \tag{61}$$

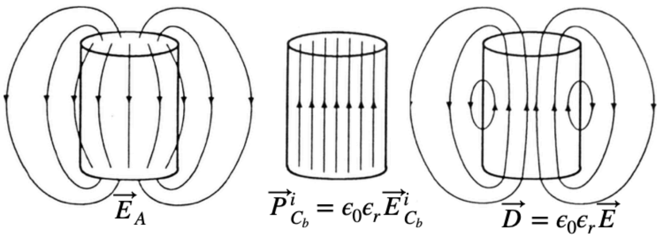

FIG. 3: From left to right, 3D sketch of the $\vec{E}_A$, $\vec{P}^i_{C_b}$ and $\vec{D}$ fields inside and outside a cylindrical bar electret (reproduced from the solution manual of [19]). Assuming $\vec{P}^i_{C_b}$ is constant within the electret and along the cylindrical $z-$axis of the bar in the positive direction. Note, the impressed polarization is only defined inside the voltage source.

$$\oint_P \vec{B}\cdot d\vec{l} - \mu_0\frac{d}{dt}\int_S \vec{D}\cdot d\vec{a} = 0 \tag{62}$$

$$\oiint_S \vec{B}\cdot d\vec{a} = 0, \tag{63}$$

$$\oint_P \vec{E}\cdot d\vec{l} + \frac{d}{dt}\int_S \vec{B}\cdot d\vec{a} = -I_{m_{enc}}, \tag{64}$$

From these integral equations it is straightforward to derive the modified boundary conditions as follows (subscript "in" refers to inside the bar electret and subscript "out" refers to outside the bar electret),

$$\vec{D}^{\perp}_{out} = \vec{D}^{\perp}_{in}, \tag{65}$$

$$\vec{B}^{\parallel}_{out} = \vec{B}^{\parallel}_{in}, \tag{66}$$

$$\vec{B}^{\perp}_{out} = \vec{B}^{\perp}_{in}. \tag{67}$$

$$\vec{E}^{\parallel}_{A_{out}} - \vec{E}^{\parallel}_{A_{in}} = 0, \tag{68}$$

and since $\vec{E}^{i\parallel}_{C_{out}} = 0$ as it does not exist outside the voltage source, then by the left hand rule,

$$\vec{E}^{i\parallel}_{C_{in}} = -\vec{\kappa}^i_{mb}\times\hat{n}, \tag{69}$$

Axial Boundaries

On the top and bottom axial boundaries, as shown in fig.4, we can use eqn.(65) to get a relation ship between $\vec{E}_A$ and $\vec{E}^i_C$. Inside the voltage source at the boundary, $\vec{E}_A = E_A\hat{z}$ and $\vec{E}^i_C = E^i_C\hat{z}$, therefore $D_{in} = \epsilon_0\epsilon_r(E^i_C + E_A)\hat{z}$. On the outside of the boundary $D_{out} = -\epsilon_0 E_A\hat{z}$ as $\vec{E}_A$ points in the opposite direction and is in free space. Matching this condition gives the following ratio of amplitudes at the boundary,

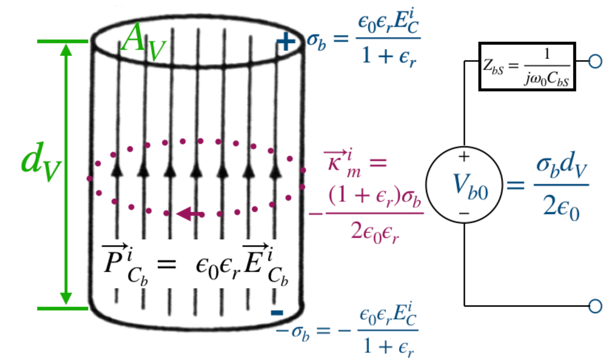

FIG. 4: Putative AC bar electret of cross sectional area A_V and length d_V. The associated impressed force per unit charge $\vec{E}^i_{C_b}$ supplies the work to seperate the bound charges σ_b in a similar way to a battery, resulting in an electret with permanent polarisation $\vec{P}^i_{C_b}$, which can be modelled as an impressed magnetic current at the boundary, $\vec{\kappa}^i_m$. The Thevenin equivalent circuit is pictorially shown with the generated emf, $\mathcal{E}$, and effective source impedance, Z_{bS}, which is calculated in the text.

$$E_A = -\frac{\epsilon_r E^i_C}{1+\epsilon_r} \tag{70}$$

Therefore the $\vec{D}$-field above and below the axial boundaries is continuous and can be calculated in terms of the impressed electric field $E^i_C\hat{z}$ to be,

$$\vec{D}_{ax} = \frac{\epsilon_0\epsilon_r E^i_C}{1+\epsilon_r}\hat{z} \tag{71}$$

with the electric field above and below the top boundary given by,

$$\vec{E}_{Aax_{out}} = \frac{\epsilon_r E^i_C}{1+\epsilon_r}\hat{z} \tag{72}$$

$$\vec{E}_{Aax_{in}} = -\frac{\epsilon_r E^i_C}{1+\epsilon_r}\hat{z} \tag{73}$$

with the total polarization inside the voltage source due to the impressed source and dielectric response calculated from equation (58)to be,

$$\vec{P}_{ax_{in}} = 2\frac{\epsilon_0\epsilon_r E^i_C}{1+\epsilon_r}\hat{z} \tag{74}$$

These fields are consistent with a bound charge on the top and bottom axial surfaces of,

$$\sigma_{b\pm} = \pm 2\frac{\epsilon_0\epsilon_r E^i_C}{1+\epsilon_r} \tag{75}$$

The displacement current density, $\vec{J}_D = \frac{\partial E_C^i}{\partial t}\frac{\epsilon_0\epsilon_r}{1+\epsilon_r}\hat{z}$ through the voltage source is continuous, and if it has a cross sectional area of A_V the effective displacement current through the voltage source is given by,

$$I_d = A_V \frac{\epsilon_0\epsilon_r}{1+\epsilon_r}\frac{\partial E_C^i}{\partial t} = \frac{A_V}{2}\frac{\partial \sigma_b}{\partial t} \tag{76}$$

Radial Boundaries

On the radial boundary we can determine that the impressed surface magnetic current density from equation (69) to be,

$$\vec{\kappa}_m^i = -E_{C_{in}}^i \hat{\phi} = -\frac{(1+\epsilon_r)\sigma_b}{2\epsilon_0\epsilon_r}, \tag{77}$$

and the electric fields, $\vec{E}_A$ from equation (68) to be,

$$\vec{E}_{Arad_{in}} = \vec{E}_{Arad_{out}} = -\frac{\epsilon_r E_C^i}{1+\epsilon_r}\hat{z} = -\frac{\sigma_b}{2\epsilon_0}\hat{z} \tag{78}$$

To calculate the emf we need to integrate around the radial boundary, in the quasi static limit we set $\frac{\partial \vec{B}}{\partial t} = 0$ and therefore the emf may be calculated from,

$$\mathcal{E} = -I_{m_{enc}} = \oint_P \vec{E}_{C_b}^i \cdot d\vec{l}, \tag{79}$$

where $I_{m_{enc}} = \int_S \vec{J}_m^i \cdot d\vec{a}$, is the enclosed magnetic current. defining the length of the voltage source as d_V, the induced emf is given by,

$$\mathcal{E} = E_C^i d_V = \frac{(1+\epsilon_r)\sigma_b d_V}{2\epsilon_0\epsilon_r} \tag{80}$$

Even though equation (80) calculates the emf supplied by the voltage source with respect to the bound surface charge σ_b, because we have modelled the system as a polarizable material, this will not be the voltage available at the terminals due to the finite impedance. For example, outside the bound charge voltage source the system appears like a Hertzian dipole with a reduced electric field of $\frac{\sigma_b}{2\epsilon_0}\hat{z}$ compared to the free charge voltage source of $\frac{\sigma_f}{\epsilon_0}\hat{z}$. Thus for the same charge density at the terminals of the voltage source, the bound charge source exhibits an electric field of a factor of two less than the free charge source.

This can be explained because the bound charge medium is no-conductive so the total electric field, $\vec{E}$ inside the bound charge voltage source will not be zero, while in the free case $\vec{E} = 0$ inside the voltage source. This occurs because the free-charge voltage source may have an internal free current flowing to charge the voltage terminals, while the dielectric voltage source must have a displacement current, given by eqn. (76) to charge the terminals with bound charges, which requires $\vec{E}$ to be non-zero. This is also why the D-field in fig.(3) appears continuous in this system. Thus, the free charge voltage source acts like an ideal voltage source with no impedance, because the total electric field inside is zero while the current flows freely. Inside the bound charge voltage source a finite total electric field, $\vec{E}_{in}$, exists of,

$$\vec{E}_{in} = \frac{\vec{D}_{ax}}{\epsilon_0\epsilon_r} = \frac{\vec{E}_C^i}{1+\epsilon_r} = \frac{\sigma_b}{2\epsilon_0\epsilon_r}\hat{z}. \tag{81}$$

Thus, one can calculate the voltage drop across the bound voltage source as,

$$V_d = \frac{\sigma_b d_V}{2\epsilon_0\epsilon_r} \tag{82}$$

Then assuming the charge density oscillates harmonically such that $\sigma_b = \sigma_{b0}e^{j\omega_0}$, then source impedance may be calculated by

$$Z_{bS} = \frac{V_d}{I_d} = \frac{1}{j\omega_0 C_{bS}} \quad C_{bS} = \frac{\epsilon_0\epsilon_r A_V}{d_V}, \tag{83}$$

and the output voltage at the terminals, V_{b0}, can be calculated to be.

$$V_{b0} = \mathcal{E} - V_d = \frac{\sigma_b d_V}{2\epsilon_0}, \tag{84}$$

which is consistent with the electric field generated outside the voltage source, given in equations (78) and (72).

This system is a Hertzian dipole with similar characteristics to the free charge system. The magnetic field generated will be of similar form and can be calculated in the quasi static limit via Ampere's law through the displacement current.

DISCUSSION

First, It should be emphasized here that the effective magnetic current boundaries defined in the voltage sources have nothing to do with magnetic monopoles, and is solely due to the voltage created and the charge distribution caused by the external force per unit charge supplied to the system (equivalent to an impressed electric field), which means from an electrodynamics point of view the system is non-conservative. The fact is, to properly describe a voltage source, a forcing function must be inputted to Faraday's law. To model this properly we had to use the well known force relations with in the voltage sources, for both the bound charge and free charge case. Solving the bound charge case highlighted the fact that these force equations can be thought of extensions to the constitutive relationships between the respective fields in the system. The constitutive relationships in the bound charge system are already known, for example the fields in electrets have been discussed before[19]. However, there has never been a proper analysis of the electret as a non-conservative system. By implementing the impressed electrical field technique commonly used by engineers, we have been able to relate the permanent polarisation of an electret to an impressed electric field. In this way we have shown that the permanent polarization effectively adds to the Lorentz force where the total electric field (or force per unit charge) was redefined through the extension of the constitutive relations. Because this extension is non-conservative, part of the total electric field must be defined by an electric vector potential. Reflecting on the free charge system, by analogy we can consider the relationship between the electric fields given by eqns. (29) and (30) as a constitutive relationship for the non-conservative free charge voltage source.

CONCLUSION

We have explored the electrodynamics of impressed bound and free charge voltage sources. This was represented by a force per unit charge, which converts external energy into electromagnetic energy and may be considered as a non-conservative electric field vector with an electric vector potential. The source term is necessarily impressed into Maxwell's equations as an equivalent magnetic current, due to the energy conversion, the resulting charge distribution and emf produced by the voltage source, results in a modification of Faraday's law. Recently, an analogy to the electret bound-charge voltage source model detailed in this paper has been used to explain modifications of electrodynamics due to dark matter axions. The model implements a similar impressed electric field technique described by a magnetic current boundary source [17], and is what motivated this work. This work also shows that the axion modifications can add to the constitutive relations. This effect has been shown to be related to the Witten effect [26].

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