Exercises to be completed during the Block Course

Day 1

(1) Construct a Table similar to the one shown in the lecture for a fixed number of trials $N = 6$ and a success probability $p = 0.3$. From the table, find the following:
  (a) The Central Interval for $1 - \alpha = 0.68$ and $1 - \alpha = 0.90$
  (b) The Smallest Interval for $1 - \alpha = 0.68$ and $1 - \alpha = 0.90$

(2) Prove that the mode of the likelihood function for a Binomial distribution is given by $p^* = r/N$.

(3) Show that the variance of a flat probability distribution, $P(x) = \frac{1}{b-a}$ defined in the interval (a, b) is

$$V[x] = \frac{(b-a)^2}{12}$$

(4) Consider a Bernoulli process with a fixed success probability p. Show that the expectation value for the number of trials until you have a first success is $1/p$.

Day 2

(1) Calculate the skew of the Poisson distribution

(2) For a Poisson process with mean 1.5, what is the probability to see 6 or more events ? What is the probability of exactly 0 events ?

(3) You have observed a part of the sky for 10^3 s and observed 10 events.
  (a) Starting from Bayes Theorem and using a flat prior, calculate the probability distribution you would get for the rate of events.
  (b) What is the most probable value for the rate ?
  (c) Find the 68 % smallest interval.
  (d) Your colleague performed the experiment in a slightly different way. He waited until he saw the 10th event and then stopped his measurements. Call the time at which the 10th event arrived T_2. What is the probability distribution for the rate given T_2 ? What value of T_2 would have to be measured for you to get the same result for the most probable value of the rate ?

Day 3

(1) Show that the correlation matrix for a set of independent variables is diagonal.

(2) In liquid Argon, 40000 photons in the visible range are produced per MeV of deposited energy. Imagine that you have a detector that has an overall efficiency of 0.001 for detecting these photons. Your detector produces a signal of 1 Volt/detected photon. What is the resulting resolution of the detector as a function of incident energy (resolution is defined as the standard deviation measured in energy units, MeV) ?

(3) Convolution of Gaussians: Suppose you have a true distribution which follows a Gaussian distribution:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(x-x_0)^2}{2\sigma_x^2}}$$

and the measured quantity, y follows a Gaussian distribution around the value x.

$$P(y|x) = \frac{1}{\sqrt{2\pi}\sigma_y} e^{-\frac{(y-x)^2}{2\sigma_y^2}} \; .$$

What is the predicted distribution for the observed quantity y ?

(4) Derive the mean, variance and mode for the χ^2 distribution for one data point.

(5) Using the Central Limit Theorem, argue that the χ^2 distribution for large enough number of data points approaches a Gauss probability distribution. What are the parameters of the Gauss distribution ?

Day 4

(1) Prove that the Box-Muller method produces two uncorrelated Gauss distributed random numbers

(2) Use the inverse transform method to generate random numbers according to the probability distribution

$$P(y)dy = \frac{1}{y^2}dy \qquad\qquad y > 0.01 \; .$$

Day 5

(1) Try out the following algorithm to generate random numbers uniformly on the surface of a sphere:
   (a) generate random numbers uniformly within a unit cube.
   (b) reject values with radius r> 1.
   (c) for the other values, project the values along a radius to $r = 1$.
   (d) Prove that the density on the surface of the sphere is uniform.

(2) Integrate the function $[\cos(50x) + \sin(20x)]^2$ from $0 \leq x \leq 1$ using the hit-or-miss technique. The correct value is approximately $I = 0.9652$. How close is your result? How does your uncertainty scale with the number of samples you used?

Day 6

(1) write a Metropolis algorithm to generate random numbers according to the Unit Normal distribution. Try different proposal distributions and see what happens.

(2) Consider the Metropolis algorithm for a target density distribution that is proportional to the Boltzmann factor:

$$f(E) \propto e^{-E/kT} \ .$$

The temperature of the system is fixed. Show that detailed balance is preserved by the Metropolis algorithm for this probability density