Exercises - Monte Carlo Methods

(1) Produce a linear congruential generator which generates uniform random real numbers between [0,1). Generate a long sequence of numbers and calculate the mean and variance.

(2) Calculate analytically the cumulative distribution function and the probability density distribution for the product of 3 independent and identically distributed (iid) random numbers flat between [0,1).

(3) Compare your expression in problem 2 with numerically generated products of three uniform random numbers. Perform a Kolmogorov-Smirnov test to see if your generated distribution agrees with the analytic result.

(4) We discussed in the lectures how to generate random numbers according to an exponential distribution. For $n = 1, 10, 100$, compare the analytic expression for the waiting time for the n^{th} event in a constant unit-rate process:

$$P_n(t)dt = \frac{t^{n-1}e^{-t}}{\Gamma(n)}dt \qquad (t \geq 0)$$

with simulations of the waiting times generated by simulating the process.

(5) Generate random numbers according to the probability distribution:

$$p(x) = \frac{2}{\pi R^2}\sqrt{R^2 - x^2} \qquad -R < x < R$$

using the acceptance-rejection technique. Define a covering function and find the efficiency analytically and numerically.

(6) For the example in the previous exercises, sample x according to $f(x) = \frac{1}{2R}$ and give each event a weight $p(x)/f(x)$.

(a) Generate $10^2, 10^3, 10^4$ samples from $f(x)$ and compare the distribution of weighted events to the expected distribution by making a histogram, summing the weights in the bins and comparing to the expectation for each bin.

(b) Analyze what you expect analytically for the variance of the weighted x distribution in the bins.
(c) Generate 100 sets of samples and compare the observed variance to your estimate.

(7) Code an Ehrenfest model with a permeable barrier which preferentially passes particles in one direction, and find the equilibrium distribution for the number of molecules on each side for different r. Take as transition probability matrix:

$$\begin{aligned} P_{ij} &= r\frac{a-i}{2a} & j &= i+1 \\ &= \frac{a+i}{2a} & j &= i-1 \\ &= \frac{(a-i)(1-r)}{2a} & j &= i \end{aligned}$$

(8) Code a random walk in the plane where, in addition to a random angle, the step size is variable and follows an exponential distribution. I.e.,

$$P(s) = e^{-s}$$

where s is the step size and

$$P(\theta) = \frac{1}{2\pi} \; .$$

Make a distribution of the distance from the origin after 10^5 steps and compare the results to the Rayleigh distribution that we derived in the lecture and discuss. You will need to generate many random walks each of which takes 10^5 steps, extract the distance from the origin for each, and plot the resulting distribution.

(9) (Only required for the TUM students) Solve the Traveling Salesman problem for 25 cities numerically by running a Simulated Annealing algorithm based on the Metropolis Markov Chain algorithm. Find an optimum path starting from the nominal visiting sequence (1,2,...,25). Choose a proposal function and state it clearly. Also define your starting temperature and your annealing sequence. Show your optimal path graphically.