Section 2

A Modern Computational Technology

- Conversion to nested sums, nested integrals

arXiv:1211.5204 [hep-th], 1408.5590 [hep-ph]

Y. Sumino

(Tohoku Univ.)

Conversion to nested sums, nested integrals

☆ Contents

1. Introduction: background, motivation
2. Examples, applications
3. Main idea and strategy of conversion

A goal of (perturbative) calculations: combination of nested sums, nested integrals

$$\sum_{n_1>n_2>\cdots>n_N>0} f(n_1)f(n_2)\cdots f(n_N)$$

$$\int_{x_1>x_2>\cdots>x_N>0} dx_1 dx_2 \cdots dx_N\, f(x_1)f(x_2)\cdots f(x_N)$$

Ex: Harmonic polylogs (HPL), multiple polylogs, multiple zeta values (MZV), etc.

$\Longrightarrow$ Reduction to a small set of basis functions/values possible

Here we present a method to convert non-nested sum/integral to nested ones.

$$F(\nu) = \sum_{n_1=1}^{\infty/f_1(\nu)} \cdots \sum_{n_N=1}^{\infty/f_N(\nu)} \frac{\lambda_1^{n_1}\cdots\lambda_N^{n_N}}{\Pi_i\,(\nu + c_i + a_{i1}n_1 + \cdots + a_{iN}n_N)^{p_i}}$$

implemented in Mathematica package “Wa”

Simple examples:

$$C = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$: Catalan const.

$$\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{(k+m)^2(2m+4k+1)} = 4 - 2C\pi - \frac{\pi^2}{6} - 4\log 2 + \frac{21}{4}\zeta(3)$$

Z-sum

$$\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{(-1)^{k+m}}{(k+1)^2(2m+1)} = -8\,\boxed{Z(\infty; 2,1; -i, i)} + 4C\log 2 - \frac{\pi^3}{8} - \frac{\pi^2}{12}$$

$$\| $$

$$\sum_{\infty > n_1 > n_2 > 0} \frac{(-i)^{n_1} i^{n_2}}{n_1^2\, n_2} = -\mathrm{Im}\left[\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right] - \frac{\pi^3}{128} + \frac{\pi}{32}\log^2 2$$

$$\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{(1+k+\nu)^2(1+m+n+\nu)(m+n+1)}$$

$$= \frac{2}{\nu} Z(\nu; 3; 1) + \frac{1}{\nu} Z(\nu; 1,2; 1,1) - \frac{\pi^2}{6} Z(\nu; 1; 1) \;= \frac{2}{\nu}\sum_{n=1}^{\nu}\frac{1}{n^3} + \cdots$$

$$f(x) = \int_0^1 dy\; \log(1-y)\log(1-xy)\log(1+xy)\left[\log(1-x^2y)\right]^2$$ $\longrightarrow$ HPLs

Application: Computation of NR bound state spectrum

$$\left(-\frac{\Delta}{M} - C_F\frac{\alpha_s}{r}\right)\psi_{nl} = E_n\,\psi_{nl}$$

$$\Delta E_{nl} = \langle\psi_{nl}|\Delta V|\psi_{nl}\rangle + \langle\psi_{nl}|\Delta V\cdot\bar{G}_{nl}\cdot\Delta V|\psi_{nl}\rangle + \cdots$$

$$G(\vec{x},\vec{y};E) \equiv \sum\!\!\!\!\!\!\int_\alpha \frac{\psi_\alpha(\vec{x})\psi_\alpha^*(\vec{y})}{E-E_\alpha} = \sum_{l=0}^{\infty}(2l+1)\,x^l y^l\,P_l(\cos\theta)\,G_l(x,y;k)$$

$$x = |\vec{x}|,\qquad y = |\vec{y}|,\qquad \cos\theta = \frac{\vec{x}\cdot\vec{y}}{x\,y},\qquad k = \sqrt{-ME}$$

Repr. by Laguerre polynomials:

$$G_l(x,y;k) = \frac{Mk}{2\pi}(2k)^{2l}e^{-k(x+y)}\sum_{m=0}^{\infty}\frac{L_m^{2l+1}(2kx)L_m^{2l+1}(2ky)\,m!}{(m+l+1-\nu)(m+2l+1)!}$$

$$\nu = \frac{C_F\alpha_s M}{2k},\qquad \sum_{m=0}^{\infty}L_m^{2l+1}(\rho)t^m = \frac{\exp(-\frac{\rho t}{t+1})}{(1-t)^{2l+2}}$$

Reduced Green fn.:

$$\bar{G}_{nl} = \lim_{E\to E_n}\left(G - \frac{\psi_{nl}\psi_{nl}^*}{E-E_n}\right)$$

$$\Rightarrow\quad \langle\psi_{nl}|\Delta V\cdot\bar{G}_{nl}\cdot\Delta V|\psi_{nl}\rangle \ni \sum_{k=1}^{\infty}\frac{(n-l+k-1)!}{(n+l+k)!\,k^3} \equiv A(n,l)$$

$$\Longrightarrow \quad \langle \psi_{nl} | \Delta V \cdot \bar{G} \cdot \Delta V | \psi_{nl} \rangle \ni \sum_{k=1}^{\infty} \frac{(n-l+k-1)!}{(n+l+k)!\, k^3} \equiv A(n,l)$$

$$\frac{(n-l+k-1)!}{(n+l+k)!} = \prod_{m=-l}^{+l} \frac{1}{n+k+m} = \sum_{m=-l}^{+l} \frac{R(l,m)}{n+k+m} \qquad R(l,m) = \frac{(-1)^{l-m}}{(l+m)!\,(l-m)!}$$

$$A(n,l) = \sum_{k=1}^{\infty} \sum_{m=-l}^{+l} \frac{R(l,m)}{(k+n+m)\, k^3}$$

$$\sum_{k=1}^{\infty} \frac{1}{(k+\nu)\, k^3} = \frac{\zeta(3)}{\nu} - \frac{\zeta(2)}{\nu^2} + \frac{S_1(\nu)}{\nu^3}, \qquad S_1(\nu) = \sum_{k=1}^{\nu} \frac{1}{k}$$

: Harmonic sum

$$= \sum_{m=-l}^{+l} R(l,m) \left[\frac{\zeta(3)}{n+m} - \frac{\zeta(2)}{(n+m)^2} + \frac{S_1(n+m)}{(n+m)^3} \right]$$

$$\sum_{m=-l}^{+l} \frac{R(l,m)}{(n+m)^a} = \frac{1}{a!} \left(-\frac{\partial}{\partial k} \right)^a \left. \frac{\Gamma(n-l+k)}{\Gamma(n+l+k+1)} \right|_{k=0}$$

for $a \geq 0$

$$= \frac{(n+l)!}{(n-l-1)!} [\zeta(3) - \zeta(2)\{S_1(n+l) - S_1(n-l-1)\}] + \sum_{m=-l}^{+l} \frac{R(l,m)}{(n+m)^3} S_1(n+m)$$

includes only known transcendental numbers and finite sums

How to convert:

$$F(\nu) = \sum_{n_1=1}^{\infty/f_1(\nu)} \cdots \sum_{n_N=1}^{\infty/f_N(\nu)} \frac{\lambda_1^{n_1} \cdots \lambda_N^{n_N}}{\prod_i (\nu + c_i + a_{i1}n_1 + \cdots + a_{iN}n_N)^{P_i}}$$

Main idea: resum sequence of differences iteratively

cf. $\Delta f(k) = f(k+1) - f(k) \quad \Longrightarrow \quad f(\nu) = \sum_{k=1}^{\nu-1} \Delta f(k) + f(1)$

$$= \sum_{k=1}^{\nu-1} \left(\sum_{m=1}^{k-1} \Delta^2 f(m) + \Delta f(1) \right) + f(1)$$

$$\vdots$$

Shift invariance of summand:

You can find a set of integers $(\Delta_0, \Delta_1, \cdots, \Delta_N)$ s.t. for $\forall i,$

$$\nu + c_i + a_{i1}n_1 + \cdots + a_{iN}n_N = (\nu + \Delta_0) + c_i + a_{i1}(n_1 + \Delta_1) + \cdots + a_{iN}(n_N + \Delta_N)$$

Example:

$$S=\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{k+m}}{(2+m)^2(1+k+m)^2}=\sum_{m=1}^{\infty}\frac{(-1)^m f(m)}{(2+m)^2}$$

$$f(m)=\sum_{k=1}^{\infty}\frac{(-1)^k}{(1+k+m)^2}$$

$$f(m)=[f(m)+f(m-1)]-[f(m-1)+f(m-2)]+\cdots$$

$$+(-1)^m[f(2)+f(1)]-(-1)^m f(1)$$

$$=\sum_{j=2}^{m}(-1)^{m-j}[f(j)+f(j-1)]-(-1)^m f(1)$$

$$k=k'+1 \quad \sum_{k'=0}^{\infty}\frac{(-1)^{k'+1}}{(1+k'+j)^2}$$

$$f(j)+f(j-1)=\sum_{k=1}^{\infty}\frac{(-1)^k}{(1+k+j)^2}+\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+j)^2}$$

$$=\left(\sum_{k=1}^{\infty}-\sum_{k=0}^{\infty}\right)\frac{(-1)^k}{(1+k+j)^2}=-\frac{1}{(1+j)^2}$$

Bulk part cancels and only surface term remains.

Example:

$$S=\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{k+m}}{(2+m)^2(1+k+m)^2}=\sum_{m=1}^{\infty}\frac{(-1)^m f(m)}{(2+m)^2}$$

$$f(m)=\sum_{k=1}^{\infty}\frac{(-1)^k}{(1+k+m)^2}$$

$$\therefore\ f(m)=(-1)^m\sum_{j=2}^{m}\frac{(-1)^{1-j}}{(1+j)^2}-(-1)^m\sum_{k=1}^{\infty}\frac{(-1)^k}{(2+k)^2}$$

$$-(-1)^m f(1)$$

$$\therefore S=\sum_{m=1}^{\infty}\frac{1}{(2+m)^2}\sum_{j=2}^{m}\frac{(-1)^{1-j}}{(1+j)^2}-\sum_{m=1}^{\infty}\frac{1}{(2+m)^2}\sum_{k=1}^{\infty}\frac{(-1)^k}{(2+k)^2}$$

nested sum — product of single sums