Introduction to Lattice Gauge Theory

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Outline of the Course

- Lecture 1: Bosons: Basic constructs, scalar fields, gauge fields (Monday)
- Lecture 2: Fermions: Doubling, chiral symmetry, choices (Tuesday)
- Lecture 3: Renormalization: Effective field theory approach (Thursday)

Lattice Fermion Fields

Simplest Discretization

- We seek a discrete version of the Dirac Lagrangian

$$-\bar{\psi}(x)(\partial\!\!\!/ + m)\psi(x)$$

- First thing anyone thinks of is $\partial_\mu\psi(x) \to (2a)^{-1}[\psi(x+ae_\mu)-\psi(x-ae_\mu)]$:
  - in momentum space, FT$\{(2a)^{-1}[\psi(x+ae_\mu)-\psi(x-ae_\mu)]\} = (a)^{-1} i \sin p_\mu a$.
- Propagator:

$$\frac{a}{\sum_\mu i\gamma_\mu \sin(p_\mu a) + ma} = \frac{a[-\sum_\mu i\gamma_\mu \sin(p_\mu a) + ma]}{\sum_\mu \sin^2(p_\mu a) + (ma)^2}$$

which scales correctly at the 16 points with $p_\mu a \in \{0, \pi\}$.

- As on a previous slide, let's compute $G(x^4, \boldsymbol{p})$.

$$G(t,\omega) = \int_{-\pi}^{\pi} \frac{dp}{2\pi} \frac{(b - i\gamma^4 \sin p)e^{ipt}}{\sin^2 p + \omega^2}$$

$$= \int_{C_{\pi/2}} \frac{dp}{2\pi} \frac{(b - i\gamma^4 \sin p)e^{ipt}}{\sin^2 p + \omega^2}$$

poles at $p = \pm iE,\ \pm iE + \pi$

with $+iE$ and $iE + \pi$ enclosed by $C_{\pi/2}$

$\sinh^2 E = \omega^2$ (or $\cosh 2E = 1 + 2\omega^2$)

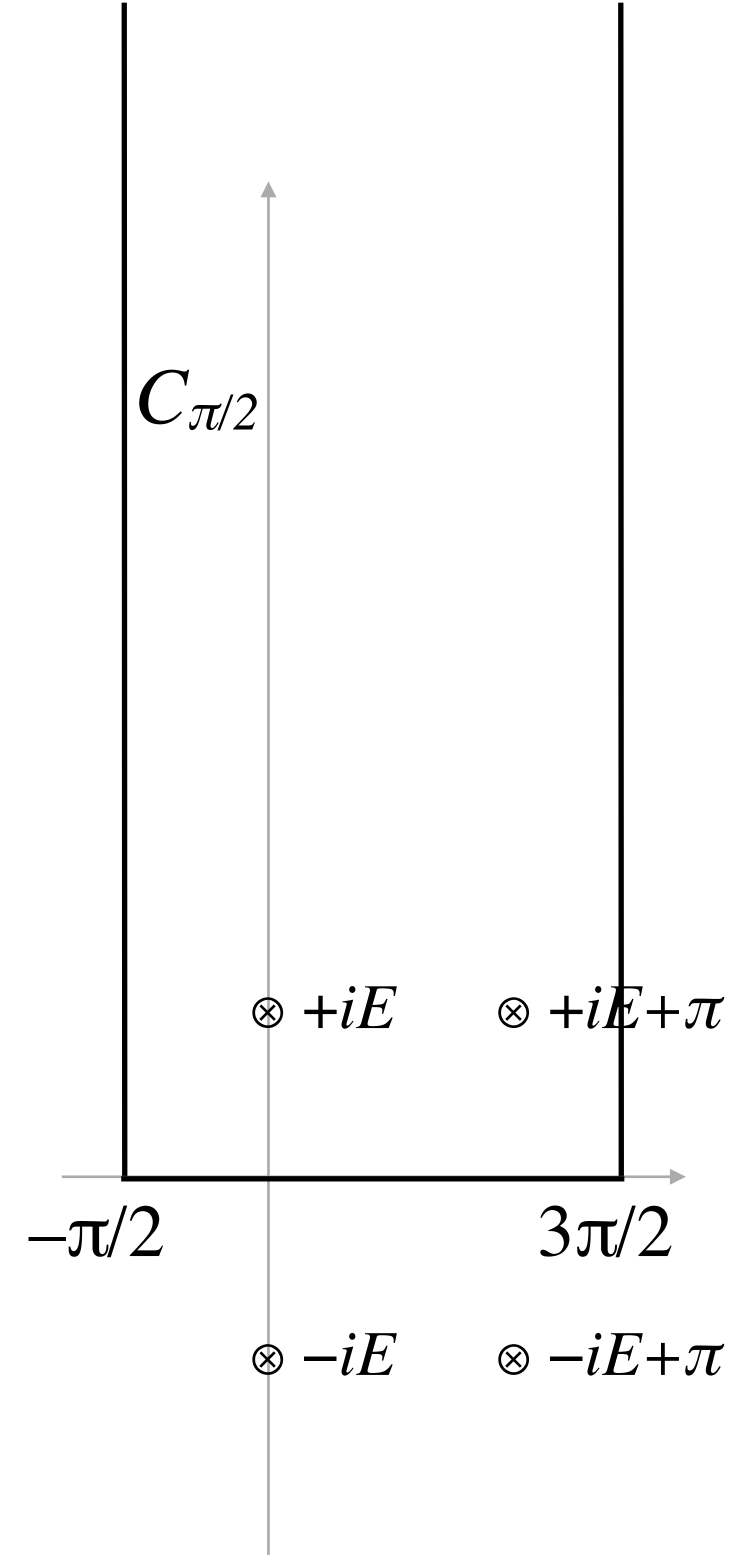
$$G(t,\omega) = \frac{2\pi i}{2\pi}\left[e^{ipt} \left(\frac{d(\sin^2 p + \omega^2)}{dp} \right)^{-1} \right]_{p=iE}$$

$$+ \frac{2\pi i}{2\pi}\left[e^{ipt} \left(\frac{d(\sin^2 p + \omega^2)}{dp} \right)^{-1} \right]_{p=iE+\pi}$$

$aS_\mu(p) = \sin(p_\mu a)$

$$= \frac{ma \mp \gamma^4 Ea - ia\boldsymbol{\gamma}\cdot\boldsymbol{S}(\boldsymbol{p})}{\sinh 2Ea} e^{-E|t|}$$

$$+ (-1)^{t/a} \frac{ma \pm \gamma^4 Ea - ia\boldsymbol{\gamma}\cdot\boldsymbol{S}(\boldsymbol{p})}{\sinh 2Ea} e^{-E|t|}$$



upper sign $t > 0$

lower sign $t < 0$

Doubling Problem

- The result, noting $\sinh^2 Ea = \sum_i \sin^2(p_i a) + m^2a^2 = a^2\boldsymbol{S}^2(\boldsymbol{p}) + m^2a^2$

$$G(t,\boldsymbol{p}) = \frac{ma \mp \gamma^4 Ea - ia\boldsymbol{\gamma}\cdot\boldsymbol{S}(\boldsymbol{p})}{\sinh 2Ea} e^{-E|t|} + (-1)^{t/a} \frac{ma \pm \gamma^4 Ea - ia\boldsymbol{\gamma}\cdot\boldsymbol{S}(\boldsymbol{p})}{\sinh 2Ea} e^{-E|t|}$$

$aS_\mu(p) = \sin(p_\mu a)$

- Remarks:
  - energy $Ea \ll 1$ near eight values of $\boldsymbol{p}$: $p_i \in \{0, \pi/a\}$;
  - first term: (anti)particle forward (backward) in time—Dirac fermion;
  - second term: (anti)particle backward (forward) in time.
- Extra states and funny properties part of so-called “fermion doubling problem.”

Fermion Path Integrals

Grassmann Variables

- Take a collection of fermions $\{\hat{\psi}^\dagger_\alpha, \hat{\psi}_\beta\} = \delta_{\alpha\beta}, \{\hat{\psi}_\alpha, \hat{\psi}_\beta\} = \{\hat{\psi}^\dagger_\alpha, \hat{\psi}^\dagger_\beta\} = 0.$

- Define "empty" and "full" states by

$$\hat{\psi}_\alpha|\text{empty}\rangle = 0, \qquad \hat{\psi}^\dagger_\alpha|\text{full}\rangle = 0;$$

$$\langle\text{empty}|\hat{\psi}^\dagger_\alpha = 0, \qquad \langle\text{full}|\hat{\psi}_\alpha = 0.$$

- Define field eigenstates by

$$|\psi\rangle = e^{\sum_\alpha \hat{\psi}^\dagger_\alpha \psi_\alpha}|\text{empty}\rangle, \qquad |\breve{\psi}\rangle = e^{\sum_\alpha \hat{\psi}^\dagger_\alpha \breve{\psi}_\alpha}|\text{full}\rangle;$$

$$\langle\psi| = \langle\text{empty}|e^{-\sum_\alpha \hat{\psi}^\dagger_\alpha \psi_\alpha}, \qquad \langle\breve{\psi}| = \langle\text{full}|e^{-\sum_\alpha \hat{\psi}^\dagger_\alpha \breve{\psi}_\alpha};$$

- These satisfy $\hat{\psi}_\alpha|\psi\rangle = \psi_\alpha|\psi\rangle$, $\hat{\psi}^\dagger_\alpha|\breve{\psi}\rangle = \breve{\psi}_\alpha|\breve{\psi}\rangle$, and (proofs as exercise)

$$1 = \int \prod_\alpha d\psi_\alpha |\psi\rangle\langle\psi| = \int \prod_\alpha d\breve{\psi}_\alpha |\breve{\psi}\rangle\langle\breve{\psi}|$$

- Take a quadratic Hamiltonian $\hat{H} = \sum_{\alpha\beta} \hat{\psi}^{\dagger}_{\alpha} M_{\alpha\beta} \hat{\psi}_{\beta}$ with $M_{\beta\alpha} = M^{*}_{\alpha\beta}$. Then

$$\langle \psi_N | e^{-\hat{H}\tau} | \psi_0 \rangle = \int \prod_{n=1}^{N-1} \mathscr{D}\psi_n \mathscr{D}\breve{\psi}_n \mathscr{D}\breve{\psi}_0 \, \langle \psi_N | \breve{\psi}_{N-1} \rangle \langle \breve{\psi}_{N-1} | e^{-a\hat{H}} | \psi_{N-1} \rangle \cdots \times$$

$$\langle \psi_{n+1} | \breve{\psi}_n \rangle \langle \breve{\psi}_n | e^{-a\hat{H}} | \psi_n \rangle \cdots \langle \psi_1 | \breve{\psi}_0 \rangle \langle \breve{\psi}_0 | e^{-a\hat{H}} | \psi_0 \rangle$$

- Some simple algebra shows:

$$\langle \psi_{n+1} | \breve{\psi}_n \rangle = e^{-\breve{\psi}_n \cdot \psi_{n+1}}$$

$$\langle \breve{\psi}_n | e^{-a\hat{\psi}^{\dagger} \cdot M \cdot \hat{\psi}} | \psi_n \rangle = e^{\breve{\psi}_n \cdot \psi_n - a\breve{\psi}_n \cdot M \cdot \psi_n} + \mathrm{O}(a^2)$$

- Assembling all the factors:

$$Z = \int \mathscr{D}\psi_0 \, \langle \psi_0 | e^{-\hat{H}\tau} | \psi_0 \rangle = \int \prod_n \mathscr{D}\psi_n \mathscr{D}\breve{\psi}_n \, e^{-S}$$

$$S = a \sum_{n=0}^{N-1} \left\{ \breve{\psi}_n \cdot \frac{\psi_{n+1} - \psi_n}{a} + \breve{\psi}_n \cdot M \cdot \psi_n \right\}$$

- Similarly, $\langle \breve{\psi}_N | e^{-\hat{H}\tau} | \breve{\psi}_0 \rangle$ leads to (show as exercise)

$$\langle \breve{\psi}_{n+1} | \psi_n \rangle = e^{\breve{\psi}_{n+1} \cdot \psi_n}$$

$$\langle \psi_n | e^{-a \hat{\psi}^{\dagger} \cdot M \cdot \hat{\psi}} | \breve{\psi}_n \rangle = e^{-\breve{\psi}_n \cdot \psi_n - a \breve{\psi}_n \cdot M \cdot \psi_n} + \mathrm{O}(a^2)$$

$$Z = \int \mathscr{D} \breve{\psi}_0 \, \langle \breve{\psi}_0 | e^{-\hat{H}\tau} | \breve{\psi}_0 \rangle = \int \prod_n \mathscr{D} \psi_n \mathscr{D} \breve{\psi}_n \, e^{-S}$$

$$S = a \sum_{n=0}^{N-1} \left\{ - \frac{\breve{\psi}_{n+1} - \breve{\psi}_n}{a} \cdot \psi_n + \breve{\psi}_n \cdot M \cdot \psi_n \right\}$$

- Useful ingredient to derivations:

$$\hat{\psi}_\alpha \, \hat{\psi}^{\dagger} \cdot \psi = \psi_\alpha + \hat{\psi}^{\dagger} \cdot \psi \, \hat{\psi}_\alpha$$

$$\hat{\psi}_\alpha (\hat{\psi}^{\dagger} \cdot \psi)^n |\text{empty}\rangle = n \psi_\alpha (\hat{\psi}^{\dagger} \cdot \psi)^{n-1} |\text{empty}\rangle$$

and similarly for $\hat{\psi}^{\dagger}_{\alpha}yy\hat{\psi}yyy\breve{\psi}$, $\hat{\psi}^{\dagger}{}_{\alpha} (\hat{\psi} \cdot \breve{\psi})^n |\text{full}\rangle$.

Dirac Field

- Dirac field annihilates particles and creates antiparticles: $\frac{1}{2}(1 \pm \gamma^4)\,\hat{\psi}_x$;
  - use $\langle \breve{\psi}_N | e^{-\hat{H}\tau} | \breve{\psi}_0 \rangle$ ($\langle \psi_N | e^{-\hat{H}\tau} | \psi_0 \rangle$) for upper (lower) sign.
- Then the action becomes ($\bar{\psi} = \breve{\psi}\gamma^4$):

$$S = a^d \sum_x \left\{ \sum_\mu \bar{\psi}_x \left[\gamma_\mu \frac{\psi_{x+ae_\mu} - \psi_{x-ae_\mu}}{2a} - \frac{a}{2} \frac{\psi_{x+ae_\mu} + \psi_{x-ae_\mu} - 2\psi_x}{a^2} \right] + m_0 \bar{\psi}_x \psi_x \right\}$$

choosing M has to give spatial and temporal kinetic terms the same form.

- Note the extra term: looks like a second-order derivative.
- Known as the Wilson action.

gamma matrices

Wilson Fermion Propagator

Wilson Fermions

- Including the Wilson terms solves the “doubling” problem

$$G(t,\boldsymbol{p}) = \int_{-\pi/a}^{\pi/a} \frac{dp_4}{2\pi} \frac{a e^{ip_4 t}}{i\gamma^4 \sin p_4 a + i\sum_i \gamma^i \sin p_i a + ma + a^2 \frac{1}{2}\sum_\mu \hat{p}_\mu^2},$$

$$= \int_{-\pi/a}^{\pi/a} \frac{dp_4}{2\pi} \frac{a e^{ip_4 t}\,[-i\gamma^4 \sin p_4 a - i\sum_i \gamma^i \sin p_i a + ma + a^2 \frac{1}{2}\sum_\mu \hat{p}_\mu^2]}{\sin^2 p_4 a + \sum_i \sin^2 p_i a + (ma + a^2 \frac{1}{2}\sum_\mu \hat{p}_\mu^2)^2},$$

- This expression only has poles at $p_4 = \pm iE$ ($\cosh E$ on next slide):
  - poles at $p_4 = \pm iE + \pi/a$ have been projected out;
  - low-energy solutions for p_i near 0 only.

Wilson Fermion Energy Spectrum

$$G(t,\boldsymbol{p}) = \int_{-\pi/a}^{\pi/a} \frac{dp_4}{2\pi} \frac{a e^{ip_4 t}}{i\gamma^4 \sin p_4 a + i\sum_i \gamma^i \sin p_i a + ma + a^2 \frac{1}{2}\sum_\mu \hat{p}_\mu^2},$$

$$= \frac{a e^{-Et}}{2\sinh(Ea)} \frac{\gamma_4 \sinh(Ea) - i\sum_i \gamma_i \sin(p_i a) + m_0 a + \frac{1}{2}a^2\hat{\boldsymbol{p}}^2 + 1 - \cosh(Ea)}{1 + m_0 a + \frac{1}{2}a^2\hat{\boldsymbol{p}}^2},$$

$$\cosh(Ea) = 1 + \frac{1}{2}\frac{\sum_i \sin^2(p_i a) + (m_0 a + \frac{1}{2}a^2\hat{\boldsymbol{p}}^2)^2}{1 + m_0 a + \frac{1}{2}a^2\hat{\boldsymbol{p}}^2}$$

show as exercise
hint: express
temporal Wilson
term in $\cos$ form

- Wilson fermion action is almost completely satisfactory:
  - vacuum polarization leads to $\beta_0 = \frac{11}{3}N_c - \frac{2}{3}n_f$;
  - triangle diagram now gives the expected anomaly—possible, because the Wilson term breaks chiral symmetry.
- Must finely tune m_0 to cancel explicit breaking of the Wilson term to reach spontaneously broken (Nambu-Goldstone) vacuum:

$$M_\pi^2 = (m_0 - m_{\text{crit}})B \to 0$$

- Alas, makes renormalization of composite operators similarly nasty: operators of different chirality can mix.
- For two (or and even number of) quarks any of these problems can be alleviated by adding a “twisted” mass term: twist in isospin at boundary.

Staggered Fermions

(Banks-Kogut-)Susskind Fermions

- In in Hamiltonian LGT, another way of alleviating the doubling/chiral problem:
    - Kogut & Susskind put (anti)particles on even (odd) sites;
    - Banks, Kogut, & Susskind used 1 component/site in $d = 1$ (+time);
    - Susskind studied 1 component/site in $d = 3$ (+time);
- Kawamoto & Smit and Sharatchandra, Thun, & Weisz generalized Susskind fermions to 4 Euclidean dimensions.
- Complicated history, funny name: staggered fermions.

Naïve Fermions

- The naive action is

$$S = \frac{a^3}{2}\sum_{x,\mu}\bar{\Upsilon}(x)\gamma_\mu\left[U_\mu(x)\Upsilon(x+\hat{\mu}) - U^\dagger_{x-\mu,\mu}\Upsilon(x-\hat{\mu})\right] + m_0 a\sum_x \bar{\Upsilon}(x)\Upsilon(x)$$

  with Grassmann variables Υ^i_α, $\bar{\Upsilon}^i_\alpha$ on each site.

- Invariant under $SU_{\mathrm{color}}(3)$, translations, hypercubic rotations, and $U_V(n_f)\times U_A(n_f)$.

- The naïve action also has a remarkable “doubling” symmetry [Karsten & Smit]:

$$\Upsilon \mapsto B_\mu\Upsilon, \quad \bar{\Upsilon} \mapsto \bar{\Upsilon}B_\mu^{-1}, \quad B_\mu = i\gamma_\mu\gamma_5(-1)^{x_\mu/a}$$

- Generates a Clifford group Γ_4: $\{B_\mu, B_\nu\} = 2\delta_{\mu\nu}$.

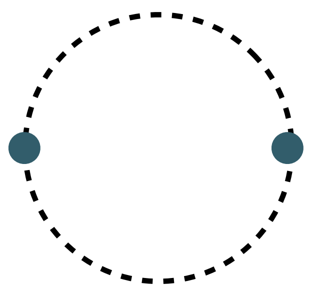
Ramifications

- Doubling symmetries
  - map $p \mapsto p + \pi^A/a$, where π^A is a corner of the Brillouin zone;
  - shuffle Dirac indices.
- The 16 states are really there. In loops—
  - running coupling: $\beta_0 = \frac{11}{3}N_c - \frac{2}{3}(16n_f)$;
  - axial anomaly:

$$\mathfrak{A}_{\text{naive lat}} = (1-4+6-4+1)\mathfrak{A} = 0$$

  because chiral symmetry exact.

- Gluons really see them: $aS_\mu(p) = \sin(p_\mu a)$

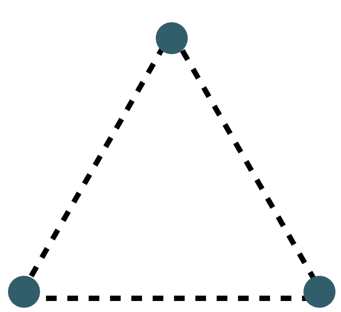


$$= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{d^4k}{2\pi}\,\mathrm{tr}\left[\frac{\cos(p+\frac{1}{2}k)_\mu}{i\not{S}(p+k)+m}\gamma_\mu\frac{\cos(p+\frac{1}{2}k)_\nu}{i\not{S}(k)+m}\gamma_\nu\right]$$

$$= \int_{-\pi/2}^{3\pi/2}\int_{-\pi/2}^{3\pi/2}\int_{-\pi/2}^{3\pi/2}\int_{-\pi/2}^{3\pi/2} \qquad \text{same periodic function}$$

which can be broken in 16 equal regions:

- $[-\pi/2,\pi/2]^4$; $[-\pi/2,\pi/2]^3\times[\pi/2,3\pi/2]$ (four such); $[-\pi/2,\pi/2]^2\times[\pi/2,3\pi/2]^2$ (six such); $[-\pi/2,\pi/2]\times[\pi/2,3\pi/2]^3$ (four such), $[\pi/2,3\pi/2]^4$.



- Anomaly diagram: sixteen regions contribute the same up to sign: $+1-4+6-4+1 = 0$. In lattice field theory, an exact symmetry cannot break via counting problems (aka UV divergences).

Staggering Magic

- But the lattice allows a transformation [Kawamoto & Smit]:

$$\Upsilon(x) \mapsto \psi(x) = \Omega(x)\Upsilon(x), \quad \Omega(x) = \gamma_1^{n_1}\gamma_2^{n_2}\gamma_3^{n_3}\gamma_4^{n_4},$$
$$\bar{\Upsilon}(x) \mapsto \bar{\psi}(x) = \bar{\Upsilon}(x)\Omega^{-1}(x), \quad n = x/a.$$

where Ω clobbers the Dirac matrices:

$$\Omega^{-1}(x)\gamma_\mu\Omega(x \pm \hat{\mu}) = (-1)^{\sum_{\rho<\mu} n_\rho} =: \eta_\mu(x).$$

- The naïve action assumes a simpler form (plus mass term):

$$S = \frac{a^3}{2}\sum_{x,\mu}\bar{\psi}(x)\eta_\mu(x)\left[U_\mu(x)\psi(x+\hat{\mu}) - U^\dagger_{x-\mu,\mu}\psi(x-\hat{\mu})\right]$$

- Also attained by diagonalizing a maximal subgroup of doubling symmetry [Sharatchandra, Thun, & Weisz].

- Dirac index now trivial: $\langle \Upsilon(x)\bar{\Upsilon}(y)\rangle_U = \Omega(x)\Omega^{-1}(y)\langle \chi(x)\bar{\chi}(y)\rangle_U$, where χ has one component per color-flavor.

- The (kinetic) action for χ is

$$S = \frac{a^3}{2}\sum_{x,\mu}\bar{\chi}(x)\eta_\mu(x)\left[U_\mu(x)\chi(x+\hat{\mu}) - U^\dagger_{x-\mu,\mu}\chi(x-\hat{\mu})\right]$$

- Dirac index gone.

- Other symmetries are entangled, *e.g.*, translations become shifts:

$$\begin{aligned} S_\mu : \chi(x) &\mapsto \zeta_\mu(x)\chi(x+\hat{\mu}), \quad \bar{\chi}(x) \mapsto \zeta_\mu(x)\bar{\chi}(x+\hat{\mu}) \\ U_\nu(x) &\mapsto U_\nu(x+\hat{\mu}), \forall\nu; \quad \zeta_\mu(x) = (-1)^{\sum_{\rho>\mu} n_\rho} \end{aligned}$$

- Clifford again: $\{S_\mu, S_\nu\} = 2\delta_{\mu\nu}$.

- Shifts and spatial inversion do not commute.

Species Interpretation

- After "staggering" there are enough degrees of freedom for four species of Dirac fermion (16 ÷ 4 = 4):
  - called "taste" because they aren't used for u, d, s, c flavors;
  - the non-anomalous axial symmetry is a taste non-singlet.
- Perturbative and nonperturbative calculations show that such a structure emerges in the continuum limit:
  - anomaly in a taste-singlet axial current;
  - agreement with various tests of chiral symmetry, e.g., eigenvalue spectrum in accord with random matrix theory.

- If $i\lambda_s$ is an eigenvalue, so is $-i\lambda_s$, with eigenvector $f_{-s} = \gamma_P^5(x) f_s$ where $\gamma_P^5(x) = (-1)^{(x_1+x_2+x_3+x_4)/a}$.

P: "pseudoscalar taste"

- Zero modes: some gauge fields are found to have quartets of near-zero eigenvalues, two with $\lambda_s > 0$, two with $\lambda_s < 0$:
  - these $i\mathring{\lambda}_i$ have $|\mathring{\lambda}_i| \sim \Lambda^3 a^2$.
- They behave like the zero modes expected on some gauge fields in continuum gauge theory.

Nielsen-Ninomiya Theorem

N&N, *PLB* **105** (1981) 219; *NPB* **185** (1981) 20; *NPB* **193** (1981) 173

D. Friedan, *Commun. Math. Phys.* **85** (1982) 481

Content of the Theorem

- In even spacetime dimensions, no free lattice fermion system exists that is simultaneously
    - has a bilinear Hamiltonian $H = \psi^\dagger M \psi$ (of course, $H = H^\dagger$);
    - is local, *i.e.*, continuous in momentum space;
    - is translation invariant (in $\boldsymbol{x}$ space);
    - has chiral internal symmetries a.k.a, $\{\not{D}, \gamma^5\} = 0$.
- A “no-go” theorem.

Statement of the Theorem

following D. Friedan, *CMP* **85** (1982) 481

- Suppose the Hamiltonian takes the form

$$\hat{H} = \int \frac{d^3 p}{(2\pi)^3} \hat{\psi}^\dagger_\alpha(\boldsymbol{p}) K_{\alpha\beta}(\boldsymbol{p}) \hat{\psi}_\beta(\boldsymbol{p})$$

  where $K(\boldsymbol{p})$ is the Fourier transform of M, and α, β are internal indices.

- Study the spectrum of $K(\boldsymbol{p})$.

- Massless fermion requires zero eigenvalues, at $\boldsymbol{p} = \boldsymbol{p}_a$, in $K(\boldsymbol{p})$.

- To look relativistic, near $\boldsymbol{p}_a$ it must be that $K(\boldsymbol{p}) = i\Gamma^4\boldsymbol{\Gamma}\cdot(\boldsymbol{p} - \boldsymbol{p}_a) + \mathrm{O}((p - p_a)^3)$, where $\Gamma^4\Gamma^j$ are a representation of the d-dim Clifford algebra ($d = 3$).

- Count chirality of modes at all locations $\boldsymbol{p}_a$. They total zero.

Proof

following D. Friedan, *CMP* **85** (1982) 481

- The eigenvalues of $K(\boldsymbol{p})$ near $\boldsymbol{p}_a$ are $\pm|\boldsymbol{p} - \boldsymbol{p}_a|$.
- Now track how these eigenvalues change $\boldsymbol{p}$ as goes through $\boldsymbol{p}_a$:
  - right-handed particles (left-handed antiparticles) ascend;
  - left-handed particles (right-handed antiparticles) descend.
- The index is the number of the first kind (over two), minus the number of the second kind.
- But $K(\boldsymbol{p})$ is periodic and continuous: what goes up must come down.

Evading the Theorem

- Put up with extra states, *i.e.*, worry only about vector-like theories (like QCD).
- Make the spectrum discontinuous (SLAC): worse unphysical behavior.
- Make $K(\boldsymbol{p})$ an infinite matrix.
- Don't have a Hamiltonian (or transfer operator) at $a \neq 0$.
- Allow a special exception to the anticommutator: $\{D\!\!\!\!/, \gamma^5\} = aD\!\!\!\!/\gamma^5 D\!\!\!\!/ \neq 0$:
    - called the Ginsparg-Wilson relation.

Chiral Fermions

e.g., M. Lüscher, arXiv:hep-th/0102028

Ginsparg-Wilson Magic

- Suppose we have an operator satisfying the Ginsparg-Wilson relation:

$$\gamma^5 \not{D} + \not{D}\gamma^5 = a\not{D}\gamma^5\not{D}, \qquad \text{and } \not{D}^\dagger = \gamma^5\not{D}\gamma^5$$

  See also Hasenfratz, Laliena, & Niedermayer.

- Then we can define a gauge-field-dependent $\hat{\gamma}^5 = \gamma^5(1 - a\not{D})$:

$$\hat{\gamma}^{5\,\dagger} = \hat{\gamma}^5, \qquad (\hat{\gamma}^5)^2 = 1$$

- Take chiral transformations for two definitions of chirality [Lüscher]:

$$\begin{aligned} \psi_{R/L} &= \tfrac{1}{2}(1 \pm \hat{\gamma}^5)\psi, & \psi &\mapsto \exp[i\alpha\hat{\gamma}^5/2]\psi, \\ \bar{\psi}_{R/L} &= \bar{\psi}\tfrac{1}{2}(1 \mp \gamma^5), & \bar{\psi} &\mapsto \bar{\psi}\exp[i\alpha\gamma^5/2]. \end{aligned}$$

- The eigenvalues of a G-W operator lie on a circle: $\Lambda_s = a^{-1}(1 - e^{-i\delta_s a})$.

- $\Lambda_s = a^{-1}(1 - e^{-i\delta_s a})$: eigenvector $f_{-s} = \gamma^5 f_s$ has $\delta_{-s} = -\delta_s$ unless $\delta_s = 0,\ \pi$.

- Zero modes: some gauge fields have zero eigenvalues with $\gamma^5 f_0 = \pm f_0$; they yield non-zero terms in $\mathrm{Tr}\,\gamma^5$. But $\mathrm{Tr}\,\gamma^5 = 0$, so each zero mode of chirality ± 1 must have a partner with chirality ∓ 1:

    - they have $\delta_s = \pi$, $\Lambda_s = 2/a$.

- Lüscher's chiral transformation introduces a non-trivial Jacobian:

$$\begin{aligned}\mathrm{Det}\exp[i\alpha\hat{\gamma}^5/2] &= \exp\{i\alpha\,\mathrm{Tr}[\gamma^5(1 - a\not{D})]/2\} \\ &= \exp[i\alpha(n_+ - n_-)],\end{aligned}$$

  where $n_\pm$ is the number of zero modes with chirality ± 1.

- Suggests a definition of topological charge density:

$$q(x) = -a^{-4}\,\mathrm{tr}[\gamma^5(1 - \tfrac{1}{2}a\not{D})]$$

Domain-Wall Fermions

- One implementation of the Ginsparg-Wilson relation [D.B. Kaplan] starts with a 5-dimensional spacetime.
- Take Wilson fermions with “mass” term $a^{-1}[\theta(s) - \theta(-s)]$.
- In a finite system, use antiperiodic b.c. in s.
- Then a right-(left-)handed Weyl fermion is localized at $s = 0$, $s = N_s - 1$, which combine to form a Dirac fermion.
- In practice, small chiral symmetry breaking $\sim \exp(-N_s)$;
  - if the mode at is thrown away, and the cost of making $K(\boldsymbol{p})$ an infinite matrix.

Overlap Fermions

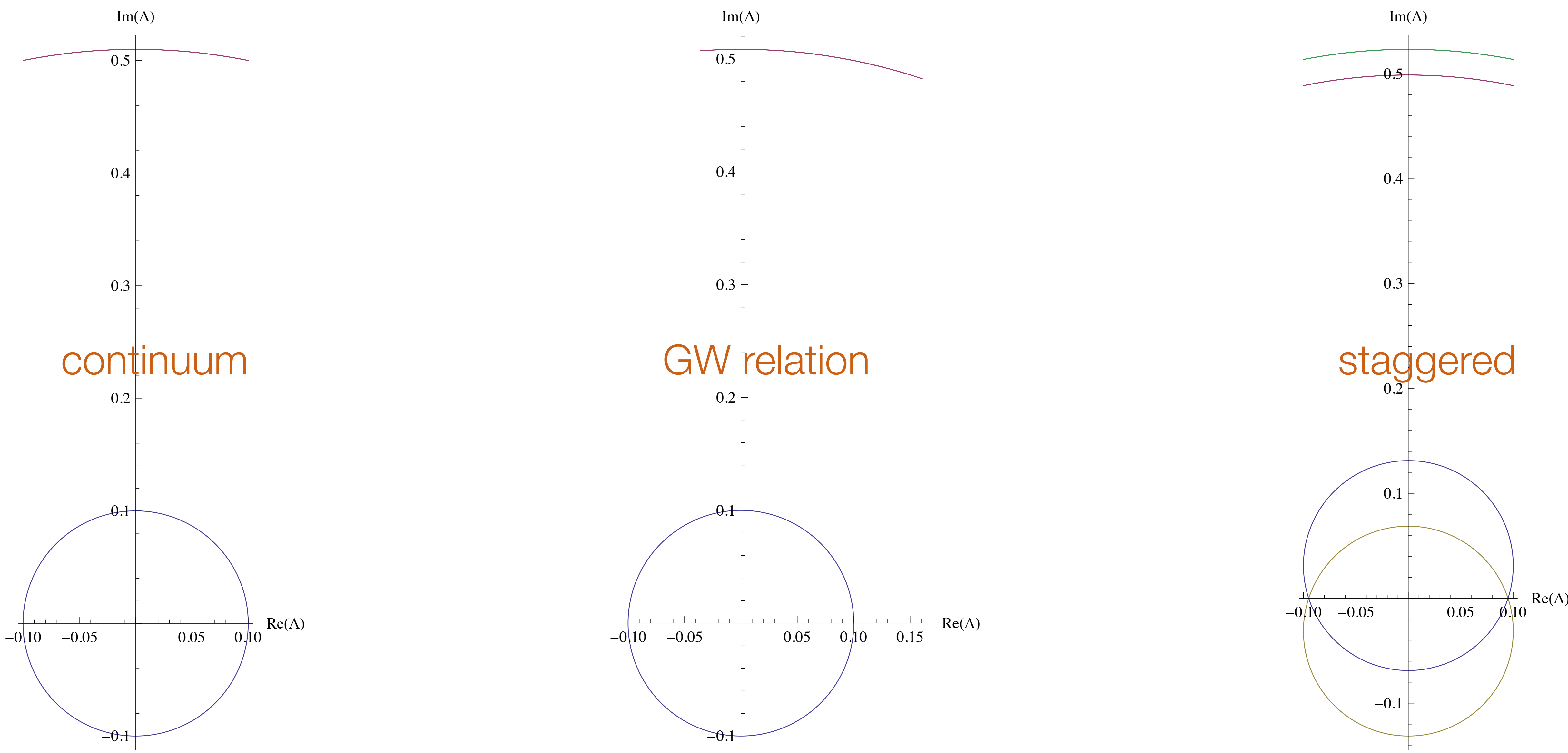
- Narayanan & Neuberger pursued an approach to obtaining the phase in a chiral gauge theory, which involved an infinite matrix:
  - chiral gauge theory: right- and left-handed fermions in different irreps, so determinant can have a gauge-field dependent phase.
- Also based on Wilson fermions.
- If you give up on the phase, the construction simplifies and suggests the Dirac operator [Neuberger]:

$$\not{D} = \frac{1}{a}\left(1 - \frac{1 - a\not{D}_{\mathrm{W}}}{|1 - a\not{D}_{\mathrm{W}}|}\right)$$

- Can verify (as exercise) that this operator satisfies the GW relation.

- Lüscher symmetry seems to violate *CP*, transforming ψ and $\bar{\psi}$ differently.
- Considering Lüscher and his charge conjugate, Mandula uncovered a huge symmetry group, in which *CP* acts as an automorphism.
- Extra generators vanish in the spacetime continuum limit,
  - but do not go away in the limit of a spatial lattice with continuous time,
  - so GW-satisfying operators do not have a transfer operator or Hamiltonian.
- This topic seems never to be discussed (9 citations for Mandula's paper).

- Dependence on θ of eigenvalues adding a mass term $me^{i\gamma_5\theta}$ (left-and right-handed modes feel topology in equal and opposite way):



- All three behave qualitatively the same, suggesting correct sensitivity to topologically non-trivial gauge fields.

Questions?

Solutions to Some Exercises

Correlation Functions

- One idea: insert complete sets of states of the Hamiltonian (or transfer operator):

$$Z = \mathrm{Tr}\, e^{-\hat{H}\tau} = \sum_n \langle n|e^{-\hat{H}\tau}|n\rangle = \sum_n e^{-E_n\tau} \stackrel{\text{large }\tau}{\rightarrow} e^{-E_0\tau}$$

$$\langle Q(x)\rangle = \frac{1}{Z}\mathrm{Tr}\, Q(\hat{x})e^{-\hat{H}\tau} = \frac{1}{Z}\sum_n e^{-E_n\tau}\langle n|Q(\hat{x})|n\rangle \stackrel{\text{large }\tau}{\rightarrow} \langle 0|Q(\hat{x})|0\rangle$$

$$\begin{aligned}
\langle Q_1(t_1)Q_2(t_2)\rangle_c = {} & \frac{1}{Z}\sum_{nn'}\langle n|e^{-\hat{H}(\tau-t_1)}\hat{Q}_1(t_1)|n'\rangle\langle n'|e^{-\hat{H}(t_1-t_2)}\hat{Q}_2(t_2)e^{-\hat{H}t_2}|n\rangle \\
& -\frac{1}{Z}\sum_n\langle n|e^{-\hat{H}(\tau-t_1)}\hat{Q}_1(t_1)e^{-\hat{H}t_1}|n\rangle\frac{1}{Z}\sum_{n'}\langle n'|e^{-\hat{H}(\tau-t_2)}\hat{Q}_2(t_2)e^{-\hat{H}t_2}|n'\rangle \\
& \stackrel{\text{large }\tau}{\rightarrow} \sum_{n'\neq 0}\langle 0|\hat{Q}_1(t_1)|n'\rangle\langle n'|\hat{Q}_2(t_2)|0\rangle e^{-(E_{n'}-E_0)(t_1-t_2)} + \sum_{n\neq 0}\langle 0|\hat{Q}_2(t_2)|n\rangle\langle n|\hat{Q}_1(t_1)|0\rangle e^{-(E_n-E_0)(\tau+t_2-t_1)} \\
& \stackrel{\text{large }\tau,(t_1-t_2)}{\rightarrow} \langle 0|\hat{Q}_1(t_1)|1\rangle\langle 1|\hat{Q}_2(t_2)|0\rangle e^{-(E_1-E_0)(t_1-t_2)} + \langle 0|\hat{Q}_2(t_2)|1\rangle\langle 1|\hat{Q}_1(t_1)|0\rangle e^{-(E_1-E_0)(\tau+t_2-t_1)}
\end{aligned}$$

- Try this for a three-point function, too.

Dirac Matrix Conventions

- Euclidean indices, $1, 2, 3, 4$; metric $\delta^{\mu\nu} = \mathrm{diag}(1,1,1,1)$.
- Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$; all γ^μ are Hermitian.
- Chiral $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$; also Hermitian.
- Spin $[\gamma^\mu, \gamma^\nu] = -2i\sigma^{\mu\nu}$; $\sigma^{\mu\nu}$ is also Hermitian.
- Continue back to Minkowski with $g^{\mu\nu} = \mathrm{diag}(-1,1,1,1)$,
  - and time components $x^0 = -ix^4$; $\gamma^0 = -i\gamma^4$; $p^0 = E = -ip^4$.

$$\gamma^4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix},$$

$$\gamma^5 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

return