Lecture : Perturbative calculations Y. Sumino

PLAN

Sec. 1. Pre-modern approach to bound state problem ← White board

Sec. 2. A modern computational technology (technical) } slides

Sec. 3. Understanding quark mass and interquark force

Sec. 1. Pre-modern approach : from Feynman diagrams to bound state theory

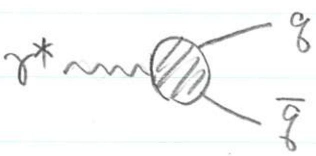
§1-1 Leading Coulomb Singularities

(Refs. hep-ph/9910424 Sec. 3
PRD43 (1991) 1500 Sec. 3)

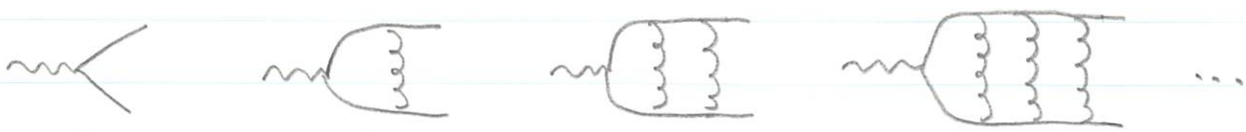
When q and $\bar{q}$ form a non-relativistic (NR) bound state, naive PT breaks down.

e^+e^-, $c\bar{c}$, $b\bar{b}$, $t\bar{t}$, ...
(Ps) (J/ψ) (Υ)

Amplitude $\gamma^* \to q\bar{q}$

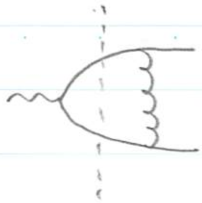


$$\beta = \sqrt{1 - \frac{4m^2}{s}} \ll 1$$



1 α_s/β $(\alpha_s/\beta)^2$ $(\alpha_s/\beta)^3$

Breakdown of PT at $\beta \lesssim \alpha_s$ even if $\alpha_s \ll 1$.

Im  $= \int d\Phi_2(q\bar{q})$  $\times$ 

cf. Cutkosky rules

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\frac{\beta}{16\pi}\int d\cos\theta \times 1 \times \alpha_s/\beta^2 \sim \alpha_s/\beta.$$

$$p_1^\mu = E\,(1,\ \beta\sin\theta\cos\phi,\ \beta\sin\theta\sin\phi,\ \beta\cos\theta) \qquad E = \frac{\sqrt{s}}{2}$$

$$p_1'^\mu = E\,(1,\ 0,\ 0,\ \beta)$$

$$k^\mu = E\,(0,\ \beta\sin\theta\cos\phi,\ \beta\sin\theta\sin\phi,\ \beta(\cos\theta-1))$$

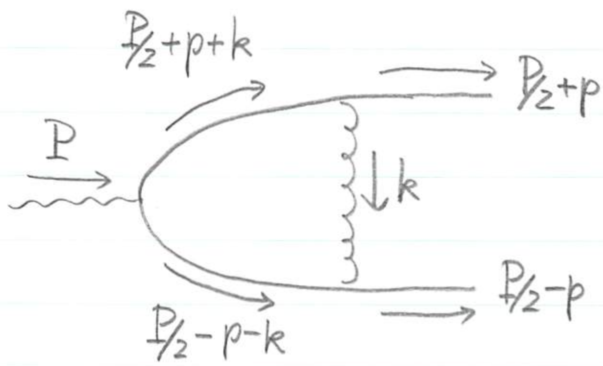
$$\Rightarrow \quad k^2 = -|\vec{k}|^2 \propto -E^2\beta^2$$

Analyticity of diagram implies real part is also $\sim \alpha_s/\beta$.

<u>Relevant kinematics</u>

$$\frac{\sqrt{s}}{2} - m \sim \beta^2$$

$$|\vec{p}| \sim \beta$$



$$P^\mu = (\sqrt{s}, \vec{0}), \quad p^\mu = (0, \vec{p})$$

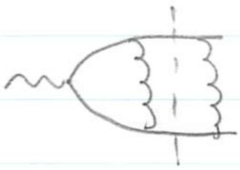
$$\begin{cases} k^0 \sim \beta^2 \\ |\vec{k}| \sim \beta \end{cases} \rightarrow k^2 \sim \beta^2$$

q or $\bar{q}$ propagator $\sim 1/\beta^2$

$$\because \left(\frac{P^\mu}{2} \pm p^\mu \pm k^\mu\right)^2 - m^2 = \left(\frac{\sqrt{s}}{2} \pm k^0\right)^2 - |\vec{p}+\vec{k}|^2 - m^2$$

 $$= \int \underbrace{\frac{d^4k}{(2\pi)^4}}_{\beta^2\times\beta^3} \frac{1}{\beta^2}\frac{1}{\beta^2}\frac{\alpha_s}{\beta^2} \sim \frac{\alpha_s}{\beta}$$

2-loop ladder diagram

 $= \displaystyle\int d\Phi_2(q\bar{q}) \times$  $\times$  $\sim (\alpha_s/\beta)^2$

$\beta \qquad \alpha_s/\beta \qquad \alpha_s/\beta^2$

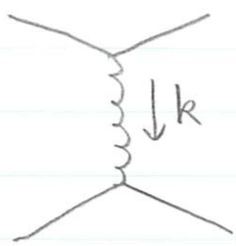
Ladder diagrams contain leading Coulomb singularities $\sim(\alpha_s/\beta)^n$.
On the other hand, other diagrams (crossed ladder diagrams) contain only sub-leading singularities $\alpha_s^{n+\ell}/\beta^n \ (\ell \geq 1)$.

[Physical meaning ?]

Gauge independence

$$\underbrace{\mathcal{M}^{(\text{full})}(\alpha_s, \beta)}_{\text{gauge indep.}} = \sum_n \underbrace{C_n}_{\text{gauge indep.}} (\alpha_s/\beta)^n + (\text{non-leading})$$

Check:



$$= \bar{u}_f \gamma^\mu u_i \frac{-i}{k^2+i\varepsilon}\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right] \bar{v}_i \gamma^\nu v_f$$

$$u_i, u_f = \begin{bmatrix} O(1) \\ O(\beta) \end{bmatrix}, \quad v_i, v_f = \begin{bmatrix} O(\beta) \\ O(1) \end{bmatrix} \qquad \gamma^0 = \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix} \quad \gamma^i = \begin{pmatrix} & \sigma^i \\ -\sigma^i & \end{pmatrix}$$

$$\bar{u}_f \gamma^\mu u_i = \delta^{\mu 0}\, \bar{u}_f \gamma^0 u_i + O(\beta)$$

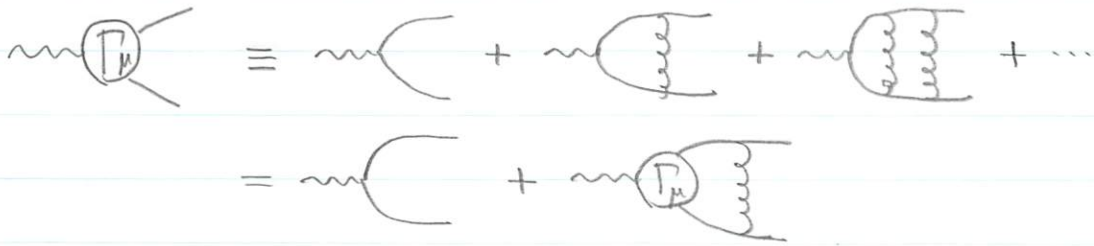
$$\bar{v}_i \gamma^\nu v_f = \delta^{\nu 0}\, \bar{v}_i \gamma^0 v_f + O(\beta)$$

$$k^2 = -|\vec{k}|^2, \quad k^0 = 0$$

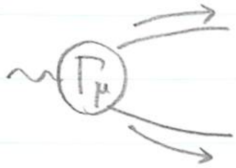
$$= \bar{u}_f \gamma^0 u_i \frac{i}{|\vec{k}|^2} \bar{v}_i \gamma^0 v_f + O(\beta)$$

Leading term is gauge-indep. and Coulomb potential.

§1-2 Resummation of Leading Singularities



Keep only $(\alpha_s/\beta)^n$ on both sides.



$$\Gamma^\mu(E, p^0, \vec{p}) = \quad \begin{aligned} &\tfrac{P}{2}+p = \left(m+\tfrac{E}{2}+p^0, \vec{p}\right) \\ &\tfrac{P}{2}-p = \left(m+\tfrac{E}{2}-p^0, -\vec{p}\right) \end{aligned}$$

$$P^\mu = (2m+E, \vec{0})$$

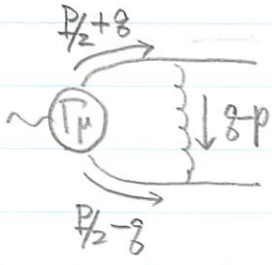
$$S_F\left(\tfrac{P}{2}+p\right) = \frac{i(\not{P}/2+\not{p}+m)}{(P/2+p)^2-m^2+i0} = \frac{1+\gamma^0}{2}\underbrace{\frac{i}{E/2+p^0-\vec{p}^2/2m+i0}}_{\sim\beta^2}\left[1+O(\beta)\right]$$

$$S_F\left(-\tfrac{P}{2}+p\right) = \cdots = \frac{1-\gamma^0}{2}\frac{i}{E/2-p^0-\vec{p}^2/2m+i0}\left[1+O(\beta)\right]$$

Self-consistent eq.

$$\Gamma^\mu(E,p^0,\vec{p}) = \gamma^\mu + (ig)^2 C_F \int\frac{d^4q}{(2\pi)^4}\frac{i}{E/2+q^0-\vec{q}^2/2m+i0}\frac{i}{E/2-q^0-\vec{q}^2/2m+i0}$$
$$\times\gamma^0\frac{1+\gamma^0}{2}\Gamma^\mu(E,q^0,\vec{q})\frac{1-\gamma^0}{2}\gamma^0\frac{i}{|\vec{p}-\vec{q}|^2}$$

$$T_F^a T_F^a = C_F \mathbb{1}, \qquad C_F = \frac{N_c^2-1}{2N_c}$$



Sandwich both sides between $\frac{1+\gamma^0}{2}$ and $\frac{1-\gamma^0}{2}$

Let $\frac{1+\gamma^0}{2}\Gamma^\mu(E,p^0,\vec{p})\frac{1-\gamma^0}{2} = \frac{1+\gamma^0}{2}\gamma^\mu\frac{1-\gamma^0}{2}\times\Gamma(E,p^0,\vec{p})$, then

$$\Gamma(E,p^0,\vec{p}) = 1 - i\cdot 4\pi\alpha_s\cdot C_F\int\frac{d^4q}{(2\pi)^4}\frac{1}{E/2+q^0-\vec{q}^2/2m+i0}\,\frac{1}{E/2-q^0-\vec{q}^2/2m+i0}\,\frac{1}{|\vec{p}-\vec{q}|^2}\,\Gamma(E,q^0,\vec{q})$$

RHS is indep. of p^0 ($\because k^0\sim\beta^2 \ll |\vec{k}|\sim\beta$) $\Rightarrow$ LHS $\Gamma(E,\vec{p})$

On RHS $\Gamma(E,q^0,\vec{q}) \to \Gamma(E,\vec{q})$, then integrate $\int dq^0$

$$\Gamma(E,\vec{p}) = 1 - C_F\cdot 4\pi\alpha_s\int\frac{d^3\vec{q}}{(2\pi)^3}\frac{1}{E-\vec{q}^2/m+i0}\,\frac{1}{|\vec{p}-\vec{q}|^2}\,\Gamma(E,\vec{q})$$

Define $\tilde{G}(E,\vec{p})$ by $\Gamma(E,\vec{p}) = -(E-\vec{p}^2/m+i0)\,\tilde{G}(E,\vec{p})$

$$-(E-\frac{\vec{p}^2}{m}+i0)\,\tilde{G}(E,\vec{p}) - C_F\cdot 4\pi\alpha_s\int\frac{d^3\vec{q}}{(2\pi)^3}\frac{1}{|\vec{p}-\vec{q}|^2}\,\tilde{G}(E,\vec{q}) = 1$$

Fourier tr.

$$-(E+i0+\frac{\vec{\nabla}^2}{m})\,G(E,\vec{x}) - C_F\frac{\alpha_s}{r}\,G(E,\vec{x}) = \delta^3(\vec{x})$$

Let $\hat{H} = \frac{\hat{\vec{p}}^2}{m} - C_F\frac{\alpha_s}{r}$, then

$$G(E,\vec{x}) = \langle\vec{x}|\,\frac{-1}{E-\hat{H}+i0}\,|\vec{y}=\vec{0}\rangle = -\sum\!\!\!\!\!\!\!\int_n \frac{\psi_n(\vec{x})\,\psi_n^*(\vec{y}=\vec{0})}{E-E_n+i0}$$

In summary,

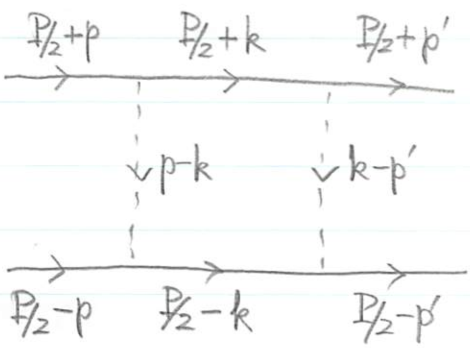
 $\approx -\left(\frac{1+\gamma^0}{2}\gamma^\mu\frac{1-\gamma^0}{2}\right)(E-\frac{\vec{p}^2}{m}+i0)\,\tilde{G}(E,\vec{p})$

$$\tilde{G}(E,\vec{p}) = \langle\vec{p}|\,\frac{-1}{E-\hat{H}+i0}\,|\vec{x}=\vec{0}\rangle$$

$$= -\sum\!\!\!\!\!\!\!\int_n \frac{\phi_n(\vec{p})\,\psi_n^*(\vec{0})}{E-E_n+i0}$$

Q. Why crossed diagrams have no $(\alpha_s/\beta)^n$?

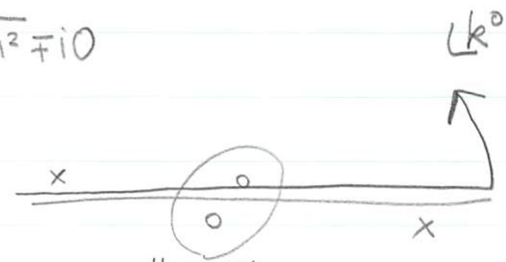
Easier in Coulomb gauge.



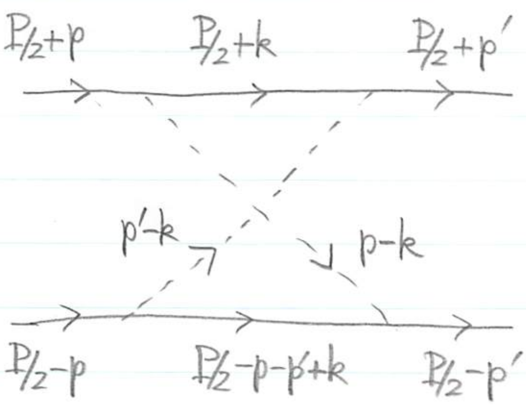
$$\int d^4k \frac{N}{[(P/2+k)^2-m^2+i0][(P/2-k)^2-m^2+i0]\,|\vec p-\vec k|^2\,|\vec p'-\vec k|^2}$$

$$P^0/2+k^0=\pm\sqrt{\vec k^2+m^2}\mp i0$$

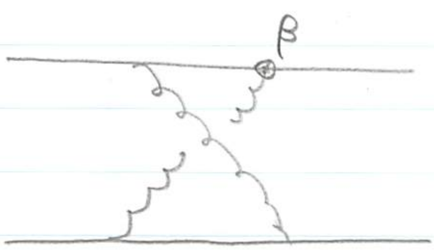
$$P^0/2-k^0=\pm\sqrt{\vec k^2+m^2}\mp i0$$



all particles almost on-shell



Transverse gluon



$\gamma^i \sim O(\beta)$ cf. Lorentz force $\vec F = q\vec v\times\vec B$

$$D_{\mu\nu}=\delta_{\mu0}\delta_{\nu0}\frac{i}{|\vec k|^2}+\left(\delta_{ij}-\frac{k_ik_j}{|\vec k|^2}\right)\frac{i}{k^2+i0}$$

Coulomb $\mu=i,\ \nu=j$ Transverse

Extra suppression of β

$$D_{00}(k)=\frac{i}{|\vec k|^2} \xleftrightarrow{\text{F.T.}} \delta(t)\times\frac{1}{r}$$

instantaneous