I. NON-RELATIVISTIC QED

1. Consider the following Lagrangian density that describes a non-relativistic fermion (electron) interacting with photons

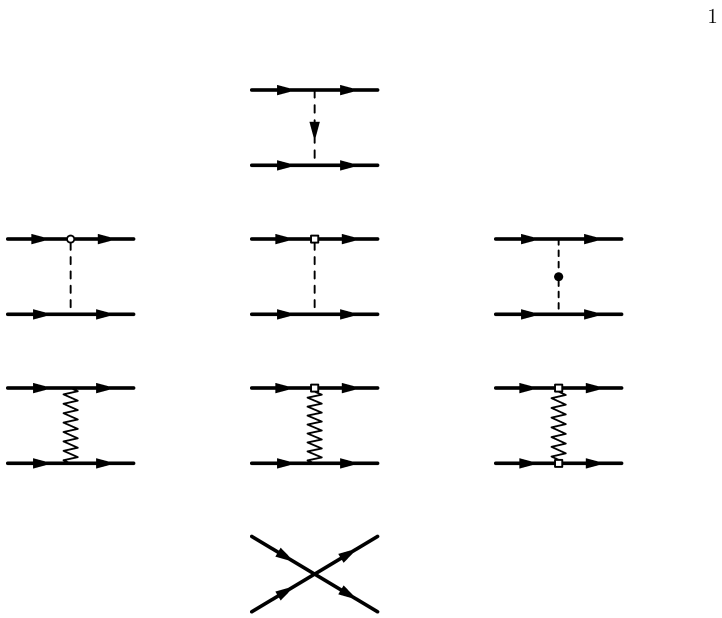
$$\mathcal{L} = \varphi^\dagger(t,\mathbf{x})\Big\{ iD_0 + \frac{c_k}{2m}\mathbf{D}^2 + \frac{c_F}{2m}\boldsymbol{\sigma}\cdot e\mathbf{B} + \frac{c_D}{8m^2}\left(\mathbf{D}\cdot e\mathbf{E} - e\mathbf{E}\cdot\mathbf{D}\right)$$
$$+i\,\frac{c_S}{8m^2}\boldsymbol{\sigma}\cdot\left(\mathbf{D}\times e\mathbf{E} - e\mathbf{E}\times\mathbf{D}\right)\Big\}\varphi(t,\mathbf{x})\,, \tag{1}$$

where $iD_0 = i\partial_0 - eA_0$, $i\mathbf{D} = i\boldsymbol{\nabla} + e\mathbf{A}$ is the covariant derivative.
a) Obtain the Feynman rules of this Lagrangian.
b) Compute in the low energy limit the process: $e^-\gamma \to e^-\gamma$.

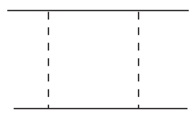
2. Consider the following Lagrangian density that describes a non-relativistic electron and positron interacting with photons

$$\begin{aligned}\mathcal{L} \;=\; &\varphi^\dagger(t,\mathbf{x})\Big\{ iD_0 + \frac{c_k}{2m}\mathbf{D}^2 + \frac{c_F}{2m}\boldsymbol{\sigma}\cdot e\mathbf{B} + \frac{c_D}{8m^2}\left(\mathbf{D}\cdot e\mathbf{E} - e\mathbf{E}\cdot\mathbf{D}\right)\\
&+i\,\frac{c_S}{8m^2}\boldsymbol{\sigma}\cdot\left(\mathbf{D}\times e\mathbf{E} - e\mathbf{E}\times\mathbf{D}\right)\Big\}\varphi(t,\mathbf{x})\\
&+\chi_c^\dagger(t,\mathbf{x})\Big\{ iD_0^c + \frac{c_k}{2m}\mathbf{D}_c^2 - \frac{c_F}{2m}\boldsymbol{\sigma}\cdot e\mathbf{B} - \frac{c_D}{8m^2}\left(\mathbf{D}_c\cdot e\mathbf{E} - e\mathbf{E}\cdot\mathbf{D}_c\right)\\
&-i\,\frac{c_S}{8m^2}\boldsymbol{\sigma}\cdot\left(\mathbf{D}_c\times e\mathbf{E} - e\mathbf{E}\times\mathbf{D}_c\right)\Big\}\chi_c(t,\mathbf{x})\\
&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{d_2}{m^2}F_{\mu\nu}D^2F^{\mu\nu}\\
&-\frac{d_s}{m^2}\psi^\dagger\psi\chi_c^\dagger\chi_c(t,\mathbf{x}_2) + \frac{d_v}{m^2}\psi^\dagger\boldsymbol{\sigma}\psi\chi_c^\dagger\boldsymbol{\sigma}\chi_c(t,\mathbf{x}_2)\,,\end{aligned} \tag{2}$$

where $iD_0^c = i\partial_0 + eA_0$, $i\mathbf{D}^c = i\boldsymbol{\nabla} - e\mathbf{A}$.
a) Compute the following set of diagrams where the dashed and zigzag lines represent the A_0 and $\mathbf{A}$ fields respectively, while the continuous lines represent the fermion and antifermion fields. The first diagram would give the Coulomb potential. For the A_0 the circle is the vertex proportional to c_D, the square to c_S (spin dependent) and the dashed dot to d_2 (the vacuum polarization), while for $\mathbf{A}$ the square is the vertex proportional to c_F and the other vertex appear from the covariant derivative in the kinetic term. The last diagram is proportional to d_s and d_v.



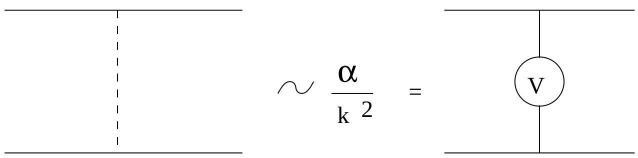
b) Relate the following diagram, where the dashed lines correspond to a longitudinal photon, with the one loop diagram that appears in the next exercise. Consider this scattering process in the center of mass frame, and work in the Coulomb gauge.

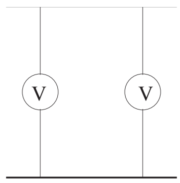


3. Consider the following Lagrangian

$$L = \int d^3\mathbf{x} \left(\psi^\dagger \left\{ i\partial^0 + \frac{\boldsymbol{\nabla}^2}{2m_1} \right\} \psi + \chi_c^\dagger \left\{ i\partial^0 + \frac{\boldsymbol{\nabla}^2}{2m_2} \right\} \chi_c \right.$$
$$\left. - \int d^3\mathbf{x}_1 d^3\mathbf{x}_2 \psi^\dagger(t,\mathbf{x}_1)\chi_c^\dagger(t,\mathbf{x}_2) V(\mathbf{x}) \chi_c(t,\mathbf{x}_2)\psi(t,\mathbf{x}_1) \right),$$

where $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$, ψ is a fermion and χ_c an anti-fermion.
Compute the following diagrams in the center of mass frame, and, for the first one, show its relation with the left-hand side:





4. Compute the transition probability (decay rate) of the Hydrogen atom in a quantum state n to another state $n' < n$ plus one photon. The physics of this process can be described by the following Lagrangian:

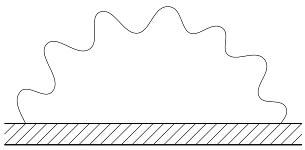
$$L = \int d^3\mathbf{x} \varphi^\dagger(t,\mathbf{x}) \left(i\partial_0 + \frac{\mathbf{D}^2}{2m} + \frac{\alpha}{|\mathbf{x}|} \right) \varphi(t,\mathbf{x}) - \int d^3\mathbf{x} \frac{1}{4} F_{\mu\nu}F^{\mu\nu}, \qquad (3)$$

where $\varphi(x)$ describes the Hydrogen atom field and $i\mathbf{D} = i\boldsymbol{\nabla} + e\mathbf{A}$ is the covariant derivative (hint: to simplify the problem use the fact that the transition energy is much smaller than the inverse Bohr radius).

5. Redo the previous exercise with the Lagrangian

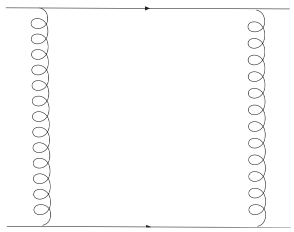
$$L = \int d^3\mathbf{x}\varphi^\dagger(t,\mathbf{x})\left(i\partial_0 + \frac{\boldsymbol{\nabla}^2}{2m} + \frac{\alpha}{|\mathbf{x}|} + e\mathbf{x}\cdot\mathbf{E}\right)\varphi(t,\mathbf{x}) - \int d^3\mathbf{x}\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \quad (4)$$

and also compute the logarithmic dependent contribution to the energy shift of the following diagram



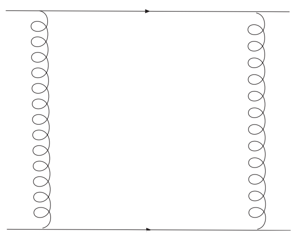
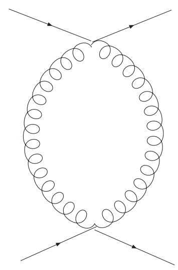
II. NON-RELATIVISTIC QCD

1. 1) Compute the logarithmic divergent term of the following diagram in QCD with the quarks at rest $P^\mu = (m, \mathbf{0})$ and deduce the associated contribution to d_{ss}, d_{vs}, d_{sv} and d_{vv}.

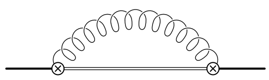


2. Obtain the Feynman rules of NRQCD at $\mathcal{O}(1/m)$.

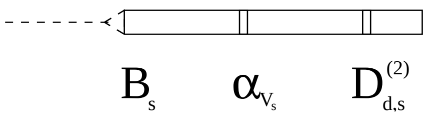
3. Obtain the logarithmic behavior of the following diagrams (plus crossed) of $\mathcal{O}(1/m^2)$ in NRQED/NRQCD with static heavy quark propagators. Work in the Coulomb gauge. In the first diagram one of the gluons is transverse and the other is longitudinal. In the second diagram both gluons are transverse. Deduce the associated corrections to the potential.

4. Compute the ultraviolet (logarithmic) behavior of the following diagram in pNRQCD. Absorb those divergences in the potentials and deduce the ultrasoft renormalization group equation of the potentials.



5. Compute the logarithmic divergent term of the following diagram in pNRQCD and



obtain its contribution to the renormalization group equation of

$$\nu_p \frac{d}{d\nu_p} B_s \,. \tag{5}$$

6. Using that

$$\mathrm{Im} \int_0^\infty \mathrm{d}t \, e^{-t/\alpha_s} \, \delta B[m_{\mathrm{OS}}](t) \sim \Lambda_{QCD} \tag{6}$$

obtain the coefficients b and c_1 if (we define $u = \frac{\beta_0 t}{4\pi}$)

$$\delta B[m_{\mathrm{OS}}](t(u)) = N_m \nu \frac{1}{(1-2u)^{1+b}} \left(1 + c_1(1-2u) + c_2(1-2u)^2 + \cdots\right). \tag{7}$$