NON-RELATIVISTIC EFFECTIVE FIELD THEORIES: RENORMALIZATION GROUP

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Motivation

Resummation of logarithms in Quantum Field Theories (a long tale)

Sudakov logarithms
Resummation of Logarithms in Deep Inelastic scattering
Resummation of logarithms in HQET
Electroweak logarithms
and so on ...

BUT!!!

WHAT ABOUT THE FIRST QUANTUM-FIELD-THEORY LOG?

THE LAMB SHIFT

$$\delta E \sim m\alpha^4 + m\alpha^5 \ln\alpha + (???)m\alpha^6 \ln^2\alpha + \cdots$$

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Summing logs in non-relativistic systems

Large logs understood as ratios of scales: $\ln(mv/m) \sim \ln\alpha$, $\ln(mv^2/(mv)) \sim \ln\alpha$.

Resummation of logs: $(\alpha \ln)^n$.

$$
\begin{aligned}
\delta E &\sim m\alpha^4 + m\alpha^5 \ln\alpha + m\alpha^6 \ln^2\alpha + \cdots \\
&+ m\alpha^5 + m\alpha^6 \ln\alpha + m\alpha^7 \ln^2\alpha + m\alpha^8 \ln^3\alpha + \cdots \\
\Gamma(V_Q(nS) \to e^+e^-) &\sim m\alpha^3(1 + \alpha^2 \ln\alpha + \alpha^3 \ln^2\alpha + \cdots) \\
\Gamma(P_Q(nS) \to \gamma\gamma) &\sim m\alpha^3(1 + \alpha^2 \ln\alpha + \alpha^3 \ln^2\alpha + \cdots) \\
\frac{\Gamma(V_Q(nS) \to e^+e^-)}{\Gamma(P_Q(nS) \to \gamma\gamma)} &\sim 1 + \alpha^2 \ln\alpha + \alpha^3 \ln^2\alpha + \cdots \\
&\quad +\alpha^3 \ln\alpha + \alpha^4 \ln^2\alpha + \cdots
\end{aligned}
$$

Relevant for...

Determination of the bottom and charm masses

Spectroscopy: Hyperfine splitting of heavy quarkonium: η_b, B_c, η_c, ...

Decays of heavy quarkonium

Heavy quarkonium sum rules

t-$\bar{t}$ production near threshold: m_t, α_s, Higgs-top coupling

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&+ m\alpha^5 + m\alpha^6 \ln\alpha + m\alpha^7 \ln^2\alpha + m\alpha^8 \ln^3\alpha + \cdots \\
\Gamma(V_Q(nS) \to e^+e^-) &\sim m\alpha^3(1 + \alpha^2 \ln\alpha + \alpha^3 \ln^2\alpha + \cdots) \\
\Gamma(P_Q(nS) \to \gamma\gamma) &\sim m\alpha^3(1 + \alpha^2 \ln\alpha + \alpha^3 \ln^2\alpha + \cdots) \\
\frac{\Gamma(V_Q(nS) \to e^+e^-)}{\Gamma(P_Q(nS) \to \gamma\gamma)} &\sim 1 + \alpha^2 \ln\alpha + \alpha^3 \ln^2\alpha + \cdots \\
&\quad +\alpha^3 \ln\alpha + \alpha^4 \ln^2\alpha + \cdots
\end{aligned}
$$

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Renormalization group in NRQCD (LL) (Soft running)

Aim: to obtain the running of the NRQCD matching coefficients: $\left(\alpha_s \ln \frac{m}{\nu}\right)^n$

Relevant for:

- pNRQCD in the perturbative regime
- pNRQCD in the nonperturbative regime
- "Standard" NRQCD

$$\mathcal{L}_{NRQCD} = \bar{\Psi} i\gamma^0 D_0 \Psi + \bar{\Psi}\left\{ \frac{\mathbf{D}^2}{2m} + c_F\, g \frac{\mathbf{\Sigma}\cdot\mathbf{B}}{2m} + c_D\, g \frac{\gamma^0 \left(\mathbf{D}\cdot\mathbf{E} - \mathbf{E}\cdot\mathbf{D}\right)}{8m^2} \right.$$

$$\left. + i c_S\, g \frac{\gamma^0 \mathbf{\Sigma}\cdot\left(\mathbf{D}\times\mathbf{E} - \mathbf{E}\times\mathbf{D}\right)}{8m^2} + \frac{\mathbf{D}^4}{8m^3} \right\}\Psi$$

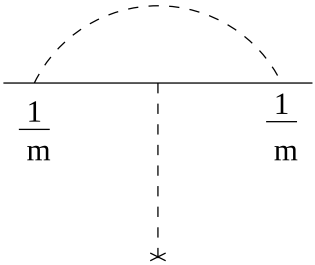
$$-\frac{1}{4} c_1 F_{\mu\nu}F^{\mu\nu} + \frac{c_2}{m^2} g F_{\mu\nu} D^2 g F^{\mu\nu} + \frac{c_3}{m^2} g^3 f_{ABC} F^A_{\mu\nu} F^B_{\mu\alpha} F^C_{\nu\alpha}$$

$$\delta\mathcal{L}_{NRQCD} = \frac{d_{ss}}{m_1 m_2}\psi_1^\dagger\psi_1\chi_2^\dagger\chi_2 + \frac{d_{sv}}{m_1 m_2}\psi_1^\dagger\boldsymbol{\sigma}\psi_1\chi_2^\dagger\boldsymbol{\sigma}\chi_2$$

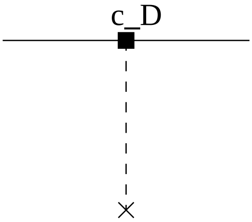
$$+\frac{d_{vs}}{m_1 m_2}\psi_1^\dagger \mathrm{T}^a\psi_1\chi_2^\dagger\mathrm{T}^a\chi_2 + \frac{d_{vv}}{m_1 m_2}\psi_1^\dagger\mathrm{T}^a\boldsymbol{\sigma}\psi_1\chi_2^\dagger\mathrm{T}^a\boldsymbol{\sigma}\chi_2\,.$$

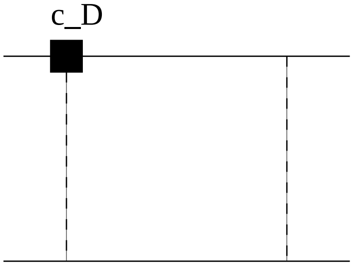
Typically, $c_i \sim 1 + \sum_n A_n \left(\alpha_s \ln \frac{m}{\nu}\right)^n$ $\qquad d_i \sim \alpha_s \left(1 + \sum_n B_n \left(\alpha_s \ln \frac{m}{\nu}\right)^n\right)$

$\nu_p \gg |\mathbf{p}|$: quark-antiquark relative three-momentum.
$\nu_s \gg |\mathbf{k}|$: gluon three-momentum, transfer momentum between the quark and antiquark.
$m \gg \nu_p \sim \nu_s$
Matching coefficients: $c(\nu_s)$, $d(\nu_s, \nu_p)$
LL $\rightarrow c(\nu_s)$, $d(\nu_s)$
Running ν_s LL: HQET; $1/m$ expansion, $\frac{i}{q^0 + i\epsilon}$

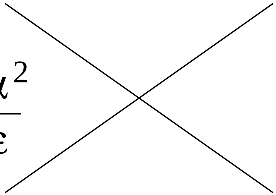


$$\sim \frac{1}{m^2} \frac{\alpha}{\epsilon}$$





$$\sim \frac{c\_D}{m^2} \frac{\alpha^2}{\epsilon}$$



$$\nu_s \frac{d}{d\nu_s} c_D = \frac{\alpha_s}{4\pi}\left[\frac{4C_A}{3}c_D - \left(\frac{2C_A}{3}+\frac{32C_f}{3}\right)c_k^2 - \frac{10C_A}{3}c_F^2 + \frac{8T_F n_f}{3}c_1^{hl}\right],$$

$$\begin{aligned}
\nu_s \frac{d}{d\nu_s} d_{ss} &= -2C_f\left(C_f - \frac{C_A}{2}\right)\alpha_s^2 c_k^2\,, \\
\nu_s \frac{d}{d\nu_s} d_{sv} &= 0\,, \\
\nu_s \frac{d}{d\nu_s} d_{vs} &= 4\left(C_f - C_A\right)\alpha_s^2 c_k^2 + \frac{3}{2}\alpha_s^2 C_A c_D\,, \\
\nu_s \frac{d}{d\nu_s} d_{vv} &= -\frac{C_A}{2}\alpha_s^2 c_F^2\,.
\end{aligned}$$

We define $z = \left[\frac{\alpha_s(\nu_s)}{\alpha_s(m)}\right]^{\frac{1}{\beta_0}} \simeq 1 - 1/(2\pi)\alpha_s(\nu_s)\ln(\frac{\nu_s}{m})$, $\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_f$

$$\begin{aligned}
c_F(\nu_s) &= z^{-C_A}\,, \\
c_S(\nu_s) &= 2z^{-C_A} - 1\,, \\
c_D(\nu_s) &= \frac{9C_A}{9C_A + 8T_F n_f}\left\{ -\frac{5C_A + 4T_F n_f}{4C_A + 4T_F n_f} z^{-2C_A} + \frac{C_A + 16C_f - 8T_F n_f}{2(C_A - 2T_F n_f)} \right. \\
&\quad + \frac{-7C_A^2 + 32C_A C_f - 4C_A T_F n_f + 32C_f T_F n_f}{4(C_A + T_F n_f)(2T_F n_f - C_A)} z^{4T_F n_f/3 - 2C_A/3} \\
&\quad \left. + \frac{8T_F n_f}{9C_A}\left[z^{-2C_A} + \left(\frac{20}{13} + \frac{32}{13}\frac{C_f}{C_A}\right)\left[1 - z^{\frac{-13C_A}{6}}\right]\right]\right\}\,,
\end{aligned}$$

Bauer-Manohar

$$
\begin{aligned}
d_{ss}(\nu_s) &= d_{ss}(m) + 4C_f\left(C_f - \frac{C_A}{2}\right)\frac{\pi}{\beta_0}\alpha_s(m)\left[z^{\beta_0} - 1\right] , \\
d_{sv}(\nu_s) &= d_{sv}(m) , \\
d_{vs}(\nu_s) &= d_{vs}(m) - (C_f - C_A)\frac{8\pi}{\beta_0}\alpha_s(m)\left[z^{\beta_0} - 1\right] \\
&-\frac{27C_A^2}{9C_A + 8T_F n_f}\frac{\pi}{\beta_0}\alpha_s(m)\left\{-\frac{5C_A + 4T_F n_f}{4C_A + 4T_F n_f}\frac{\beta_0}{\beta_0 - 2C_A}\left(z^{\beta_0 - 2C_A} - 1\right)\right. \\
&+\frac{C_A + 16C_f - 8T_F n_f}{2(C_A - 2T_F n_f)}\left(z^{\beta_0} - 1\right) + \frac{-7C_A^2 + 32C_A C_f - 4C_A T_F n_f + 32C_f T_F n_f}{4(C_A + T_F n_f)(2T_F n_f - C_A)} \\
&\quad\times\frac{3\beta_0}{3\beta_0 + 4T_F n_f - 2C_A}\left(z^{\beta_0 + 4T_F n_f/3 - 2C_A/3} - 1\right) \\
&+\frac{8T_F n_f}{9C_A}\left[\frac{\beta_0}{\beta_0 - 2C_A}\left(z^{\beta_0 - 2C_A} - 1\right) + \left(\frac{20}{13} + \frac{32}{13}\frac{C_f}{C_A}\right)\right. \\
&\quad\left.\left.\times\left(\left[z^{\beta_0} - 1\right] - \frac{6\beta_0}{6\beta_0 - 13C_A}\left[z^{\beta_0 - \frac{13C_A}{6}} - 1\right]\right)\right]\right\} , \\
d_{vv}(\nu_s) &= d_{vv}(m) + \frac{C_A}{\beta_0 - 2C_A}\pi\alpha_s(m)\left\{z^{\beta_0 - 2C_A} - 1\right\} .
\end{aligned}
$$

Pineda

One equation for the soft running.

Renormalization group in pNRQCD (LL) (Ultrasoft running)

Aim: to obtain the running of the pNRQCD matching coefficients: $(\alpha_s \ln)^n$, $\alpha_s (\alpha_s \ln)^n$

Relevant for:

- Spectrum: heavy quarkonium and QED.
- Currents: electromagnetic decays.
- Currents: Normalization of bottomonium sum rules.
- Currents: Normalization of t-$\bar{t}$ production near threshold.

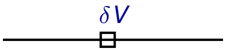
$$
\begin{aligned}
\mathcal{L}_{pNRQCD} &= \mathrm{Tr}\Big\{ S^\dagger \left(i\partial_0 - V_s^{(0)}(\mathbf{x}) \right) S + O^\dagger \left(iD_0 - V_o^{(0)}(\mathbf{x}) \right) O \Big\} \\
&+ gV_A(\mathbf{x}) \mathrm{Tr}\left\{ O^\dagger \mathbf{x}\cdot\mathbf{E}\, S + S^\dagger \mathbf{x}\cdot\mathbf{E}\, O \right\} + g\frac{V_B(\mathbf{x})}{2} \mathrm{Tr}\left\{ O^\dagger \mathbf{x}\cdot\mathbf{E}\, O + O^\dagger O \mathbf{x}\cdot\mathbf{E} \right\} \\
&- \mathrm{Tr}\Big\{ S^\dagger \left(\frac{\mathbf{p}^2}{m} + \sum_n \frac{V_s^{(n)}(\mathbf{x})}{m^n} \right) S - O^\dagger \left(\frac{\mathbf{p}^2}{m} + \sum_n \frac{V_o^{(n)}(\mathbf{x})}{m^n} \right) O \Big\},
\end{aligned}
$$

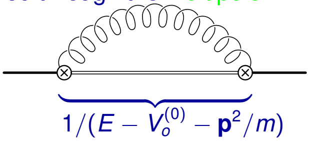
$$
V_s^{(0)} \equiv -C_F \frac{\alpha_{V_s}}{r}., \qquad \frac{V_s^{(1)}}{m} \equiv -\frac{C_F C_A D_s^{(1)}}{2mr^2}.
$$

$$
\begin{aligned}
\frac{V_s^{(2)}}{m^2} &= -\frac{C_F D_{1,s}^{(2)}}{2m^2} \left\{ \frac{1}{r}, \mathbf{p}^2 \right\} + \frac{C_F D_{2,s}^{(2)}}{2m^2} \frac{1}{r^3} \mathbf{L}^2 + \frac{\pi C_F D_{d,s}^{(2)}}{m^2} \delta^{(3)}(\mathbf{r}) \\
&+ \frac{4\pi C_F D_{S^2,s}^{(2)}}{3m^2} \mathbf{S}^2 \delta^{(3)}(\mathbf{r}) + \frac{3 C_F D_{LS,s}^{(2)}}{2m^2} \frac{1}{r^3} \mathbf{L}\cdot\mathbf{S} + \frac{C_F D_{S_{12},s}^{(2)}}{4m^2} \frac{1}{r^3} S_{12}(\hat{\mathbf{r}}),
\end{aligned}
$$

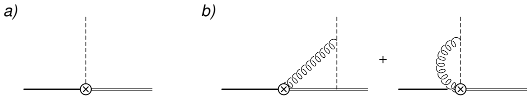
where $S_{12}(\hat{\mathbf{r}}) \equiv 3\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_1\, \hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$ and $\mathbf{S} = \boldsymbol{\sigma}_1/2 + \boldsymbol{\sigma}_2/2$.

$\nu_p \gg |\mathbf{p}|$: quark-antiquark relative three-momentum.
$\nu_{us} \gg |\mathbf{k}|$: gluon three-momentum.
$|\mathbf{p}| \gg \nu_{us} \gg \mathbf{p}^2/m$
Matching coefficients:
$\tilde{V}(d(\nu_p,\nu_s,m),c(\nu_s,m),\nu_s,\nu_{us},r) = \tilde{V}(\nu_p,m,\nu_{us},r) \equiv \tilde{V}(\nu_p,\nu_{us})\,.$
$\nu_s \frac{d}{d\nu_s}\tilde{V} = 0$; $\nu_s = 1/r$
LL: $\nu_p \frac{d}{d\nu_p}\tilde{V} = 0$
ν_{us}. The computation can be formally organized through the multipole expansion.



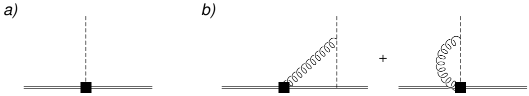


$V_A \mathbf{r} \cdot g\mathbf{E} \sim$



singlet–octet vertex

$d^{abc} V_B \mathbf{r} \cdot g\mathbf{E}^c \sim$



octet–octet vertex

LL: Pineda, Soto; NLL: Brambilla, Garcia-Tormo, Soto, Vairo

Corrections to the Green Function

$$G_s(E) = P_s \frac{1}{H - H_I - E} P_s = G_s^{(0)} + \delta G_s$$

From the potential:

$$\delta G_s \sim \frac{1}{H_s - E} \delta V \frac{1}{H_s - E}$$

From ultrasoft gluons:

$$\begin{aligned} \delta G_s &\sim \frac{1}{H_s - E} \int \frac{d^3\mathbf{k}}{(2\pi)^{D-1}} \mathbf{r} \frac{k}{k + H_o - E} \mathbf{r} \frac{1}{H_s - E} \\ &\sim \frac{1}{H_s - E} \mathbf{r} (H_o - E)^3 \left\{ \frac{1}{\epsilon} + \gamma + \ln \frac{(H_o - E)^2}{\nu^2} + C \right\} \mathbf{r} \frac{1}{H_s - E} \end{aligned}$$

$$
\begin{aligned}
\nu_{us}\frac{d}{d\nu_{us}}\alpha_{V_s} &= \nu_{us}\frac{d}{d\nu_{us}}\alpha_{V_o} = \frac{2}{3}\frac{\alpha_s}{\pi}V_A^2\left(\left(\frac{C_A}{2}-C_f\right)\alpha_{V_o}+C_f\alpha_{V_s}\right)^3, \\
\nu_{us}\frac{d}{d\nu_{us}}\alpha_s &= -\beta_0\frac{\alpha_s^2}{2\pi}, \\
\nu_{us}\frac{d}{d\nu_{us}}V_A &= 0\,.
\end{aligned}
$$

Soto, Pineda(LL)
Brambilla, Garcia-Tormo, Soto, Vairo(NLL); Pineda, Stahlhofen(NLL)

$$
\begin{aligned}
\nu_{us}\frac{d}{d\nu_{us}}C_A D_s^{(1)} &= \frac{16}{3}\frac{\alpha_s}{\pi}V_A^2 c_k\left[\left(\frac{C_A}{2}-C_f\right)\alpha_{V_o}+C_f\alpha_{V_s}\right] \\
&\quad\times\left[2C_f\alpha_{V_s}+\left(\frac{C_A}{2}-C_f\right)\alpha_{V_o}\right], \\
\nu_{us}\frac{d}{d\nu_{us}}D_{d,s}^{(2)} &= \frac{16}{3}\frac{\alpha_s}{\pi}V_A^2 c_k^2\left(\frac{C_A}{2}-C_f\right)\alpha_{V_o}, \\
\nu_{us}\frac{d}{d\nu_{us}}D_{1,s}^{(2)} &= \frac{8}{3}\frac{\alpha_s}{\pi}V_A^2 c_k^2\left[\left(\frac{C_A}{2}-C_f\right)\alpha_{V_o}+C_f\alpha_{V_s}\right],
\end{aligned}
$$

Pineda(LL, NLL)

RG equations within an strict expansion in α

$$\nu_{us}\frac{d}{d\nu_{us}}\alpha_{V_s} = \frac{2}{3}\frac{\alpha_s(\nu_{us})}{\pi}\left(\frac{C_A}{2}\right)^3\alpha_s^3(r^{-1}),$$

$$\nu_{us}\frac{d}{d\nu_{us}}\alpha_{V_o} = \frac{2}{3}\frac{\alpha_s(\nu_{us})}{\pi}\left(\frac{C_A}{2}\right)^3\alpha_s^3(r^{-1}),$$

$$\nu_{us}\frac{d}{d\nu_{us}}C_A D_s^{(1)} = \frac{16}{3}\frac{\alpha_s(\nu_{us})}{\pi}\frac{C_A}{2}\left(C_f+\frac{C_A}{2}\right)\alpha_s^2(r^{-1}),$$

$$\nu_{us}\frac{d}{d\nu_{us}}D_{1,s}^{(2)} = \frac{8}{3}\frac{\alpha_s(\nu_{us})}{\pi}\frac{C_A}{2}\alpha_s(r^{-1}),$$

$$\nu_{us}\frac{d}{d\nu_{us}}D_{d,s}^{(2)} = \frac{16}{3}\frac{\alpha_s(\nu_{us})}{\pi}\left(\frac{C_A}{2}-C_f\right)\alpha_s(r^{-1}).$$

Initial conditions ($\nu_{us} = 1/r$):

$$
\begin{aligned}
\alpha_{V_s}(r^{-1}) &= \alpha_{\rm s}(r^{-1})\left\{1 + (a_1 + 2\gamma_E\beta_0)\,\frac{\alpha_{\rm s}(r^{-1})}{4\pi}\right.\\
&\quad \left. + \left[\gamma_E\,(4a_1\beta_0 + 2\beta_1) + \left(\frac{\pi^2}{3} + 4\gamma_E^2\right)\beta_0^2 + a_2\right]\frac{\alpha_{\rm s}^2(r^{-1})}{16\,\pi^2}\right\},\\
D_s^{(1)}(r^{-1}) &= \alpha_{\rm s}^2(r^{-1}),\\
D_{1,s}^{(2)}(r^{-1}) &= \alpha_{\rm s}(r^{-1}),\\
D_{2,s}^{(2)}(r^{-1}) &= \alpha_{\rm s}(r^{-1}),\\
D_{d,s}^{(2)}(r^{-1}) &= \alpha_{\rm s}(r^{-1})(2 + c_D(r^{-1}) - 2c_F^2(r^{-1}))\\
&\quad + \frac{1}{\pi}\left[d_{vs}(r^{-1}) + 3d_{vv}(r^{-1}) + \frac{1}{C_f}(d_{ss}(r^{-1}) + 3d_{sv}(r^{-1}))\right],\\
D_{S^2,s}^{(2)}(r^{-1}) &= \alpha_{\rm s}(r^{-1})c_F^2(r^{-1}) - \frac{3}{2\pi C_f}(d_{sv}(r^{-1}) + C_f d_{vv}(r^{-1})),\\
D_{LS,s}^{(2)}(r^{-1}) &= \frac{\alpha_{\rm s}(r^{-1})}{3}(c_S(r^{-1}) + 2c_F(r^{-1})),\\
D_{S_{12},s}^{(2)}(r^{-1}) &= \alpha_{\rm s}(r^{-1})c_F^2(r^{-1}),\\
\alpha_{V_o}(r^{-1}) &= \alpha_{\rm s}(r^{-1}),\\
V_A(r^{-1}) &= 1
\end{aligned}
$$

The RG improved potentials for the singlet read:

$$
\begin{aligned}
\alpha_{V_s}(\nu_{us}) &= \alpha_{V_s}(r^{-1}) + \frac{C_A^3}{6\beta_0}\alpha_s^3(r^{-1})\log\left(\frac{\alpha_s(r^{-1})}{\alpha_s(\nu_{us})}\right),\\
D_s^{(1)}(\nu_{us}) &= D_s^{(1)}(r^{-1}) + \frac{16}{3\beta_0}\left(\frac{C_A}{2}+C_f\right)\alpha_s^2(r^{-1})\log\left(\frac{\alpha_s(r^{-1})}{\alpha_s(\nu_{us})}\right),\\
D_{1,s}^{(2)}(\nu_{us}) &= D_{1,s}^{(2)}(r^{-1}) + \frac{8C_A}{3\beta_0}\alpha_s(r^{-1})\log\left(\frac{\alpha_s(r^{-1})}{\alpha_s(\nu_{us})}\right),\\
D_{2,s}^{(2)}(\nu_{us}) &= D_{2,s}^{(2)}(r^{-1}),\\
D_{d,s}^{(2)}(\nu_{us}) &= D_{d,s}^{(2)}(r^{-1}) + \frac{32}{3\beta_0}\left(\frac{C_A}{2}-C_f\right)\alpha_s(r^{-1})\log\left(\frac{\alpha_s(r^{-1})}{\alpha_s(\nu_{us})}\right),\\
D_{S^2,s}^{(2)}(\nu_{us}) &= D_{S^2,s}^{(2)}(r^{-1}),\\
D_{LS,s}^{(2)}(\nu_{us}) &= D_{LS,s}^{(2)}(r^{-1}),\\
D_{S_{12},s}^{(2)}(\nu_{us}) &= D_{S_{12},s}^{(2)}(r^{-1}).
\end{aligned}
$$

One equation for the ultrasoft running.

OBSERVABLE: NNLL heavy quarkonium mass $O(m\alpha^{4+n}\ln^n\alpha)$ (Pineda)

$$
\delta E^{\rm pot}_{n,l,j}(\nu_{us}) = E_n\alpha_s^2\left\{-\frac{2C_A}{3\beta_0}\left[\frac{C_A^2}{2}+4C_AC_f\frac{1}{n(2l+1)}+2C_f^2\left(\frac{8}{n(2l+1)}-\frac{1}{n^2}\right)\right]\right.
$$
$$
\times\log\left(\frac{\alpha_s(\nu_{us})}{\alpha_s}\right)+\frac{C_f^2\delta_{l0}}{3n}\left(-\frac{16}{\beta_0}\left[C_f-\frac{C_A}{2}\right]\log\left(\frac{\alpha_s(\nu_{us})}{\alpha_s}\right)\right.
$$
$$
\left.-\frac{3}{2}(1+c_D-2c_F^2)-\frac{3}{2\pi\alpha_s}\left[d_{vs}+3d_{vv}+\frac{1}{C_f}(d_{ss}+3d_{sv})\right]\right)
$$
$$
-\frac{4}{3}\frac{C_f^2\delta_{l0}\delta_{s1}}{n}\left\{z^{-2C_A}-1+\frac{3}{2}\frac{C_A}{\beta_0-2C_A}\left[z^{-\beta_0}-z^{-2C_A}\right]\right\}
$$
$$
\left.-\frac{(1-\delta_{l0})\delta_{s1}}{l(2l+1)(l+1)n}C_{j,l}\frac{C_f^2}{2}\right\},
$$

where $E_n=-mC_f^2\alpha_s^2/(4n^2)$, $\nu_s=2a_n^{-1}$ where $2a_n^{-1}=\frac{mC_f\alpha_s(2a_n^{-1})}{n}$, and

$$
C_{j,l}=\begin{cases}
-\frac{(l+1)}{2\,l-1}\left\{4(2l-1)\left(z^{-C_A}-1\right)+\left(z^{-2C_A}-1\right)\right\} & ,\ j=l-1\\
-4\left(z^{-C_A}-1\right)+\left(z^{-2C_A}-1\right) & ,\ j=l\\
\frac{l}{2\,l+3}\left\{4(2l+3)\left(z^{-C_A}-1\right)-\left(z^{-2C_A}-1\right)\right\} & ,\ j=l+1.
\end{cases}
$$

Check with $O(m\alpha^5\ln\alpha)$ known logs: Brambilla, Vairo, Soto, Pineda; Kniehl, Penin; Hoang, Manohar and Stewart.

Renormalization group in pNRQCD (NLL) (potential running)

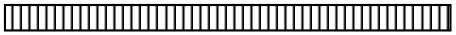
ν_p enters into the game.
The running on ν_p can be obtained from pNRQCD (there is no running of ν_p from NRQCD to pNRQCD). It can be obtained by Quantum mechanics computations. Example: iteration of the potentials. Divergent integrals in $|\mathbf{p}|$ and r.

$$h_s = c_k \frac{\mathbf{p}^2}{m} - C_f \frac{\alpha_{V_s}}{r} - c_4 \frac{\mathbf{p}^4}{4m^3} - \frac{C_f C_A D_s^{(1)}}{2mr^2} - \frac{C_f D_{1,s}^{(2)}}{2m^2}\left\{\frac{1}{r},\mathbf{p}^2\right\} + \frac{C_f D_{2,s}^{(2)}}{2m^2}\frac{1}{r^3}\mathbf{L}^2$$
$$+ \frac{\pi C_f D_{d,s}^{(2)}}{m^2}\delta^{(3)}(\mathbf{r}) + \frac{4\pi C_f D_{S^2,s}^{(2)}}{3m^2}\mathbf{S}^2\delta^{(3)}(\mathbf{r}) + \frac{3C_f D_{LS,s}^{(2)}}{2m^2}\frac{1}{r^3}\mathbf{L}\cdot\mathbf{S} + \frac{C_f D_{S_{12},s}^{(2)}}{4m^2}\frac{1}{r^3}S_{12}(\hat{\mathbf{r}}),$$

where $C_f = (N_c^2-1)/(2N_c)$ and $c_k = c_4 = 1$ (we only use c_4 for tracking of the contribution due to this term). The propagator of the singlet is (formally)

$$\frac{1}{E-h_s}.$$

At leading order (within an strict expansion in α_s) the propagator of the singlet reads



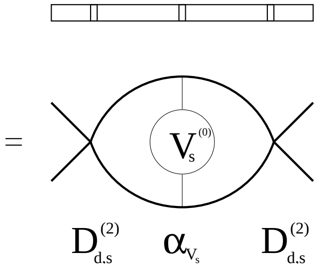
$$= G_c(E) = \frac{1}{E-h_s^{(0)}} = \frac{1}{E-\mathbf{p}^2/m - C_f\alpha_s/r}.$$

If we were interested in computing the spectrum at $O(m\alpha_s^6)$, one should consider the iteration of subleading potentials (δh_s) in the propagator as follows:

$$G_c(E)\delta h_s G_c(E)\cdots\delta h_s G_c(E)\,.$$

In general, these contributions will produce logarithmic divergences due to potential loops. These divergences can be absorbed in the matching coefficients, $D_{d,s}^{(2)}$ and $D_{S^2,s}^{(2)}$, of the local potentials (proportional to the $\delta^{(3)}(\mathbf{r})$) providing with the renormalization group equations of these matching coefficients in terms of ν_p. Let us explain how it works in detail. Since the singular behavior of the potential loops appears for $\mathbf{p}^2/m \gg \alpha_s/r$, a perturbative expansion in α_s is licit in $G_c(E)$, which can be approximated by

$$= G_c^{(0)}(E) = \frac{1}{E-\mathbf{p}^2/m}\,.$$



$$\langle \mathbf{r}=0| \frac{1}{E-\mathbf{p}^2/m} C_f \frac{\alpha_{V_s}}{r} \frac{1}{E-\mathbf{p}^2/m} |\mathbf{r}=0\rangle$$

$$\sim \int \frac{\mathrm{d}^d p'}{(2\pi)^d} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{m}{\mathbf{p}'^2 - mE} C_f \frac{4\pi\alpha_{V_s}}{\mathbf{q}^2} \frac{m}{\mathbf{p}^2 - mE} \sim -C_f \frac{m^2 \alpha_{V_s}}{16\pi} \frac{1}{\epsilon},$$

where $D = 4 + 2\epsilon$ and $\mathbf{q} = \mathbf{p} - \mathbf{p}'$. This divergence is absorbed in $D_{d,s}^{(2)}$ contributing to its running at NLL order as follows

$$\nu_p \frac{d}{d\nu_p} D_{d,s}^{(2)}(\nu_p) \sim \alpha_{V_s}(\nu_p) D_{d,s}^{(2)2}(\nu_p) + \cdots .$$

and one equation for the potential running

Correlation of cutoffs

$|\mathbf{p}| \gg \nu_{us} \gg \mathbf{p}^2/m \to \nu_{us} = {\nu_p}^2/m$ (Luke, Manohar, Rothstein)
We can not lower ν_{us} further. Fight between two terms.

$$\frac{1}{\mathbf{p}^2/m + k}$$

$$\begin{aligned} \tilde{V}(c(1/r), d(\nu_p, 1/r), 1/r, \nu_p^2/m, r) &\simeq \tilde{V}(c(\nu_p), d(\nu_p, \nu_p), \nu_p, \nu_p^2/m, \nu_p) \\ &+ \ln(\nu_p r) r \frac{d}{dr} \tilde{V}\Big|_{1/r=\nu_p} + \cdots . \end{aligned}$$

One equation for the soft running,
one equation for the ultrasoft running,
and one equation for the potential running,
which rules them all and at the hard scale binds them.

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Nonrelativistic Sum rules (b-$\bar{b}$, c-$\bar{c}$), t-$\bar{t}$ production near threshold

Determination of m_b, m_t, α_s, Higgs-top yukawa coupling, ...

$$J^\mu = \bar{Q}\gamma^\mu Q = B_1 \psi^\dagger \boldsymbol{\sigma} \chi + \cdots ,$$

$$B_1 = 1 + a_1\alpha_s + a_2\alpha_s^2 + \cdots$$

B_1 at NNNLO: Beneke, Signer, Smirnov; Czarnecki, Melnikov; Beneke et al.
B_1, B_0 at NLL: Pineda; Hoang, Stewart
B_1/B_0 at NNLL: Penin, Pineda, Smirnov, Steinhauser
B_1, B_0 at NNLL (partial): Pineda, Signer

$$(q_\mu q_\nu - g_{\mu\nu})\Pi(q^2) = i\int d^4x e^{iqx} \langle \mathrm{vac}|J_\mu(x)J_\nu(0)|\mathrm{vac}\rangle$$

$$\Pi(q^2) \sim B_1^2 \langle \mathbf{r}=\mathbf{0}|\frac{1}{E-H}|\mathbf{r}=\mathbf{0}\rangle$$

$$G(0,0,E) = \sum_{m=0}^{\infty} \frac{|\phi_{0m}(0)|^2}{E_{0m} - E + i\epsilon - i\Gamma_t} + \frac{1}{\pi}\int_0^\infty dE' \frac{|\phi_{0E'}(0)|^2}{E_{0E'} - E + i\epsilon - i\Gamma_t}$$

A NNLL renormalization group improved expression of $M(V_Q(nS))$ is also needed in order to obtain expressions for the t-$\bar{t}$ production near threshold with NNLL accuracy.

Relation of the vacuum polarization with $\sigma_{t\bar{t}}$, non-relativistic sum rules and $\Gamma(V_Q(nS) \rightarrow e^+e^-)$

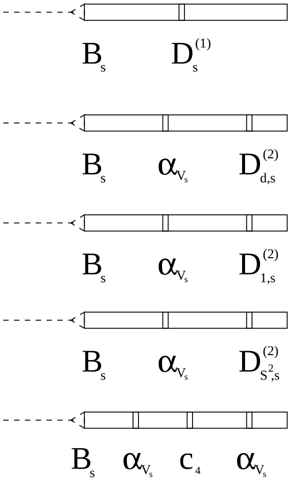
$$\Gamma(V \rightarrow e^+e^-) \sim \frac{1}{m^2} B_1^2 |\phi(\mathbf{0})|^2$$

$$\sigma_{t-\bar{t}} \sim B_1(\nu)^2 \mathrm{Im} G(0,0,\sqrt{s}) + \cdots$$

$$M_n \equiv \frac{12\pi^2 e_b^2}{n!} \left(\frac{d}{dq^2}\right)^n \Pi(q^2)\Big|_{q^2=0} = \int_0^\infty \frac{ds}{s^{n+1}} R_{b\bar{b}}(s),$$

$$M_n = 48\pi e_b^2 N_c \int_{-\infty}^{\infty} \frac{dE}{(E+2m_b)^{2n+3}} \left(B_1^2 - B_1 d_1 \frac{E}{3m_b}\right) \mathrm{Im}\, G(0,0,E)$$

Matching coefficient of the electromagnetic current at NLL



$$\nu_p \frac{d}{d\nu_p} B_s = -\frac{C_A C_f}{2} D_s^{(1)} - \frac{C_f^2}{4} \alpha_s \left\{ \alpha_s - \frac{4}{3} s(s+1) D^{(2)}_{S^2,s} - D^{(2)}_{d,s} + 4 D^{(2)}_{1,s} \right\},$$

$$b_1(m) = 1 - 2C_f \frac{\alpha_s(m)}{\pi}, \qquad b_0(m) = 1 + \left(\frac{\pi^2}{4} - 5 \right) \frac{C_f}{2} \frac{\alpha_s(m)}{\pi}.$$

The solution reads (Pineda; Hoang-Stewart)

$$B_s(\nu_p) = b_s(m) + A_1 \frac{\alpha_s(m)}{w^{\beta_0}} \ln(w^{\beta_0}) + A_2 \alpha_s(m) \left[z^{\beta_0} - 1 \right] + A_3 \alpha_s(m) \left[z^{\beta_0 - 2C_A} - 1 \right.$$
$$+ A_4 \alpha_s(m) \left[z^{\beta_0 - 13C_A/6} - 1 \right] + A_5 \alpha_s(m) \ln(z^{\beta_0}),$$

where $\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F n_f$, $z = \left[\frac{\alpha_s(\nu_p)}{\alpha_s(m)} \right]^{\frac{1}{\beta_0}}$ and $w = \left[\frac{\alpha_s(\nu_p^2/m)}{\alpha_s(\nu_p)} \right]^{\frac{1}{\beta_0}}$.

The coefficients A_i read

$$
\begin{aligned}
A_1 &= \frac{8\pi C_f}{3\beta_0^2}\left(C_A^2 + 2C_f^2 + 3C_f C_A\right)\,, \\
A_2 &= \frac{\pi C_f[3\beta_0(26C_A^2 + 19C_A C_f - 32C_f^2) - C_A(208C_A^2 + 651C_A C_f + 116C_f^2)]}{78\,\beta_0^2\,C_A}\,, \\
A_3 &= -\frac{\pi C_f^2\left[\beta_0(4s(s+1)-3) + C_A(15 - 14s(s+1))\right]}{6(\beta_0 - 2C_A)^2}\,, \\
A_4 &= \frac{24\pi C_f^2(3\beta_0 - 11C_A)(5C_A + 8C_f)}{13\,C_A(6\beta_0 - 13C_A)^2}\,, \\
A_5 &= \frac{-\pi C_f^2}{\beta_0^2\,(6\beta_0 - 13C_A)(\beta_0 - 2C_A)}\left\{C_A^2(-9C_A + 100C_f)\right. \\
&\left. +\beta_0 C_A(-74C_f + C_A(42 - 13s(s+1))) + 6\beta_0^2(2C_f + C_A(-3 + s(s+1)))\right\}.
\end{aligned}
$$

Leading (Czarnecki-Melnikov; Beneke-Signer-Smirnov) and subleading (Kniehl-Penin) logs correct.

Inclusive decays to leptons and photons at NLL (Pineda)

By setting $\nu_p \sim m\alpha_s$, $B_s(\nu_p)$ includes all the large logs at NLL order in any (inclusive enough) S-wave heavy-quarkonium production observable we can think of. For instance, the decays to e^+e^- and to two photons at NLL $O(\alpha^{1+n}\ln^n\alpha)$ order read

$$
\begin{aligned}
\Gamma(V_Q(nS)\to e^+e^-) &= 2\left[\frac{\alpha_{em}Q}{M_{V_Q(nS)}}\right]^2\left(\frac{m_Q C_f\alpha_s}{n}\right)^3\left\{B_1(\nu_p)(1+\delta\phi_n)\right\}^2 \\
&\simeq 2\left[\frac{\alpha_{em}Q}{M_{V_Q(nS)}}\right]^2\left(\frac{m_Q C_f\alpha_s}{n}\right)^3\left\{1+2(B_1(\nu_p)-1)+2\delta\phi_n\right\}\,, \\
\Gamma(P_Q(nS)\to\gamma\gamma) &= 6\left[\frac{\alpha_{em}Q^2}{M_{P_Q(nS)}}\right]^2\left(\frac{m_Q C_f\alpha_s}{n}\right)^3\left\{B_0(\nu_p)(1+\delta\phi_n)\right\}^2 \\
&\simeq 6\left[\frac{\alpha_{em}Q^2}{M_{P_Q(nS)}}\right]^2\left(\frac{m_Q C_f\alpha_s}{n}\right)^3\left\{1+2(B_0(\nu_p)-1)+2\delta\phi_n\right\}\,,
\end{aligned}
$$

where V and P stand for the vector and pseudoscalar heavy quarkonium, we have fixed $\nu_p = m_Q C_f\alpha_s/n$, $\alpha_s=\alpha_s(\nu_p)$, and ($\Psi_n(z)=\frac{d^n\ln\Gamma(z)}{dz^n}$ and $\Gamma(z)$ is the Euler Γ-function)

$$
\delta\phi_n = \frac{\alpha_s}{\pi}\left[-C_A+\frac{\beta_0}{4}\left(\Psi_1(n+1)-2n\Psi_2(n)+\frac{3}{2}+\gamma_E+\frac{2}{n}\right)\right].
$$