# Introduction to Lattice Gauge Theory

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Methods of Effective Field Theory and Lattice Field Theory
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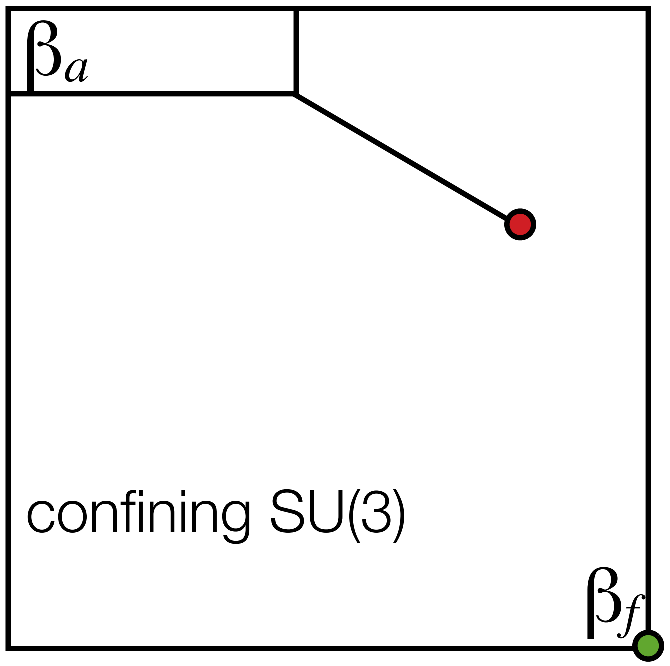
# Outline of the Course

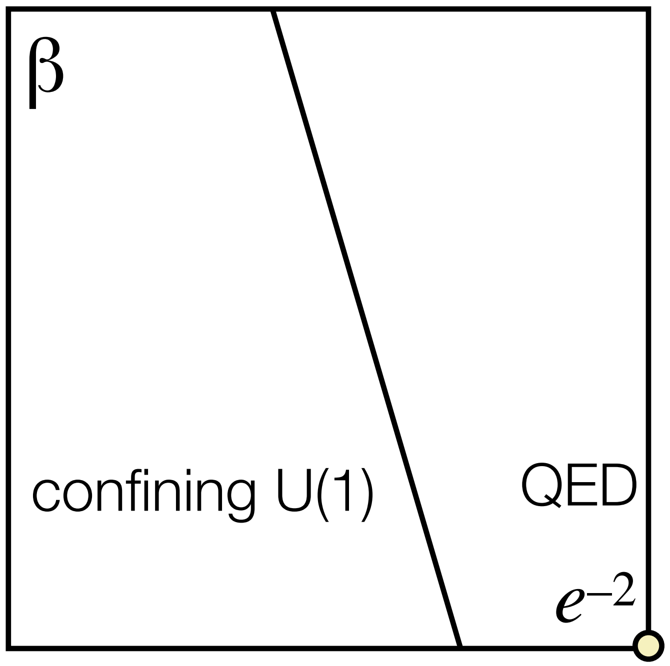
- Lecture 1: Bosons: Basic constructs, scalar fields, gauge fields (Monday)
- Lecture 2: Fermions: Doubling, chiral symmetry, choices (Tuesday)
- Lecture 3: Renormalization: Effective field theory approach (Thursday)

# Renormalization without Perturbation Theory

# Phase Transitions (recap)

- This lecture revisits phase transitions:
    - not those of the 3-dim. QCD Hamiltonian (or other QFT of interest to particle physics).
- Those of a 4-dim. (essentially classical) theory with many couplings:
    - "bulk" transitions.
- First-order transitions are obstacles to obtaining continuum qft from lattice ft.
- Second-order transitions are key.



# Two Partition Functions

- Consider a big space $\mathfrak{S}$ of all conceivable actions for a lattice field theory:
  - pick your degrees of freedom and their symmetries—imagine any allowed interaction, labeled by couplings $K$.
- Quantum continuum limit of a lattice field theory:

$$Z = \lim_{L_\mu \to \infty} \lim_{\substack{a \to 0,\, N_\mu \to \infty \\ L_\mu \text{ fixed}}} \int \prod_{x}^{N_s^3 N_4} \mathcal{D}\phi_x \, e^{-S[K(a);N_\mu]}$$

$K(a)$ denotes a trajectory, depending on $a$, leading to a renormalized QFT.

- Trajectory: in practice a sequence.

- Statistical mechanics system

$$Z = \lim_{N_\mu \to \infty} \int \prod_x^{N_s^3 N_4} \mathcal{D}\phi_x \, e^{-S[K;N_\mu]}$$

has phase transitions on a hypersurface in $\mathfrak{S}$, of low dimension.

- Suppose the stat. mech. system has a "second-order critical point":
  - characterized by infinitely long correlation lengths, or …
  - … set of eigenvalues of the transfer matrix $\mathbb{T} = \mathbb{T}_0 e^{-a/\xi}$;
  - identify mass $m = 1/\xi$; actually, $a = (a/\xi)_{\text{calc}}/m_{\text{expt}}$.
- Suppose further that these eigenvalues accumulate to form a relativistic mass shell:

$$\frac{a}{\xi_{\boldsymbol{p}}} = \sqrt{\frac{a^2}{\xi^2} + \boldsymbol{p}^2 a^2}$$

# Finding Critical Points

- Stipulate that we have an apparatus for calculating correlation functions:
  - mean field theory (a quick stat. mech. technique);
  - perturbation theory (if we have a good idea where to look);
  - high-order strong coupling expansion (perhaps with Padé extrapolation);
  - Monte Carlo calculations (perhaps with block averaging).

- Compare lattice theories with spacings $sa$ and $a$: “renormzalition-group transformation”:

- Find $K(a)$ by demanding that correlation functions match at longish distances:

$$K_m(sa) = \Psi_m(K(a))$$

- When $\xi \gg a$, iteration can freeze: $K(sa) \approx K(a) =: K^*$; expand about $K^*$:

$$\begin{aligned} K_m(sa) - K_m^* &= \frac{\partial \Psi_m}{\partial K_n}(K_n(a) - K_n^*) + O\left((K-K^*)^2\right) \\ &=: \mathscr{M}_{mn}(K_n(a) - K_n^*) + O\left((K-K^*)^2\right) \end{aligned}$$

fixed point

- In differential form, $s = 1 + dt$:

$$\frac{dK_m(a)}{dt} = \mathscr{M}_{mn}(K_n(a) - K_n^*) + O\left((K-K^*)^2\right)$$

Renormalization group equations. Actually a semigroup.

- Look at the eigenvalues $V\mathcal{M}_{mn}V^{-1} = \mathrm{diag}(D_1, D_2, \ldots)$, $\kappa_m = V_{mn}(K - K^*)_n$

| | | |
|---|---|---|
| $D_n > 0:$ | $\kappa_n = \kappa_n^0 e^{D_n t}$ | relevant |
| $D_n = 0:$ | | go one order higher |
| $D_n < 0:$ | $\kappa_n = \kappa_n^0 e^{-\lvert D_n \rvert t}$ | irrelevant |

- Marginal case $D_n = 0$:

$$\kappa_n = \frac{\kappa_n^0}{1 - \kappa_n^0 b_n t} = \frac{\kappa_n^0}{1 - \kappa_n^0 b_n \ln(a_t/a_0)} \begin{cases} \kappa_n^0 b_n > 0 & \text{marginally relevant} \\ \kappa_n^0 b_n < 0 & \text{marginally irrelevant} \end{cases}$$

- Location of the fixed point depends on details of the RG transformation.

- The number of relevant, *etc.*, couplings does not.

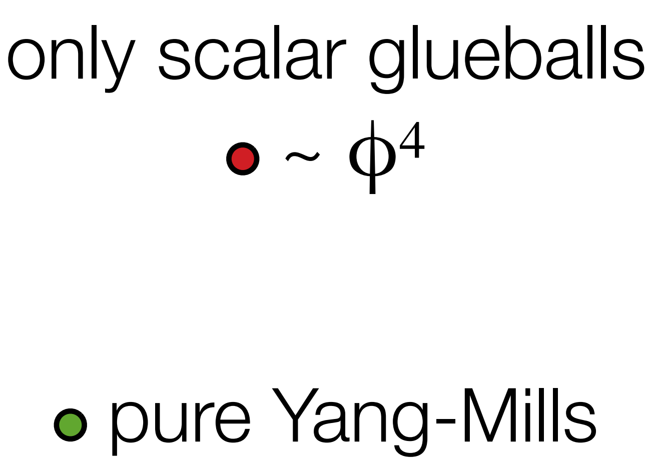
- Continuum limit taken by setting all irrelevant $\kappa_n = 0$. Critical surface.

# Renormalized Trajectory

- Generic RG trajectories near the critical surface will flow towards the fixed point, until getting close to it.  Then they are repelled.

- The limit of all outward flows is a special trajectory, the “renormalized trajectory”.  It has nonzero lattice spacing, but no cutoff effects, because the RG flow connects it to the fixed point, which is on the critical surface.

- Are there any nontrivial fixed points?



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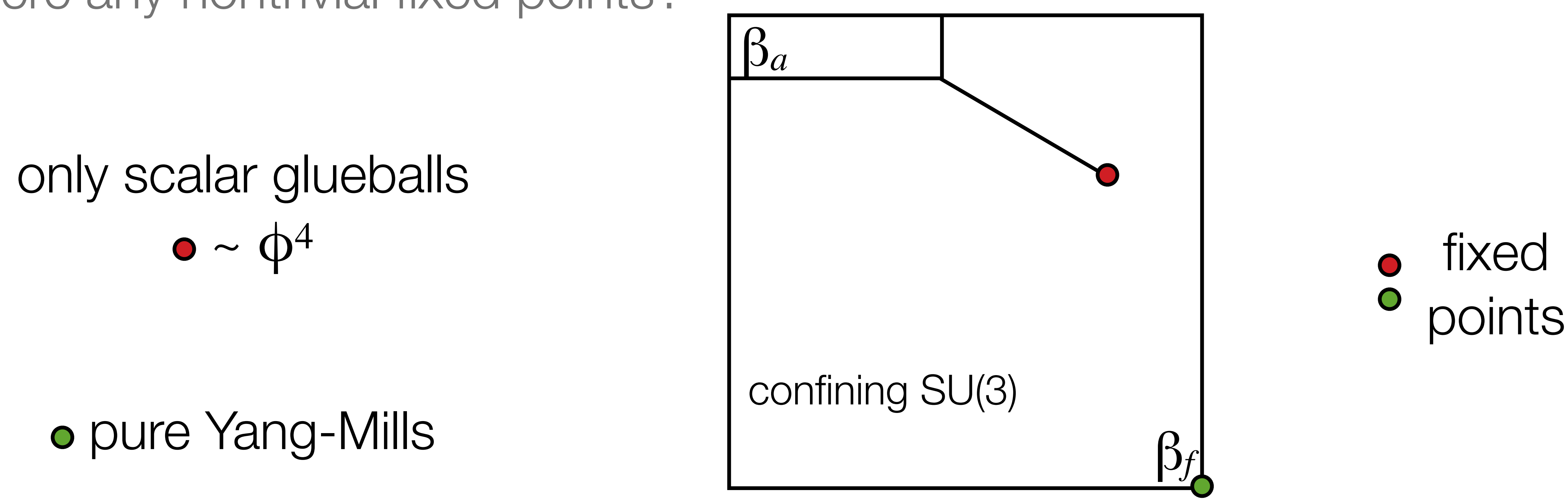

# Physical Mass Parameters

- In particle physics, we are used to thinking of mass scales:

$$\mu_n = \begin{cases} a^{-1} e^{-1/b_n \kappa_n} & \text{for marginal couplings} \\ a^{-1} \kappa_n^{1/D_n} & \text{otherwise} \end{cases} ,$$

- Different ratios of these scales specify different physics.
- The marginal(ly) relevant mass parameter is $\Lambda_{\text{QCD}}$; one with $D_n = 1$ is $m_q$.
- Different ratios of $m_q/\Lambda_{\text{QCD}}$ correspond to up, down, strange, …, quarks.

# Goldstone Oddity

P. Hasenfratz

- There is an oddity when Goldstone bosons appear at the fixed point.
- Goldstone bosons arise when symmetries are broken spontaneously, and massless particles arise.
- Then the scaling dimensions become ambiguous. For example, take
- Choose scaling

$$m_\pi^2 a^2 = \kappa\Sigma, \qquad m_\sigma^2 a^2 = \Sigma^2$$

$$\kappa = m_q a^{1+p}, \qquad \Sigma = \Lambda a^{1-p}.$$

where $\Lambda$ is another scale.

- Then $m_\pi^2 = m_q\Lambda$, $m_\sigma^2 = \Lambda^2 a^{-2p}$. If $p > 0$, $\pi$'s correlation length diverges while $\sigma$'s does not. Different universality class than standard $p = 0$.

# RG and Chiral Symmetry

- Let's start with a continuum action $S_q(q,\bar{q}) \mapsto S_q(e^{i\varepsilon\gamma^5}q,\, \bar{q}e^{i\varepsilon\gamma^5}) = S_q(q,\bar{q})$.
- Define $e^{-S(\psi,\bar{\psi})} = \int \mathcal{D}q\mathcal{D}\bar{q}\, \exp\left[-(\bar{\psi}_m - q_m)\alpha_{nm}(\psi_m - q_m) - S_q(q,\bar{q})\right]$
- Write $S(\psi,\bar{\psi}) = \bar{\psi}\not{D}\psi$ and carry out a small chiral transformation $1 - i\varepsilon\gamma^5$:

$$
\begin{aligned}
e^{-S(\psi,\bar{\psi})}\left[1 + i\varepsilon\bar{\psi}\{\gamma^5,\not{D}\}\psi\right] &= \int \mathcal{D}q\mathcal{D}\bar{q}\left[1 + i\varepsilon(\bar{\psi}-q)\{\gamma^5,\alpha\}(\psi - q)\right]e^{-(\bar{\psi}-q)\alpha(\psi-q)-S_q(q,\bar{q})} \\
&= \left[1 - i\varepsilon\frac{\partial}{\partial\psi}\{\gamma^5,\alpha^{-1}\}\frac{\partial}{\partial\bar{\psi}}\right]e^{-S(\psi,\bar{\psi})} \\
&= \left[1 + i\varepsilon\bar{\psi}\not{D}\{\gamma^5,\alpha^{-1}\}\not{D}\psi\right]e^{-S(\psi,\bar{\psi})}
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- One sees $\{\gamma^5,\not{D}\} = \not{D}\{\gamma^5,\alpha^{-1}\}\not{D}$; pick $\alpha = 2/a$.

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# Improvement

# Improved Actions

- In the above discussion, special regions had small cutoff effects: the fixed point and the renormalized trajectory.
- Now we will examine a way to approach the renormalized trajectory in the case that we all know well. The fixed point of QCD.
- Start with the Wilson plaquette gauge action and the Wilson fermion action (for definiteness).
- Consider a more general action $S = S_W + \sum_i c_i O_i$, where the $O_i$ are gauge-invariant composite operators. The $c_i$ are called improvement couplings.
- Convenient to organize the series so that the $O_i$ have dimension > 4; balance dimensions with powers of $a$.

- The $c_i O_i$ are chosen to reduce cutoff effects. For example, the quark-gluon vertex ($c_{\rm SW}$ multiplies a term that looks like $\bar{\psi}\sigma\cdot F\psi$)

$$\begin{aligned}\Gamma_\mu(p,p') &= -g_0 t^a \left\{\gamma_\mu \cos[\tfrac{1}{2}(p+p')_\mu a] - i\sin[\tfrac{1}{2}(p+p')_\mu a] \right.\\ &\qquad \left. + \tfrac{1}{2} c_{SW}\sigma_{\mu\nu}\cos[\tfrac{1}{2}k_\mu a]\sin[k_\nu a]\right\} \\ &\approx -g_0 t^a \left\{\gamma_\mu - \tfrac{i}{2}\left[(p+p')_\mu + c_{SW} i\sigma_{\mu\nu}k_\nu\right]\right\}.\end{aligned}$$

- On the mass shell, the Gordon identity shows that the O($a$) term vanishes, if the improvement coupling $c_{\rm SW} = 1$.

- Beyond tree level, we have to allow for renormalization, which will mean that $c_{\rm SW} = 1 + {\rm O}(g^2)$.

- We will also need a notion of “on shell” that is robust enough to apply to hadrons (without hadronic calculations).

- Symanzik effective field theory.

# Local Effective Lagrangian

- An outgrowth of Symanzik's work on the Callan-Symanzik equation. Posit

$$\mathscr{L}_{\text{LGT}} \doteq \mathscr{L}_{\text{Sym}} = \mathscr{L}_{\text{QCD}} + \mathscr{L}_I$$

where the symbol $\doteq$ can be read as "has the same matrix elements as".

- The right-hand side $\mathscr{L}_{\text{Sym}}$ is not a lattice field theory. It is a tool to understand lattice field theory.

- $\mathscr{L}_{\text{QCD}}$ is the renormalized QCD Lagrangian:

  - a scale μ ≲ $1/a$ separating long from short distances:

$$\mathscr{L}_{\text{QCD}} = \frac{1}{2g^2(\mu)} \operatorname{tr}[F_{\mu\nu}F^{\mu\nu}] - \bar{q}(\not{D} + m)q$$

$$g^2 = g^2(g_0^2, m_0 a; c_i; \mu a), \qquad m = m_0 Z_m(g_0^2, m_0 a; c_i; \mu a),$$

- Lattice artifacts are described by operators of dimension $\dim \mathscr{O} > 4$:

$$\mathscr{L}_I = \sum_{\mathscr{O}} a^{\dim \mathscr{O} - 4} K_{\mathscr{O}}(g^2, ma, c_i; \mu a) \mathscr{O}(\mu)$$

where $a^{\dim \mathscr{O} - 4} K_{\mathscr{O}}$ is a short-distance coefficient, while $\mathscr{O}$ is a QCD operator encoding long distances:

  - renormalized $g^2$ and $m$: $Q\bar{Q}$ static energy at $\mu = 6/a$, and pion mass.
  - like the renormalized $g^2$ and $m$, the $K_{\mathscr{O}}$ depend on all couplings.

- Deviations from the continuum limit are defined at fixed $g^2$ and $m$.
- The elegance of this approach is as follows:
  - adjust $c_i$ so $K_{\mathscr{O}}$ vanishes (approx.) in one process—
    - then it does in all processes.

- Renormalized operators $\mathscr{O}$ are sensitive to long distances only: $1/\Lambda_{\mathrm{QCD}}$, $L$, and not the short distance $a$.
- In a hadronic system, matrix elements of are of order $p^{\dim \mathscr{O} - 4}$, where $p$ is a typical momentum, $p \sim \Lambda_{\mathrm{QCD}}, \pi/L$:
  - with the coefficient: $(pa)^{\dim \mathscr{O} - 4}$, which is small when $\dim \mathscr{O} > 4$.
- Thus, treat the $\mathscr{L}_I$ as small perturbations.
- On the other hand, the short-distance coefficients $K_{\mathscr{O}}$ depend only short distances.
- For light quarks, $m_q^{-1}$ is a long distance; expand $K_{\mathscr{O}}(m_q a)$ in $m_q a$.
- For heavy quarks $m_Q^{-1}$ is a short distance; keep full $m_Q a$ dependence in $K_{\mathscr{O}}(m_Q a)$.

- To $\mathscr{L}_I$ treat as perturbations, set up the interaction picture driven by $\mathscr{L}_{\text{QCD}}$.
- Develop series, with all matrix elements taken in $\mathscr{L}_{\text{QCD}}$ eigenstates.
- Basing the description around $\mathscr{L}_{\text{QCD}}$ is useful, because it has more symmetry than $\mathscr{L}_{\text{LGT}}$.
- Simplify $\mathscr{L}_I$ by exploiting field redefinitions:

$$q \mapsto \left(1 + a^{\dim X} \varepsilon_X X\right) a$$

- Changes of integration variables cannot change on-shell matrix elements, which are (obtained from) integrals.
- Since the interaction picture is being driven by $\mathscr{L}_{\text{QCD}}$, the mass shell in question is that of (non-perturbative) QCD, even though we haven't computed hadron masses, *etc*.

- Field redefinitions in $\mathscr{L}_{\text{QCD}}$ induce higher-dimension terms, like those in $\mathscr{L}_I$:

$$\mathscr{L}_I \mapsto \mathscr{L}_I + \sum_X a^{\dim X} \left[ \bar{\varepsilon}_X \bar{q} X (\not{D} + m_q) q + \varepsilon_X \bar{q} (-\overleftarrow{\not{D}} + m_q) X q \right].$$

- Similarly, redefinition in lower-dimension terms in $\mathscr{L}_I$ changes terms of higher dimension.

- So, field redefinitions amount to changing the coefficients of a subset of operators in $\mathscr{L}_I$.

- Since the changes are arbitrary, the operators is this subset cannot have any effect on on-shell matrix elements: they are called redundant.

- When using the effective field theory to describe the underlying theory, their coefficients may be set according to convenience.

- They are easy to identify, because they vanish by the equations of motion of the Lagrangian driving the interaction picture, here $\mathscr{L}_{\text{QCD}}$.

# Illustration—Wilson Fermions

- At dimension-five there are no operators made only of gauge fields $F$.
- There are two linearly independent quark operators:

$$\mathscr{O}_5 = i\bar{q}\sigma_{\mu\nu}F^{\mu\nu}q, \qquad \mathscr{O}_5' = 2\bar{q}D^2 q$$

- The second of these can be re-written as:

$$\mathscr{O}_5' = \mathscr{O}_5 + 2\bar{q}\not{D}\left(\not{D}+m\right)q - 2m\bar{q}\not{D}q.$$

so it is redundant: the other operator, something vanishing on shell, and something that can be absorbed into wave-function normalization.

- Thus, $aK_{\sigma\cdot F}\mathscr{O}_5$ suffices to describe all on-shell dimension-five effects, as suggested by the Feynman rule.

- With an axial chiral symmetry (so, staggered or GW), $K_{\sigma\cdot F} = 0$.

- The Wilson term breaks chiral symmetry, so with Wilson fermions $K_{\sigma\cdot F} \equiv 0$.

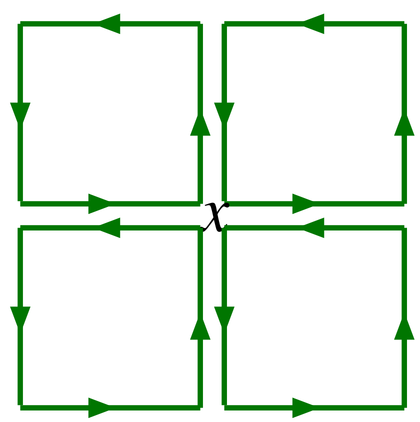
- Add a new interaction to $\mathscr{L}_{\text{LGT}}$ (*cf*. Feynman rule):

$$\mathscr{L}_{\text{SW}} = \mathscr{L}_{\text{W}} - \tfrac{i}{4} c_{\text{SW}} \bar{\psi}(x) \sigma_{\mu\nu} F^{\mu\nu}_{\text{lat}}(x) \psi(x)$$

where $F^{\mu\nu}_{\text{lat}}$ is a lattice approximant to $F^{\mu\nu}$ [Sheikholeslami & Wohlert].

- Usually called the “clover” action:

$$F^{\mu\nu}_{\text{lat}}(x) =$$




$$= F^{\mu\nu}(x) + O(a^2)$$

- One can adjust $c_{\text{SW}}$ so that $K_{\sigma\cdot F}$ vanishes, at least to order $g^{2l}$ or $a$.

# Vector and Axial Vector Currents

- Lattice currents for the flavor-changing transition $s \to u$:

$$V^\mu = \bar{\psi}_u i\gamma^\mu \psi_s - ac_V \partial_{\nu\,\mathrm{lat}} \bar{\psi}_u \sigma^{\mu\nu} \psi_s + \sum_{O_V} a^{\dim O_V - 3} c_{O_V} O_V^\mu,$$

$$A^\mu = \bar{\psi}_u i\gamma^\mu \gamma_5 \psi_s + ac_A \partial_{\mathrm{lat}}^\mu \bar{\psi}_u i\gamma_5 \psi_s + \sum_{O_A} a^{\dim O_A - 3} c_{O_A} O_A^\mu$$

- Symanzik effective field theory description:

$$V^\mu \doteq \bar{Z}_V^{-1} \mathscr{V}^\mu - aK_V \partial_\nu \bar{u} \sigma^{\mu\nu} s + \cdots,$$

$$A^\mu \doteq \bar{Z}_A^{-1} \mathscr{A}^\mu + aK_A \partial^\mu \bar{u} i\gamma_5 s + \cdots,$$

where $\bar{Z}_V$ & $K_{\sigma\cdot F}$ are short-distance coeffients, and continuum currents are $\mathscr{V}^\mu = \bar{u} i\gamma^\mu s$ and $\mathscr{A}^\mu = \bar{u} i\gamma^\mu \gamma^5 s$.

- The effective field theory says (and similarly for the vector current) that

$$\langle f_{\text{lat}}|\bar{Z}_A A^{\mu}_{\text{lat}}|i_{\text{lat}}\rangle = \langle f|\mathscr{A}^{\mu}|i\rangle + a\bar{Z}_A K_A \partial^{\mu}\langle f|\bar{u}i\gamma_5 s|i\rangle + aK_{\sigma\cdot F}\int d^4x\,\langle f|T\,\mathscr{O}_5\mathscr{A}^{\mu}|i\rangle + O(a^2),$$

  where the states on the left-hand side are eigenstates of lattice gauge theory, those on the right-hand side are eigenstates of continuum QCD.

- Two contexts where this formula is useful:
  - perturbative calculations of the short-distance coefficients:
    - reduce them systematically by powers of $g^2$;
  - small volumes, where not-so-big lattices can reach the continuum limit:
    - reduce them to pollution of order $a$, trickling down from the O($a^2$) matrix elements.

# “Proof” of Symanzik EFT

- The Symanzik effective field theory can be justified in to all orders in perturbation theory (in the gauge coupling).
- Reisz generalized the BPHZ (Bogoliubov-Parasiuk-Hepp-Zimmermann) renormalization tailored to lattice perturbation theory.
- Assuming one pole in propagators, a Feynman diagram can be expressed

$$\int \prod_{i=1}^{l} \frac{d^4 k_i}{(2\pi)^4} \mathscr{I}(\{p\},\{k\}) = I_R + I_U(\{p\}) + O(ap, am),$$

that is, a renormalization part $I_R$, a universal (a-independent) part $I_U$, and a remainder suppressed by powers of $a$.

- Remainder terms can be developed further with BPHZ oversubtractions of the loop integrands.
- Thus develop any amplitude's renormalized perturbation series, including contributions suppressed by powers of $a$, to any order in $a$ desired.
- This double series in $(g^2, a)$ is the same as one obtains from Symanzik formalism:
    - the formalism is fully justified in perturbation theory;
    - believed to hold at a nonperturbative level as well.
- (Nonperturbative effects of, say, small instantons would make contributions to the short-distance coefficients.)
- Failure here would cast doubt on all separation-of-scale methods in QCD, so a breakdown seems unlikely.

# Nonperturbative Matching

- Finite volumes effects do not alter the renormalization of a field theory.
- Small volume, $L \approx 0.5\ \mathrm{fm}$: take the continuum limit by brute force.
- There are three improvement couplings to adjust:

hyperfine splitting is sensitive to $c_{\mathrm{SW}}$
$\langle \pi_L(\mathbf{0})|A^4|0\rangle$ is sensitive to $c_A$
$\langle \pi_L(2\pi/L)|V^4|\pi_L(\mathbf{0})\rangle$ is sensitive to $c_V$

choosing finite-volume pion states as "familiar" examples.

- In practice, physical but non-laboratory states are used: compute $\langle \Phi_T|\hat{\mathbb{T}}^N|\Phi_0\rangle$ and related correlation functions, where boundary fields $\Phi_N$ & $\Phi_0$ are suitably chosen.

# Heavy Quark Methods

# Heavy Quarks

- Most impactful (arguably) lattice-QCD calculations are in the area of quark flavor physics: for example, sub-percent precision for quark masses.
- *B* meson has a mass of around 5 GeV, and *D* meson 1.8 GeV:
  - only just now getting to lattice spacings with $m_b a \ll 1.$, *e.g.*, MILC's $a = 0.045,\, 0.030\ \mathrm{fm}$ ensembles.
- Hence, understanding of heavy-quark discretization effects profit from some ideas beyond the standard Symanzik EFT:
  - actually led to the first papers on effective field theories for heavy quarks (static EFT, Eichten 1987) and non-relativistic QCD (Lepage, 1987):
    - then discretized.

# Symanzik Breakdown

- For many years (although not in Symanzik's original papers), it was assumed the EFT breaks down for $m_Q a \not\ll 1$.

- Lattice gauge theory does not break down (mathematically well defined, *etc.*), even if interpretation is less obvious.

- What happens to the Symanzik description when $m_Q a \not\ll 1$?

$$\mathscr{L}_I = \cdots + \sum_X a^{\dim X - 1} \sum_{n=3}^{\infty} K_X^{(n)} \bar{q} X \sum_{\mu=1}^{4} (-\gamma_\mu D_\mu a)^n q + \cdots ,$$

  At each $n$, term with $\mu = 4$ is large, because $(-\gamma_4 D_4 a)^n \sim (m_Q a)^n$.

- Just use the equation of motion to get rid of $D_4$ in favor of $m_Q$.

- Terms like $(m_Q a)^n$ are coefficients, not operators, so they can be absorbed into coefficients of lower-dimension operators:

$$\mathscr{L}_{\text{Sym}} = \mathscr{L}_{\text{gauge}} + \bar{q}\left(\gamma_4 D_4 + \sqrt{\frac{m_1}{m_2}}\,\boldsymbol{\gamma}\cdot\boldsymbol{D} + m_1\right)q + \mathscr{L}'_I,$$

on the same footing as the previous equation for $\mathscr{L}_{\text{Sym}}$.

- The split "QCD + small corrections" is violated by $1 - \sqrt{m_1/m_2}$ :
  - recovered as $m_Q a \to 0$.
- Alternative solutions:
  - different hopping parameters for temporal and spatial derivatives;
  - apply another effective theory to $\mathscr{L}_{\text{Sym}}$, namely HQET.

- The HQET had different short-distance coefficients than in Neubert's lectures, because we now have another short distance, the lattice spacing:

$$\begin{aligned}\mathscr{L}_{\text{Sym}} &\doteq \mathscr{L}_{\text{HQET}} \\ &= -\bar{h}(D_4 + m_1)h + \frac{1}{2m_2}\bar{h}\boldsymbol{D}^2 h + \cdots\end{aligned}$$

- Allows an understanding of heavy-quark discretization effects and a corresponding improvement program:
  - the HQET theory of cutoff effects.
- In particular, $m_1$ drops out of mass splittings and matrix elements. It can therefore be mistuned without loss.
- Usually applied to Wilson fermions, but also to all-staggered calculations such as those of the quark masses.

Questions?