Solution Exercises. Part 2

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1 Lecture 2

1.1 Exercise 1

Compute $\langle n|r^i(E_n - h_o)^2 r^i|n\rangle$. You can use as example of $\langle n|r^i(E - h_o)r^i|n\rangle$ worked out in the lecture. The virial theorem might be useful.

As we used in the lecture, $h_o = \frac{p^2}{M} + V_o = \frac{p^2}{M} + V_s + (V_o - V_s) = h_s + \Delta V$, where $\Delta V = \frac{\alpha_s N_c}{2r}$. Then,

$$\langle n|r^i(E_n-h_o)^2 r^i|n\rangle = \langle n|r^i(E_n-h_s)^2 r^i|n\rangle - \langle n|r^i\{\Delta V, E_n-h_s\}r^i|n\rangle + \langle n|r^i\Delta V^2 r^i|n\rangle \tag{1}$$

The last term is trivial to compute

$$\langle n|r^i\Delta V^2 r^i|n\rangle = \frac{\alpha_s^2 N_c^2}{4}\langle n|r^i\frac{1}{r^2}r^i|n\rangle = \frac{\alpha_s^2 N_c^2}{4} \tag{2}$$

For the first term, we do the following

$$\langle n|r^i(E_n - h_s)^2 r^i|n\rangle = \langle n|[h_s, r^i](E_n - h_s)r^i|n\rangle = -\langle n|[h_s, r^i][h_s, r^i]|n\rangle \tag{3}$$

Note that, using the canonical commutation relations, we can obtain

$$[h_s, r^i] = \frac{1}{M}[p^2, r^i] = \frac{p^j[p^j, r^i] + [p^j, r^i]p^j}{M} = -\frac{2ip^i}{M} \tag{4}$$

Therefore,

$$\langle n|r^i(E_n - h_s)^2 r^i|n\rangle = \frac{4}{M^2}\langle n|p^2|n\rangle \tag{5}$$

Now we can use the Virial theorem

$$\begin{aligned} \langle n|\frac{p^2}{M}|n\rangle &= -E_n \\ \langle n|V_s|n\rangle &= 2E_n \end{aligned} \tag{6}$$

Then,

$$\langle n|r^i(E_n - h_s)^2 r^i|n\rangle = \frac{4}{M^2}\langle n|p^2|n\rangle = -\frac{4E_n}{M} \tag{7}$$

Finally, we move to the computation of the second term

$$\begin{aligned} \langle n|r^i\{\Delta V, E_n - h_s\}r^i|n\rangle &= -\frac{\alpha_s N_c}{2}\langle n|\frac{r^i}{r}[h_s, r^i]|n\rangle + \frac{\alpha_s N_c}{2}\langle n|[h_s, r^i]\frac{r^i}{r}|n\rangle \\ &= \frac{\alpha_s N_c}{2}\langle n|[[h_s, r^i], \frac{r^i}{r}]|n\rangle = -\frac{i\alpha_s N_c}{M}\langle n|[p^i, \frac{r^i}{r}]|n\rangle \end{aligned} \tag{8}$$

We need to compute the commutator in this expression

$$[p^i, \frac{r^i}{r}] = r^i[p^i, \frac{1}{r}] + [p^i, r^i]\frac{1}{r} = -ir^i\frac{d}{dr^i}\left(\frac{1}{r}\right) - 3i\frac{1}{r} = -\frac{2i}{r} \tag{9}$$

Therefore,

$$\langle n|r^i\{\Delta V, E_n - h_s\}r^i|n\rangle = -\frac{2\alpha_s N_c}{M}\langle n|\frac{1}{r}|n\rangle = \frac{2N_c}{C_F M}\langle n|V_s|n\rangle \tag{10}$$

using again the virial theorem

$$\langle n|r^i\{\Delta V, E_n - h_s\}r^i|n\rangle = \frac{2N_c}{C_F M}\langle n|V_s|n\rangle = \frac{4N_c E_n}{C_F M} \tag{11}$$

In Summary,

$$\langle n|r^i(E_n - h_o)^2 r^i|n\rangle = -\frac{4E_n}{M} - \frac{4N_c E_n}{C_F M} + \frac{\alpha_s^2 N_c^2}{4} \tag{12}$$

1.2 Exercise 2

Consider the Lindblac equation obtained in the regime $\frac{1}{r} \gg T \gg E$ in the static limit, $M \to \infty$. Solve the equations for an arbitrary initial density of singlets and octets. Show that the entropy increases monotonically.

The equations to solve are the following

$$\frac{d\rho_s(\mathbf{r},\mathbf{r}')}{dt} = -i(V_s(\mathbf{r}) - V_s(\mathbf{r}'))\rho_s(\mathbf{r},\mathbf{r}') - \frac{\kappa(r^2 + (r')^2)}{2}\rho_s(\mathbf{r},\mathbf{r}') + \frac{\kappa \mathbf{r}\mathbf{r}'}{N_c^2 - 1}\rho_o(\mathbf{r},\mathbf{r}') \tag{13}$$

$$\frac{d\rho_o(\mathbf{r},\mathbf{r}')}{dt} = -i(V_o(\mathbf{r}) - V_o(\mathbf{r}'))\rho_o(\mathbf{r},\mathbf{r}') - \frac{\kappa(r^2 + (r')^2)}{2}\frac{N_c^2 - 2}{2(N_c^2 - 1)}\rho_o(\mathbf{r},\mathbf{r}') + \kappa \mathbf{r}\mathbf{r}'\left(\rho_s(\mathbf{r},\mathbf{r}') + \frac{N_c^2 - 4}{2(N_c^2 - 1)}\rho_o(\mathbf{r},\mathbf{r}')\right) \tag{14}$$

As expected, the evolution only relates terms with the same $\mathbf{r}$ and $\mathbf{r}'$. We can consider the following

- At the end of the day we are interested in the probability to have a singlet or an octet, in other words, in $p_x(\mathbf{r}) = \rho_x(\mathbf{r},\mathbf{r})$.
- Formally, we can regard the Lindblad equation as an equation of the type $\frac{d\rho}{dt} = \mathcal{L}[\rho(t)]$. Since the trace is conserved we know that there is an eigenvector with zero eigenvalue. Because probabilities can not be bigger than one, we know that the rest of eigenvalues are negative. Therefore, the components with $\mathbf{r}' \neq \mathbf{r}$ will all have negative eigenvalues and decay fast.

Taking this into account, we can focus on the probabilities

$$\begin{aligned} \frac{dp_s}{dt} &= -\kappa r^2 p_s + \frac{\kappa r^2}{N_c^2 - 1}p_o \\ \frac{dp_o}{dt} &= -\frac{dp_s}{dt} \end{aligned} \tag{15}$$

Let us define

$$p_t = p_s + p_o$$
$$p_d = p_s - \frac{p_o}{N_c^2 - 1} \tag{16}$$

Note that $p_t = Tr(\rho)$ and, as can be easily checked, it is a constant. Regarding p_d,

$$\frac{dp_d}{dt} = -\frac{\kappa N_c^2 r^2}{N_c^2 - 1} p_d \tag{17}$$

therefore,

$$p_d(t) = \mathrm{e}^{-\frac{\kappa N_c^2 r^2}{N_c^2 - 1} t} p_d(0) \tag{18}$$

Undoing the transformation we get

$$p_s(t) = \frac{1}{N_c^2} p_t + \frac{N_c^2 - 1}{N_c^2} \mathrm{e}^{-\frac{\kappa N_c^2 r^2}{N_c^2 - 1} t} p_d(0)$$
$$= \frac{1}{N_c^2}(p_s(0) + p_o(0)) + \frac{N_c^2 - 1}{N_c^2}\left(p_s(0) - \frac{p_o(0)}{N_c^2 - 1} p_o(0)\right) \mathrm{e}^{-\frac{\kappa N_c^2 r^2}{N_c^2 - 1} t} \tag{19}$$

$$p_o(t) = \frac{N_c^2 - 1}{N_c^2}\left(p_t - \mathrm{e}^{-\frac{\kappa N_c^2 r^2}{N_c^2 - 1} t} p_d(0)\right)$$
$$= \frac{N_c^2 - 1}{N_c^2}(p_s(0) + p_o(0)) - \frac{N_c^2 - 1}{N_c^2}\left(p_s(0) - \frac{p_o(0)}{N_c^2 - 1} p_o(0)\right) \mathrm{e}^{-\frac{\kappa N_c^2 r^2}{N_c^2 - 1} t} \tag{20}$$

Now we look at the entropy

$$S = -p_s \log p_s - p_o \log\left(\frac{p_o}{N_c^2 - 1)}\right) \tag{21}$$

Note that the octet has degeneracy $N_c^2 - 1$. Alternatively, we could have considered eight different p_o^A probabilities and impose because of the symmetries of the problem that they are all the same. To check whether the entropy increases monotonically or not, we compute its derivative

$$\frac{dS}{dt} = -\frac{dp_s}{dt}(1 + \log p_s) - \frac{dp_o}{dt}\left(1 + \log\left(\frac{p_o}{N_c^2 - 1}\right)\right)$$
$$= \kappa r^2 \left(p_s - \frac{p_o}{N_c^2 - 1}\right) \log\left(\frac{p_s(N_c^2 - 1)}{p_o}\right) = \frac{\kappa r^2 p_o}{N_c^2 - 1}(x - 1)\log x \tag{22}$$

where $x = \frac{p_s(N_c^2 - 1)}{p_o}$. Note that the function $(x - 1)\log x$ is positive for any value of x between 0 and ∞. Therefore $\frac{dS}{dt} > 0$.

2 Multiple-scale analysis example

Consider the equation

$$\frac{dv}{dt} + iEv = -\epsilon\Gamma(t)v \tag{23}$$

where ϵ is small. Using naive perturbation theory we get, at LO

$$\frac{dv_0}{dt} + iEv_0 = 0 \tag{24}$$

then

$$v_0(t) = \mathrm{e}^{-iEt} v_0(0) \tag{25}$$

At NLO

$$\frac{dv_1}{dt} + iEv_1 = -\Gamma(t) v_0 \tag{26}$$

then

$$v_1(t) = -\int_0^t dt' \Gamma(t') \mathrm{e}^{-iEt} v_0(0) \tag{27}$$

Therefore,

$$v(t) = v_0(t) + \epsilon v_1(t) + \cdots = (1 - \epsilon \int_0^t dt' \Gamma(t')) \mathrm{e}^{-iEt} \tag{28}$$

You can see that perturbation theory breaks at large times if $\int_0^t dt' \Gamma(t')$ grows linearly. Now we apply multiple-scale analysis, at LO we get

$$\frac{dv_0}{dt} + iEv_0 = 0 \tag{29}$$

therefore we get

$$v_0(t, \tau) = \mathrm{e}^{-iEt} v_0(0, \tau) \tag{30}$$

At NLO

$$\frac{dv_1}{dt} + \frac{dv_0}{d\tau} + iEv_1 = -\Gamma(t) v_0 \tag{31}$$

or, equivalently

$$\frac{dv_1}{dt} + \mathrm{e}^{-iEt} \frac{dv_0(0, \tau)}{d\tau} + iEv_1 = -\Gamma(t) \mathrm{e}^{-iEt} v_0(0, \tau) \tag{32}$$

we can separate $\Gamma(t) = \bar{\Gamma} + \delta\Gamma(t)$, such that $\bar{\Gamma} = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt' \Gamma(t')$. Then

$$\frac{dv_0(0, \tau)}{d\tau} = -\bar{\Gamma} v_0(0, \tau) \tag{33}$$

and

$$v_1(t, \tau) = -\int_0^t dt' \delta\Gamma(t') v_0(t, \tau) \tag{34}$$

In summary,

$$v_0(t, \tau) = \mathrm{e}^{-iEt - \bar{\Gamma}\tau} v_0(0, 0) \tag{35}$$

and writing everything in terms of t

$$v_0(t) = \mathrm{e}^{-iEt - \epsilon\bar{\Gamma}t} v_0(0) \tag{36}$$

And the full result including v_1

$$v(t) = \left(1 - \epsilon \int_0^t dt' \delta\Gamma(t')\right) \mathrm{e}^{-iEt - \epsilon\bar{\Gamma}t} v_0(0) + \cdots \tag{37}$$