On-shell exchanges and Lattice QCD

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Hadronic three-body Christmas
Munich Institute for Astro- and Particle Physics
Tuesday, December 21st, 2021
For three-body experts – black to move

chessskill.blogspot.com/2020/06/three-pawns-problem.html
For three-body experts - white to move
Three-body processes and hadronic spectrum

- Exotic resonances decay to three-particle final states
  - $X(3872)$, $N^*(1440)$, $a_1(1260)$, $a_1(1420)$, ...

- Interpretations
  - molecules?
  - diquark–antidiquark?
  - hybrids?
  - kinematical effects?

- Goal
  - properties of hadrons from the (Lattice) QCD
  - general three-body framework for phenomenology
Goal — three-body scattering formalism

- derive properties of hadrons from the (lattice) QCD
- having convenient three-body framework for phenomenology

\[ \mathcal{L}_{\text{QCD}} = \sum_f \overline{\Psi}_f (i \not{D} - m_f) \Psi_f - \frac{1}{4} G^a_{\mu \nu} G^{a \mu \nu} \]

\[ S_G[U] = \frac{2}{g^2} \sum_{n \in \Lambda} \sum_{\mu < \nu} \text{Re} \text{ tr} [1 - U_{\mu \nu}(n)] + \text{fermions} \]

Eigenvalues of the QCD Hamiltonian

Resonances

not only a question of computational cost. With decreasing pion mass also the kinematical thresholds for three- and four-hadron channels decrease. In particular highly excited states couple strongly to such multi-hadron final states. However, the current method is not applicable to these channels. A complete finite-volume formalism for three or even more particles would therefore be a major breakthrough for the calculation of masses and decay modes of hadron resonances. Such a formalism is already under development (see Ref. [33] and references therein).
Fundamentals of the S matrix

- Analyticity on the first Riemann sheet,
- Bound–states & resonances correspond to poles,
- Branch cuts correspond to open channels

Seeds of the S–matrix theory

- Unitarity (probability conservation)
- Analyticity (causality)
- Crossing symmetry (particles ↔ antiparticles)
- Poincaré symmetry (frame independence)
- Internal symmetries (charge, isospin, G–parity)

\[ 2 \text{Im} \begin{array}{c} \scalebox{1.5}{\textbullet} \\ \scalebox{1.5}{\textbullet} \end{array} = \begin{array}{c} \scalebox{1.5}{\textbullet} \\ \scalebox{1.5}{\textbullet} \\ \scalebox{1.5}{\textbullet} \end{array} \]

K–matrix parametrization

\[ \mathcal{M}_\ell(s) = \frac{1}{K_\ell^{-1}(s) - i\rho(s)} \]

Phase shift

\[ K_l^{-1}(s) = \frac{q^*}{8\pi E^*} \cot \delta_l(E^*) \]
Two-body quantization condition

- Relation between (Infinite Volume) phase shift and (Finite Volume) spectrum

\[ \det \left[ \mathcal{M}(s) + F^{-1}(s, P, L) \right] = 0 \]

Matrices in the angular momentum space (truncation)

Complicated but known function

An \( a_0 \) resonance in strongly coupled \( \pi \eta, KK \) scattering from lattice QCD


A \( b_1 \) resonance in coupled \( \pi \omega, \pi \phi \) scattering from lattice QCD

Woss et al. (HadSpec), Phys. Rev. D 100 (2019) 5, 054506
Path to three-body physics from QCD

- Finite volume spectrum $\rightarrow$ Quantization Condition $\rightarrow$ Three-body K-matrix
- $K$-matrix + two-body subprocesses $\rightarrow$ integral equations $\rightarrow$ 3-body amplitudes
- Final amplitudes analytically continued to the unphysical Riemann sheets
Path to three-body physics from QCD

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Final amplitudes analytically continued to the unphysical Riemann sheets

Two main frameworks

Relativistic Effective Field Theory
- generic scalar EFT,
- summation of 2PI, 3PI diagrams,
  (references in back-up slides)
- First results — three pions at $I=3$

Unitarity-based framework
- parametrization based on the S-matrix unitarity,
  (references in back-up slides)
- First results — three pions at $I=3$

Lattice QCD     FV Spectrum     Amplitudes     Particle properties
Path to three-body physics from QCD

- Finite volume spectrum → Quantization Condition → Three-body K-matrix
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    - Blanton, Sharpe, Phys. Rev. D 102 (2020) 054515
Three-body S-matrix unitarity in one slide

Phenomenology of relativistic $3 \rightarrow 3$ reaction amplitudes within the isobar approximation

Three-body S-matrix unitarity in one slide

Phenomenology of relativistic $3 \to 3$ reaction amplitudes within the isobar approximation


Three-body unitarity relation

$$2\text{Im} = \sum_{j,k} \sum_{i} \left(\frac{1}{p_j} - \frac{1}{p_k}\right)$$

Hadron spectroscopy and three-body systems
Three-body $S$-matrix unitarity in one slide

Phenomenology of relativistic $3 \rightarrow 3$ reaction amplitudes within the isobar approximation


Three-body unitarity relation
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Pair/Isobar and Spectator choice

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Pair/Isobar and Spectator choice

Partial-wave projection

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Pair/Isobar and Spectator choice

Partial-wave projection

Two Im

Three-body unitarity relation

Amputation of two-body interactions
The S-matrix unitarity approach

- Physical degrees of freedom *(domain of integration)*
- Simple parametrization with clear interpretation

\[
A = M_2 B M_2 + M_2 \int B \tau A
\]

Three-body amplitude

\[
A_{\ell m_\ell; \ell' m_{\ell'}}(\sigma', s, \sigma)
\]

- pair-spectator,
- partial waves,
- symmetrization,
The S-matrix unitarity approach

- Physical degrees of freedom \((\text{domain of integration})\)
- Simple parametrization with clear interpretation

\[
\tilde{A} = B + \int B M_2 \tau \tilde{A}
\]

\[
M_2 = K + K i \rho M_2
\]

Three-body amplitude
\[A_{\ell m_\ell; \ell' m_{\ell'}}(\sigma', s, \sigma)\]
- pair-spectator,
- partial waves,
- symmetrization,
\( \mathcal{M}_{3,pq} = A_{pq} + \mathcal{M}_{2,p} \delta_{pq} \)

\( A_{pq} = \mathcal{M}_{2,p} \mathcal{B}_{pq} \mathcal{M}_{2,q} + \mathcal{M}_{2,p} \int_k \mathcal{B}_{pk} \tau_k A_{kq} \)
**FV unitarity quantization condition**

Lattice QCD and three-particle decays of resonances

Three-body spectrum in a finite volume: the role of cubic symmetry

\[\mathcal{M}_{3,pq} = \mathcal{A}_{pq} + \mathcal{M}_{2,p} \delta_{pq}\]

\[\mathcal{A}_{pq} = \mathcal{M}_{2,p} \mathcal{B}_{pq} \mathcal{M}_{2,q} + \mathcal{M}_{2,p} \int_{k} \mathcal{B}_{pk} \tau_{k} \mathcal{A}_{kq}\]
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\[ A_{pq} = M_{2,p} B_{pq} M_{2,q} + M_{2,p} \int_k B_{pk} \tau_k A_{kq} \]

Finite volume version

\[ A^L_{pq} = M^L_{2,p} B_{pq} M^L_{2,q} + \frac{1}{L^3} M^L_{2,p} \sum_k B_{pk} \tau_k A^L_{kq} \]

\[ M^L_{3,pq} = A^L_{pq} + M^L_{2,p} L^3 \delta_{pq} \]
\[ M_{3,pq} = A_{pq} + M_{2,p} \delta_{pq} \] 
\[ A_{pq} = M_{2,p} B_{pq} M_{2,q} + M_{2,p} \int_k B_{pk} \tau_k A_{kq} \] 
\[ M_{3,pq}^L = A_{pq}^L + M_{2,p}^L L^3 \delta_{pq} \] 
\[ A_{pq}^L = M_{2,p}^L B_{pq} M_{2,q}^L + \frac{1}{L^3} M_{2,p}^L \sum_k B_{pk} \tau_k A_{kq}^L \] 
\[ A^L = \left( I - \frac{1}{L^3} M_2 B \tau \right)^{-1} M_2^L B M_2^L \]
$\mathcal{M}_{3,pq} = A_{pq} + \mathcal{M}_{2,p} \delta_{pq}$  \hspace{1cm} \text{Full amplitude: connected and disconnected parts}

$A_{pq} = \mathcal{M}_{2,p} B_{pq} \mathcal{M}_{2,q} + \mathcal{M}_{2,p} \int_k B_{pk} \tau_k A_{kq}$

$\mathcal{M}_{3,pq}^L = A_{pq}^L + \mathcal{M}_{2,p}^L L^3 \delta_{pq}$

$A_{pq}^L = \mathcal{M}_{2,p}^L B_{pq} \mathcal{M}_{2,q}^L + \frac{1}{L^3} \mathcal{M}_{2,p}^L \sum_k B_{pk} \tau_k A_{kq}^L$

$A^L = \left( \mathbb{1} - \frac{1}{L^3} \mathcal{M}_2 B \tau \right)^{-1} \mathcal{M}_2^L B \mathcal{M}_2^L$

Plugged to the first equation:

$\mathcal{M}_{3,pq}^L = \left( \mathbb{1} - \frac{1}{L^3} \mathcal{M}_2 B \tau \right)^{-1} \mathcal{M}_2^L B \mathcal{M}_2^L + L^3 \mathcal{M}_2^L$
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\[ M_{3,pq} = A_{pq} + M_{2,p} \delta_{pq} \]

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\[ A_{pq} = M_{2,p} B_{pq} M_{2,q} + M_{2,p} \int_k B_{pk} \tau_k A_{kq} \]

Finite volume version

\[ A^L_{pq} = M^L_{2,p} B_{pq} M^L_{2,q} + \frac{1}{L^3} M^L_{2,p} \sum_k B_{pk} \tau_k A^L_{kq} \]

Matrix solution

\[ M^L_{3,pq} = \left( I - \frac{1}{L^3} M_{2,B} \tau \right)^{-1} M^L_{2,B} M^L_{2} \]

Plugged to the first equation

\[ M^L_{3,pq} = \left( I - \frac{1}{L^3} M_{2,B} \tau \right)^{-1} M^L_{2,B} M^L_{2} + L^3 M^L_{2} \]

Recombining geometric series

\[ M^L_{3,} = \left[ \left( I - \frac{1}{L^3} M_{2,B} \tau \right)^{-1} \frac{1}{L^3} M^L_{2,B} + I \right] L^3 M^L_{2} \]

\[ = - \left( I + \frac{1}{L^3} M^L_{2,B} \right) \left( (M^L_{2})^{-1} + \frac{1}{L^3} B \tau \right)^{-1} \]
\[ \mathcal{M}_{3,pq} = \mathcal{A}_{pq} + \mathcal{M}_{2,p} \delta_{pq} \]

\[ \mathcal{A}_{pq} = \mathcal{M}_{2,p} \mathcal{B}_{pq} \mathcal{M}_{2,q} + \mathcal{M}_{2,p} \int_k \mathcal{B}_{pk} \tau_k \mathcal{A}_{kq} \]

Full amplitude: connected and disconnected parts

\[ \mathcal{M}_{3,pq}^{L} = \mathcal{A}_{pq}^{L} + \mathcal{M}_{2,p}^{L} L^3 \delta_{pq} \]

\[ \mathcal{A}_{pq}^{L} = \mathcal{M}_{2,p}^{L} \mathcal{B}_{pq}^{L} \mathcal{M}_{2,q}^{L} + \frac{1}{L^3} \mathcal{M}_{2,p}^{L} \sum_k \mathcal{B}_{pk} \tau_k \mathcal{A}_{kq}^{L} \]

Finite volume version

\[ \mathcal{A}^L = \left( \mathbb{1} - \frac{1}{L^3} \mathcal{M}_2 \mathcal{B} \tau \right)^{-1} \mathcal{M}_2^L \mathcal{B} \mathcal{M}_2^L \]

Matrix solution

\[ \mathcal{M}_{3,pq}^{L} = \left( \mathbb{1} - \frac{1}{L^3} \mathcal{M}_2 \mathcal{B} \tau \right)^{-1} \mathcal{M}_2^L \mathcal{B} \mathcal{M}_2^L + L^3 \mathcal{M}_2^L \]

Plugged to the first equation

\[ \mathcal{M}_3^L = \left[ \left( \mathbb{1} - \frac{1}{L^3} \mathcal{M}_2 \mathcal{B} \tau \right)^{-1} \mathcal{M}_2^L \mathcal{B} + \mathbb{1} \right] L^3 \mathcal{M}_2^L \]

Recombining geometric series

\[ \det \left( \mathcal{M}_2^L \right)^{-1} + \frac{1}{L^3} \mathcal{B} \tau \right)^{-1} = 0 \]
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Three-body spectrum in a finite volume: the role of cubic symmetry

\[ M_{3,pq} = A_{pq} + M_{2,p} \delta_{pq} \]

\[ A_{pq} = M_{2,p} B_{pq} M_{2,q} + M_{2,p} \int B_{pq} \tau_k A_{kq} \]

\[ M^L_{3,pq} = A^L_{pq} + M^L_{2,p} L^3 \delta_{pq} \]

\[ A^L = \left( I - \frac{1}{L^3} M_2 B_\tau \right)^{-1} M_2^L B M_2 \]

\[ M^L_{3,pq} = \left( I - \frac{1}{L^3} M_2 B_\tau \right)^{-1} M_2^L B M_2^L + L^3 M_2^L \]

\[ = - \left( I + \frac{1}{L^3} M_2^L B \right) \left( (M_2^L)^{-1} + \frac{1}{L^3} B_\tau \right)^{-1} \]

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\[
\mathcal{A}_{pq} = \mathcal{M}_{2,p} \mathcal{B}_{pq} \mathcal{M}_{2,q} + \mathcal{M}_{2,p} \int \mathcal{B}_{pk} \mathcal{T}_k \mathcal{A}_{kq}
\]

Full amplitude: connected and disconnected parts

Finite volume version

\[
\mathcal{M}_{3,pq}^L = \mathcal{A}_{pq}^L + \mathcal{M}_{2,p}^L L^3 \delta_{pq}
\]

\[
\mathcal{M}_{3,pq}^L = \left( \mathcal{I} - \frac{1}{L^3} \mathcal{M}_{2} \mathcal{B} \right)^{-1} \mathcal{M}_{2}^L \mathcal{B} \mathcal{M}_{2}^L
\]

Recombining geometric series

\[
\mathcal{A}^L = \mathcal{M}_{2}^L \mathcal{B} \mathcal{M}_{2}^L + L^3 \mathcal{M}_{2}^L
\]

Matrix solution

\[
\mathcal{M}_{3,pq}^L = \left( \mathcal{I} - \frac{1}{L^3} \mathcal{M}_{2} \mathcal{B} \right)^{-1} \mathcal{M}_{2}^L \mathcal{B} \mathcal{M}_{2}^L + L^3 \mathcal{M}_{2}^L
\]

Plugged to the first equation

\[
\mathcal{M}_{3}^L = \left[ \left( \mathcal{I} - \frac{1}{L^3} \mathcal{M}_{2} \mathcal{B} \right)^{-1} \mathcal{M}_{2}^L \mathcal{B} + \mathcal{I} \right] L^3 \mathcal{M}_{2}^L
\]

\[
\frac{1}{L^3} \mathcal{M}_{2} \mathcal{B} \mathcal{M}_{2}^L + \mathcal{I}
\]

Unknown K-matrix and R-matrix

Geometric function

\[
\det \left( \left( \mathcal{M}_{2}^L \right)^{-1} + \frac{1}{L^3} \mathcal{B} \right)^{-1} = 0
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\[ M_{3,pq}^L = A_{pq}^L + M_{2,p}^L L^3 \delta_{pq} \]

\[ B_{pq}^L = B_{pq} M_{2,q}^L + \frac{1}{L^3} M_{2,p}^L \sum B_{pk} \tau_k A_{kq}^L \]

\[ A^L = \left( I - \frac{1}{L^3} M_2 B \tau \right)^{-1} M_2^L B M_2^L \]

\[ M_{3,pq}^L = \left( I - \frac{1}{L^3} M_2 B \tau \right) \cdot M_2^L B \cdot M_2^L \]

\[ M_3^L = \left[ \left( I - \frac{1}{L^3} M_2 B \tau \right)^{-1} \frac{1}{L^3} M_2^L B + I \right] L^3 M_2^L \]

\[ = \left( \left( I - \frac{1}{L^3} M_2 B \tau \right)^{-1} \right) \left( \left( M_2^L \right)^{-1} + \frac{1}{L^3} B \tau \right)^{-1} \]

\[ \det \left( \left( M_2^L \right)^{-1} + \frac{1}{L^3} B \tau \right)^{-1} = 0 \]

**Unknown K-matrix and R-matrix**

**Geometric function**

**Not very useful**

**Recombining geometric series**

**Full amplitude: connected and disconnected parts**

**Matrix solution**

Plugged to the first equation
REFT three-body formalism

\[
C_L(E, \bar{P}) = \ldots + \left( B_1 \right) + \left( B_2 \right) + \ldots + \left( B_3 \right) + \left( B_4 \right) + \ldots + \ldots
\]

\[
\det \left[ K_{df,3}(s) + F_3(s, P, L)^{-1} \right] = 0
\]

Infinite volume integral equations:

\[
\mathcal{M}_3^{(u,u)} = \mathcal{D}^{(u,u)} + \mathcal{M}_{df,3}^{(u,u)}
\]
**REFT three-body formalism**

\[ C_{L}(E, \vec{P}) = \ldots + \frac{C_{B_{1}}}{B_{2}} + \frac{C_{B_{2}}}{B_{3}} + \ldots \]

**Infinite volume integral equations:**

\[ \text{det} \left[ \mathcal{K}_{df,3}(s) + F_{3}(s, \vec{P}, L)^{-1} \right] = 0 \]

\[ \mathcal{M}_{3}^{(u,u)} = \mathcal{D}^{(u,u)} + \mathcal{M}_{df,3}^{(u,u)} \]
REFT three-body formalism

\[ C_L(E, \bar{P}) = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{...}
\end{array}
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\end{array}
\end{array} \]

\[ \text{det} \left[ \mathcal{K}_{\text{df},3}(s) + F_3(s, P, L)^{-1} \right] = 0 \]

\[ M_3^{(u,u)} = D^{(u,u)} + M_{\text{df},3}^{(u,u)} \]

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REFT three–body formalism

\[ C_L(E, P) = \text{Diagram representations} + \ldots \]

\[ \det \left[ K_{df,3}(s) + F_3(s, P, L)^{-1} \right] = 0 \]

Infinite volume integral equations:

\[ M_3^{(u,u)} = D^{(u,u)} + M_{df,3}^{(u,u)} \]

\[ \text{Diagram representations} \]
Equivalence of three-body formalisms

Both formalisms equivalent in the FV and IV form

\[
\int_k \left[ \delta_{p'k} + \frac{1}{3} \int_{k'} U_{p'k'K_{df,k'k}} \right] R_{kp} = \int_{k'} \int_k U_{p'k'K_{df,k'k}} U_{kp}
\]

\[
\begin{array}{cccccc}
\mathcal{L} & \mathcal{K} & \mathcal{L} \\
{p'} & k' & p \\
\mathcal{L} & \mathcal{K} & \mathcal{L} \\
{p'} & k & p \\
\mathcal{L} & \mathcal{K} & \mathcal{L} \\
{p'} & k' & p \\
\mathcal{L} & \mathcal{K} & \mathcal{L} \\
{p'} & k & p' \\
\end{array}
= \frac{1}{9}
\begin{array}{cccccc}
\mathcal{L} & \mathcal{K} & \mathcal{L} \\
{p} & k & p \\
\mathcal{L} & \mathcal{K} & \mathcal{L} \\
{p} & k' & p \\
\mathcal{L} & \mathcal{K} & \mathcal{L} \\
{p'} & k & p \\
\mathcal{L} & \mathcal{K} & \mathcal{L} \\
{p'} & k' & p' \\
\end{array} + \frac{1}{3}
\begin{array}{cccccc}
\mathcal{L} & \mathcal{K} & \mathcal{L} \\
{p} & k & p \\
\mathcal{L} & \mathcal{K} & \mathcal{L} \\
{p} & k' & p \\
\mathcal{L} & \mathcal{K} & \mathcal{L} \\
{p'} & k & p \\
\mathcal{L} & \mathcal{K} & \mathcal{L} \\
{p'} & k' & p' \\
\end{array} + \frac{1}{3} \left[ \mathcal{L} \right]_{k} + \left[ \mathcal{L} \right]_{k'} + \left[ \mathcal{L} \right]_{p} \quad + O(K^2)
\]

\[
\det \left[ \bar{K}_{2,L}^{-1} + \bar{F} + \bar{G} - (2\omega L^3)^{-1} R^{(u,u)} (2\omega L^3)^{-1} \right] = 0
\]

In practice, differences in the regularization schemes exist in the literature.
Three–pions calculation

- Maximal isospin I=3 and S wave scattering
- Three–dynamical quarks N_f=2+1
- Pion mass: 391 MeV, Kaon mass: 550 MeV
- Lattice 20^3x256, 24^3x128

Quark bilinears

\[ \bar{\psi} \Gamma D \ldots D \psi \]

Generalized Eigenvalue Equation

\[ G_{ij}(t) = \langle \mathcal{O}_i(t) \mathcal{O}_j(0) \rangle \]

\[ M(t, t_0) = G(t_0)^{-1/2} G(t) G(t_0)^{-1/2} \]
### Three-pion calculation

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#### Spectrum

- $E_n^*(L)/m_\pi$

---

#### Fits

$$M_\ell(s) = \frac{1}{K_\ell^{-1}(s) - i\rho(s)}$$

<table>
<thead>
<tr>
<th>$p \cot \delta(p)$</th>
<th>$K_{3,iso}$</th>
<th>fit result</th>
<th>$\chi^2$/dof</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1/a_0$</td>
<td>0</td>
<td>$m_\pi a_0 = 0.300 \pm 0.007$</td>
<td>64.7/(37 - 1) = 1.80</td>
</tr>
<tr>
<td>$-1/a_0$</td>
<td>$c_1/m_\pi^2$</td>
<td>$m_\pi a_0 = 0.296 \pm 0.008$</td>
<td>64.5/(37 - 2) = 1.84</td>
</tr>
<tr>
<td>$-1/a_0$</td>
<td>0</td>
<td>$m_\pi a_0 = 0.297 \pm 0.008$</td>
<td>50.9/(37 - 1) = 1.42</td>
</tr>
<tr>
<td>$-1/a_0$</td>
<td>$c_1/m_\pi^2$</td>
<td>$m_\pi a_0 = 0.293 \pm 0.010$</td>
<td>50.7/(37 - 2) = 1.45</td>
</tr>
</tbody>
</table>

---

#### Three-body integral equations

$$\begin{align*}
E_n^*(L)/m_\pi & = \frac{1}{K_\ell^{-1}(s) - i\rho(s)} \\
& = \frac{1}{\ldots}
\end{align*}$$
Solving the REFT three-body ladder equation

- **Ladder approximation**, $B = G + (R=0)$

- **Numerical solution of the three-body EFT equations**


- **Relevant studies**

  - weakly interacting system in the $\pi^+\pi^+$ and $\pi^+\pi^+\pi^+$
    
    Hansen et al. (HadSpec), Phys. Rev. Lett. 126 (2021), 012001

  - decay $a_1(1260) \to \rho^0\pi^- \to \pi^-\pi^+\pi^-$
    
Brute force method

\[
D(\sigma_p, s, \sigma_k) = -\mathcal{M}_2(\sigma_p) G(\sigma_p, s, \sigma_k) \mathcal{M}_2(\sigma_k) - \mathcal{M}_2(\sigma_p) \int_0^{(\sqrt{s} - m)^2} \frac{d\sigma_q}{2\pi} G(\sigma_p, s, \sigma_q) \tau(\sigma_q, s) D(\sigma_q, s, \sigma_k)
\]
Brute force method

\[ D(\sigma_p, s, \sigma_k) = -\bar{M}_2(\sigma_p) G(\sigma_p, s, \sigma_k) M_2(\sigma_k) - M_2(\sigma_p) \]

\[ \int_0^{(\sqrt{s} - m)^2} \frac{d\sigma_q}{2\pi} G(\sigma_p, s, \sigma_q) \tau(\sigma_q, s) D(\sigma_q, s, \sigma_k) \]
**Brute force method**

\[
D(\sigma_p, s, \sigma_k) = -\mathcal{M}_2(\sigma_p) G(\sigma_p, s, \sigma_k) \mathcal{M}_2(\sigma_k) - \mathcal{M}_2(\sigma_p)
\]

\[
= \int_0^{(\sqrt{s} - m)^2} \frac{d\sigma_q}{2\pi} G(\sigma_p, s, \sigma_q) \tau(\sigma_q, s) D(\sigma_q, s, \sigma_k)
\]

\[\text{Amputation}\]

\[
D(\sigma_p, s, \sigma_k) = \mathcal{M}_2(\sigma_p) d(\sigma_p, s, \sigma_k) \mathcal{M}_2(\sigma_k)
\]
Brute force method

\[
D(\sigma_p, s, \sigma_k) = - \mathcal{M}_2(\sigma_p) G(\sigma_p, s, \sigma_k) \mathcal{M}_2(\sigma_k) - \mathcal{M}_2(\sigma_p) \int_0^{(\sqrt{s} - m)^2} \frac{d\sigma_q}{2\pi} G(\sigma_p, s, \sigma_q) \tau(\sigma_q, s) D(\sigma_q, s, \sigma_k)
\]

\[
d(N, \epsilon) = -G(\epsilon) - P \cdot G(\epsilon) \cdot \mathcal{M} \cdot d(N, \epsilon)
\]

\[
D(\sigma_p, s, \sigma_k) = \mathcal{M}_2(\sigma_p) d(\sigma_p, s, \sigma_k) \mathcal{M}_2(\sigma_k)
\]
Brute force method

\[ D(\sigma_p, s, \sigma_k) = -M_2(\sigma_p)G(\sigma_p, s, \sigma_k)M_2(\sigma_k) - M_2(\sigma_p) \int_0^{(\sqrt{s} - m)^2} \frac{d\sigma_q}{2\pi} G(\sigma_p, s, \sigma_q) \tau(\sigma_q, s) D(\sigma_q, s, \sigma_k) \]

Discretization

\[ d(N, \epsilon) = -G(\epsilon) - P \cdot G(\epsilon) \cdot M \cdot d(N, \epsilon) \]

Matrix inversion

\[ d(N, \epsilon) = -[1 + P \cdot G(\epsilon) \cdot M]^{-1} \cdot G(\epsilon) \]

Amputation

\[ D(\sigma_p, s, \sigma_k) = M_2(\sigma_p)d(\sigma_p, s, \sigma_k)M_2(\sigma_k) \]
Brute force method

\[ D(\sigma_p, s, \sigma_k) = -M_2(\sigma_p) G(\sigma_p, s, \sigma_k) M_2(\sigma_k) M_2(\sigma_p) \int_0^{(\sqrt{s} - m)^2} \frac{d\sigma_q}{2\pi} G(\sigma_p, s, \sigma_q) \tau(\sigma_q, s) D(\sigma_q, s, \sigma_k) \]

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Matrix inversion

\[ d(N, \epsilon) = -\left[ I + P \cdot G(\epsilon) \cdot M \right]^{-1} \cdot G(\epsilon) \]

Limits

\[ d(\sigma_p, s, \sigma_k) = \lim_{\epsilon \to 0} \lim_{N \to \infty} d(N, \epsilon) \]
Hadron spectroscopy and three-body systems

Brute force method

\[ D(\sigma_p, s, \sigma_k) = -\mathcal{M}_2(\sigma_p) G(\sigma_p, s, \sigma_k) \mathcal{M}_2(\sigma_k) - \mathcal{M}_2(\sigma_p) \int_0^{(\sqrt{s} - m)^2} \frac{d\sigma_q}{2\pi} G(\sigma_p, s, \sigma_q) \tau(\sigma_q, s) D(\sigma_q, s, \sigma_k) \]

Discretization

\[ d(N, \epsilon) = -G(\epsilon) - P \cdot G(\epsilon) \cdot \mathcal{M} \cdot d(N, \epsilon) \]

Matrix inversion

\[ d(N, \epsilon) = -\left[ \mathbb{1} + P \cdot G(\epsilon) \cdot \mathcal{M} \right]^{-1} \cdot G(\epsilon) \]

Limits

\[ d(\sigma_p, s, \sigma_k) = \lim_{\epsilon \to 0} \lim_{N \to \infty} d(N, \epsilon) \]

Amputation

\[ D(\sigma_p, s, \sigma_k) = \mathcal{M}_2(\sigma_p) d(\sigma_p, s, \sigma_k) \mathcal{M}_2(\sigma_k) \]

Double ordered limit \( \rightarrow \) single limit

\[ \epsilon \propto \eta / N \]
Numerical procedures – continuation

Interpolation

\[ d(\sigma_p, s, \sigma_k) = -G(\sigma_p, s, \sigma_k) - \int_0^{(\sqrt{s-m})^2} \frac{d\sigma_q}{2\pi} G(\sigma_p, s, \sigma_q) \tau(\sigma_q, s) M_2(\sigma_q) d(\sigma_q, s, \sigma_k) \]

Limits

Poisson summation formula \(\longrightarrow\) error

\[ \sigma(\epsilon, N) \approx |G_s \rho_2 d^{(u,u)}| e^{-\eta} \]

\[ \eta = 2\pi \epsilon N \times \text{(energy factors)} \]
Numerical procedures – continuation

Interpolation

\[ d(\sigma_p, s, \sigma_k) = -G(\sigma_p, s, \sigma_k) - \int_0^{(\sqrt{s} - m)^2} \frac{d\sigma_q}{2\pi} G(\sigma_p, s, \sigma_q) \tau(\sigma_q, s) M_2(\sigma_q) d(\sigma_q, s, \sigma_k) \]

1. Plug the solution

Limits

Poisson summation formula → error

\[ \sigma(\epsilon, N) \approx \left| G_s \rho_2 d^{(u,u)} \right| e^{-\eta} \]

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1. Plug the solution
2. Plug the BS energies

Limits

Poisson summation formula → error

\[ \sigma(\epsilon, N) \approx |G_s \rho_2 d^{(u,u)}| e^{-\eta} \]

\[ \eta = 2\pi \epsilon N \times \text{(energy factors)} \]
Numerical procedures - continuation

- Interpolation

\[
d(\sigma_p, s, \sigma_k) = -G(\sigma_p, s, \sigma_k) - \int_0^{(\sqrt{s-m})^2} \frac{d\sigma_q}{2\pi} G(\sigma_p, s, \sigma_q) \tau(\sigma_q, s) M_2(\sigma_q) d(\sigma_q, s, \sigma_k)
\]

1. Plug the solution
2. Plug the BS energies
3. Integrate (sum over discrete momenta)

- Limits

Poisson summation formula \[\rightarrow\] error

\[
\sigma(\epsilon, N) \approx \left| G_s \rho_2 d^{(u,u)} \right| e^{-\eta}
\]

\[
\eta = 2\pi\epsilon N \times \text{(energy factors)}
\]

Brute force

\[
ma = 2 \quad (E/m)^2 = 8
\]

\[
\eta = 25
\]

\[
\eta = 15
\]

\[
\eta = 5
\]
Example results

Solution above the bound-state–spectator threshold

Solution above the three-body threshold: Break-up amplitude
Example result, three-body scattering length

Agreement with the finite-volume and NREFT studies

Shallow three-body bound state appears cyclicly?
Example result, three-body scattering length

![Graph showing three-body scattering length](graph)

- Agreement with the finite-volume and NREFT studies
- Shallow three-body bound state appears cyclically?
Analytical continuation below the threshold

- Singularities unique to three-body physics

\[ R = \frac{E_{b2}}{E_{b1}} \]

- Three-body bound state poles

Contour deformation required

- We need to solve for complex isobar energies lying along the contour
Analytical continuation below the threshold

- Singularities unique to three-body physics

- Three-body bound state poles

- Contour deformation required

- We need to solve for complex isobar energies lying along the contour
Below the threshold — circular cut

- Analytic structure of the integration kernel
- OPE develops the circular cut

- **Real Particle Exchange cut**

\[
\sigma_{\pm} = \frac{1}{2} (s - \sigma_{p'} + 3m^2) \pm \frac{1}{2\sqrt{\sigma_{p'}}} \sqrt{\sigma_{p'} - 4m^2} \lambda^{1/2}(s, \sigma_{p'}, m^2)
\]

- **Crossing the real axis**

\[
\sigma_{c1} = \frac{(m^2 - s)(m^2 - \sigma_{p'} + s)}{(m^2 - \sigma_{p'} - s)}
\]
\[
\sigma_{c2} = \frac{(m^2 - s)(2m^2 - \sigma_{p'})}{\sigma_{p'}}
\]
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\[
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\[ \sigma_{c1} = \frac{(m^2 - s)(m^2 - \sigma_{p'} + s)}{(m^2 - \sigma_{p'} - s)} \]

\[ \sigma_{c2} = \frac{(m^2 - s)(2m^2 - \sigma_{p'})}{\sigma_{p'}} \]
Below the threshold — circular cut

◎ Analytic structure of the integration kernel

◎ OPE develops the circular cut

◆ Real Particle Exchange cut

\[ \sigma_{\pm} = \frac{1}{2}(s - \sigma_{p'} + 3m^2) \pm \frac{1}{2\sqrt{\sigma_{p'} - 4m^2}} \sqrt{\sigma_{p'} - 4m^2} \lambda^{1/2}(s, \sigma_{p'}, m^2) \]

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\[ \sigma_{c2} = \frac{(m^2 - s)(2m^2 - \sigma_{p'})}{\sigma_{p'}} \]
Circular cut in motion picture

Epsilon regulator from the propagator

$\epsilon = +0.3$

$\epsilon = +0.3$
Circular cut in motion picture

Epsilon regulator from the propagator

\[ \epsilon = +0.3 \]

\[ \epsilon = +0.3 \]
Circular cut in motion picture

Epsilon regulator from the propagator

$\epsilon = +0.3$

Im $\sigma_p$

Re $\sigma_p$

Im OPE

Re OPE
Circular cut in motion picture

Epsilon regulator from the propagator

Integration from 0 to the dot
Below the threshold (preliminary), $m_a=25$

Naive discretization, no contour deformation

With the contour deformation
Generalization of the unitarity formalism

- Amplitude constrained by unitarity in the physical interval

\[ \int_q \equiv \int \frac{d\Omega_q}{4\pi} \int \frac{dq}{2\pi^2 \omega_q} = \int \frac{d\Omega_q}{4\pi} \int_{\sigma_{\text{min}}}^{\frac{(\sqrt{s}-m)^2}{2\pi}} d\sigma_q \tau(s, \sigma_q) \]

- Two-body pole outside of the integration range

- Generalization needed to include coupled channels
Generalization of the unitarity formalism

- Amplitude constrained by unitarity in the physical interval

\[
\sigma_{\text{min}} = 4m^2
\]

\[
\int_{q} = \int \frac{d\Omega q}{4\pi} \int \frac{dq q^2}{2\pi^2 \omega_q} = \int \frac{d\Omega q}{4\pi} \int_{\sigma_{\text{min}}}^{q_{\text{max}}} \frac{d\sigma_q}{2\pi} \tau(s, \sigma_q)
\]

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- Generalization needed

\[ \sigma_{\text{min}} = 4m^2 \]

Bound states in the B-matrix formalism for the three-body scattering

Dawid, Szczepaniak, Phys. Rev. D 103 (2021) 1, 014009
Bound-state–particle scattering

- Toy model study – formation of the three-body bound states
- Analytic properties of the B–matrix formalism

$$\begin{bmatrix} 1 & - & - \end{bmatrix} - 1$$

Bound–state–particle scattering

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**Bound–state–particle scattering**

- Toy model study — formation of the three-body bound states
- Analytic properties of the B–matrix formalism

---

**Figure:**

Matrix representation and plots illustrating the analytic properties of the B–matrix formalism. The figure shows the results from Jackura et al. (JPAC), *Eur. Phys. J. C* 79, no. 1, 56 (2019).
Generalization of the B–matrix formalism

- **Approximation:** all multi–particle kernels are constant and real (couplings $g_{ij}$),

  \[ a_{33}(s) = \frac{g_{33}}{1 - g_{22}i\rho_2 - g_{33}\mathcal{I}(s)}, \]

  \[ a_{22}(s) = \frac{g_{22}}{1 - g_{22}i\rho_2 - g_{33}\mathcal{I}(s)}. \]

- Solutions do not satisfy unitarity below the three–body threshold
- Spurious singularities start arbitrarily close to the two–body threshold

Kernel suffering from non–physical left–hand cuts:

\[ \mathcal{I}(s) = \int_{\sigma_{\text{min}}}^{(\sqrt{s} - m)^2} \frac{\sigma_q}{2\pi} \tau(s, \sigma_q) \mathcal{M}_2(\sigma_q) \]

---

**Notes:**

- Left–hand cut
- Two–particle threshold
- Three–particle threshold

---

Three–body scattering: ladders and resonances
Mikhasenko et al. (JPAC), JHEP 08 (2019) 1, 080

Bound states in the B–matrix formalism for the three–body scattering
Dawid, Szczepaniak, Phys. Rev. D 103 (2021) 1, 014009
Generalization of the B–matrix formalism

- **Approximation:** all multi–particle kernels are constant and real (couplings $g_{ij}$),

\[
\begin{align*}
    a_{33}(s) &= \frac{g_{33}}{1 - g_{22}i\rho_2 - g_{33} \mathcal{I}(s)}, \\
    a_{22}(s) &= \frac{g_{22}}{1 - g_{22}i\rho_2 - g_{33} \mathcal{I}(s)}.
\end{align*}
\]

Solutions do not satisfy unitarity below the three-body threshold.

Spurious singularities start arbitrarily close to the two-body threshold.

Three-body threshold

\[
\mathcal{I}(s) = \int_{\sigma_{\text{min}}}^{(\sqrt{s-m^2})^2} \frac{\sigma_q}{2\pi} \tau(s, \sigma_q) M_2(\sigma_q)
\]

Two-body threshold

\[
I(s) \text{ for } \sigma_{\text{min}}/m^2 = 4
\]
Analyticity of the B–matrix equations

- Dispersion procedure ensures analyticity

Kernel free from those problems:

\[
I_d(s) = \frac{(s - s_s)^2}{\pi \pi} \int ds' \frac{\text{Im } I(s')}{(s' - s - i\epsilon)(s' - s_s - i\epsilon)^2}
\]

Spurious singularities pushed to non–physical Riemann sheets and three–body physics influences the amplitude,

General dispersion procedure for the three–body unitarity formalism is needed
N/D approximation

 Hormon - perturbative approximation of the ladder solution

\[ A_2 = \frac{N(s)}{D(s)} \]

- \( N \) - contains left hand cuts,
- \( D \) - right hand cuts

\[ N(s) = \frac{1}{\pi} \int_{-\infty}^{s_{L2}} ds' \frac{\text{Im} A_2(s')}{s' - s - i\epsilon} D(s'), \]

\[ D(s) = 1 - \frac{s}{\pi} \int_{(M+m)^2}^{\infty} ds' \frac{\rho_2(s')}{s'(s' - s - i\epsilon)} N(s') \]
**N/D approximation**

- **Non-perturbative approximation of the ladder solution**

\[ A_2 = \frac{N(s)}{D(s)} \]

- **N** - contains left hand cuts,
- **D** - right hand cuts

\[ N(s) = \frac{1}{\pi} \int_{-\infty}^{s_{L2}} ds' \frac{\text{Im} A_2(s')} {s' - s - i\epsilon} D(s'), \]

\[ D(s) = 1 + \frac{s}{\pi^2} \int_{-\infty}^{s_{2L}} ds' g(s, s') \text{Im} A_2(s') D(s') \]

Theory of low pion-pion interactions
Chew, Mandelstam, Phys. Rev. 119 (1960) 467-477
Summary

- Equivalence between three-body formalisms
- Emergence of the first realistic lattice computations
- Systematic procedure for solving the integral equations
- Generalization of the $B$-matrix formalism
- Study of the analytic properties
Thank you
BACKUP SLIDES
Efimov physics - picture

- Short range, near-resonant interaction (barely supports two-body bound state)
- Induced long-range interaction
- Discrete scale symmetry
- Borromean binding

\[ E_n \propto - \left( e^{-\frac{2\pi}{s_0}} \right)^n \]
Efimov physics - picture

- Short range, near-resonant interaction (barely supports two-body bound state)

Two bodies

Three bodies

- Induced long-range interaction
- Discrete scale symmetry
- Borromean binding

\[ E_n \propto - \left( e^{-\frac{2\pi}{s_0}} \right)^n \]
Three-body scattering in the presence of bound states

Effimov physics - picture

Helium-4 trimer (He₃)∗ (excited state)

Ozone (O₃)

1.5 nm

0.5 nm

0.13 nm

117°

15 nm

\[ E_n \propto -\left(e^{-\frac{2\pi}{s_0}}\right)^n \]

Naidon, Endo, Rept. Prog. Phys. 80 (2017) 5, 056001
Efimov physics - math

Three scalar particles in zero-range approximation; Bethe–Peierls boundary condition

\[-\frac{1}{r \Psi} \frac{\partial}{\partial r} (r \Psi) \rightarrow \frac{1}{a} \quad \text{as} \quad r \rightarrow 0\]
Efimov physics - math

✧ Three scalar particles in zero-range approximation; Bethe–Peierls boundary condition

\[-\frac{1}{r\Psi} \frac{\partial}{\partial r} (r\Psi) \longrightarrow \frac{1}{a} \quad \text{as} \quad r \to 0\]

■ Schrödinger equation in Jacobi coordinates

\[(-\nabla_{r_{12}}^2 - \nabla_{\rho_{12,3}}^2 - k^2)\Psi = 0\]

\[
\begin{align*}
\vec{r}_{ij} & = \vec{x}_j - \vec{x}_i \\
\vec{\rho}_{ij,k} & = \frac{2}{\sqrt{3}} (\vec{x}_k - \frac{\vec{x}_i + \vec{x}_j}{2})
\end{align*}
\]
Efimov physics - math

◆ Three scalar particles in zero-range approximation; Bethe–Peierls boundary condition

\[-\frac{1}{r\Psi} \frac{\partial}{\partial r} (r\Psi) \rightarrow \frac{1}{a} \text{ as } r \rightarrow 0\]

■ Schrödinger equation in Jacobi coordinates

\[
(-\nabla_{r_{12}}^2 - \nabla_{\rho_{12,3}}^2 - k^2)\Psi = 0
\]

■ Hyper-spherical coordinates

\[
\left(-\frac{\partial^2}{\partial R^2} - \frac{1}{R} \frac{\partial}{\partial R} - \frac{1}{R^2} \frac{\partial^2}{\partial \alpha^2} - k^2\right) \chi_0(R, \alpha) = 0
\]

\[
r = R \sin \alpha
\]

\[
\rho = R \cos \alpha
\]

\[
R^2 = r^2 + \rho^2 = \frac{2}{3} \left( r_{12}^2 + r_{23}^2 + r_{31}^2 \right)
\]

\[
\tilde{r}_{ij} = \tilde{x}_j - \tilde{x}_i
\]

\[
\tilde{\rho}_{ij,k} = \frac{2}{\sqrt{3}} \left( \tilde{x}_k - \frac{\tilde{x}_i + \tilde{x}_j}{2} \right)
\]
Efimov physics - math

Three scalar particles in zero-range approximation; Bethe–Peierls boundary condition

\[-\frac{1}{r \Psi} \frac{\partial}{\partial r} (r \Psi) \rightarrow \frac{1}{a} \quad \text{as} \quad r \to 0\]

Schrödinger equation in Jacobi coordinates

\[(-\nabla_{r_{12}}^2 - \nabla_{\rho_{12,3}}^2 - k^2) \Psi = 0\]

Hyper-spherical coordinates

\[
\begin{align*}
\hat{r}_{ij} &= \vec{x}_j - \vec{x}_i \\
\hat{\rho}_{ij,k} &= \frac{2}{3} \left( \vec{x}_k - \frac{\vec{x}_i + \vec{x}_j}{2} \right)
\end{align*}
\]

\[
\left(-\frac{\partial^2}{\partial R^2} - \frac{1}{R} \frac{\partial}{\partial R} - \frac{1}{R^2} \frac{\partial^2}{\partial \alpha^2} - k^2\right) \chi_0(R, \alpha) = 0
\]

\[
\begin{align*}
r &= R \sin \alpha \\
\rho &= R \cos \alpha
\end{align*}
\]

\[
R^2 = r^2 + \rho^2 = \frac{2}{3} \left( r_{12}^2 + r_{23}^2 + r_{31}^2 \right)
\]

1/r^2 potential

\[
\left(-\frac{\partial^2}{\partial R^2} - \frac{1}{R} \frac{\partial}{\partial R} + \frac{s_n^2}{R^2} - k^2\right) F_n(R) = 0
\]
Efimov physics – math

- Three scalar particles in zero-range approximation; Bethe–Peierls boundary condition

\[ -\frac{1}{r\Psi} \frac{\partial}{\partial r} (r\Psi) \longrightarrow \frac{1}{a} \quad \text{as} \quad r \to 0 \]

- Beta function of the theory
Three-body scattering in the presence of bound states

Nonrelativistic studies

The Three-Boson System with Short-Range Interactions

**NREFT approach**

$$\mathcal{L} = \psi^\dagger (i\partial_0 + \frac{\vec{v}^2}{2m})\psi + \Delta T^\dagger T - \frac{g}{\sqrt{2}} (T^\dagger \psi\psi + \text{h.c.}) + h T^\dagger T \psi^\dagger \psi + \ldots$$

**The three-body equation**

$$\begin{array}{cccc}
T & = & + & + & T
\end{array}$$

Three-body spectrum in a finite volume: the role of cubic symmetry
Bound-state system from REFT QC

Numerical exploration of three relativistic particles in a finite volume including two-particle resonances and bound states
Romero-López et al., JHEP 10 (2019) 007

Start with the quantization condition

$$\det \left[ \mathcal{K}_{df,3}(s) + F_3(s, P, L)^{-1} \right] = 0$$
Bound-state system from REFT QC

Numerical exploration of three relativistic particles in a finite volume including two-particle resonances and bound states
Romero-López et al., JHEP 10 (2019) 007

Start with the quantization condition

\[
\det \left[ \mathcal{K}_{df,3}(s) + F_3(s, P, L)^{-1} \right] = 0
\]

S-wave, \( P=0 \), and \( K_{df}=0 \)

Geometric function

\[
L^3 F^s_3 \equiv \frac{\tilde{F}^s}{3} - \tilde{F}^s \frac{1}{1/\tilde{K}^s_2 + \tilde{F}^s + \tilde{G}^s} \tilde{F}^s
\]
Bound-state system from \textit{REFT QC}

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Romero-López et al., JHEP 10 (2019) 007

\begin{itemize}
  \item Start with the quantization condition
  \[
  \text{det} \left[ \mathcal{K}_{\text{df}, 3}(s) + F_3(s, P, L)^{-1} \right] = 0
  \]
  \text{S-wave, P=0, and K}_{\text{df}}=0
  \item Geometric function
  \[
  L^3 F_3^s = \frac{\tilde{F}_s}{3} - \frac{\tilde{F}_s}{1/\tilde{K}_2^s + \tilde{F}_s + \tilde{G}_s} \tilde{F}_s
  \]
  \item Energy levels
  \item Noninteracting 3-particle states,
  \item Noninteracting dimer-particle states,
  \item Interacting 3-particle, dimer-particle, and trimer states
\end{itemize}
Bound–state system from REFT QC

Numerical exploration of three relativistic particles in a finite volume including two-particle resonances and bound states
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Start with the quantization condition

\[
\text{det} \left[ K_{df,3}(s) + F_3(s, P, L)^{-1} \right] = 0
\]

S-wave, P=0, and K_{df}=0

Geometric function

\[
L^3 F^s_3 \equiv \frac{\tilde{F}^s}{3} - \tilde{F}^s \frac{1}{1/\tilde{K}^s_2 + \tilde{F}^s + \tilde{G}^s}
\]

Energy levels

Noninteracting 3–particle states,
Noninteracting dimer–particle states,
Interacting 3–particle, dimer–particle, and trimer states

2-body quantization condition
Bound state system from REFT QC

Numerical exploration of three relativistic particles in a finite volume including two-particle resonances and bound states
Romero-López et al., JHEP 10 (2019) 007

\[ \frac{k}{m} \cot \delta_0 \]

\[ (k/m)^2 \]

\( a=2 \)

\( a=16 \)
Numerical procedure

- Discretization of the integral equation → \( N \) linear equations (Matrix equation)
- Regulation of the bound-state pole via \( \epsilon \)-prescription

\[
A_2(s) = \lim_{\epsilon \to 0^+} \lim_{N \to \infty} A_2(s; N, \epsilon)
\]

Systematics

- **Unitarity test:** \( \text{Im} A_2(s) = \rho_2(s)|A_2(s)|^2 \quad \Delta \rho_2 = 100 \times \left| \frac{\text{Im} A_2^{-1}(s; N, \epsilon) + \rho_2(s)}{\rho_2(s)} \right| \)

- **Convergence test:** \( \Delta N A_2 = 2 \times \left| \frac{A_2(s; N + 1, \epsilon) - A_2(s; N, \epsilon)}{A_2(s; N + 1, \epsilon) + A_2(s; N, \epsilon)} \right| \)

Example methods

- "Brute force"
- Explicit pole removal
- Spline-based quadratures
Numerical procedure

- Discretization of the integral equation
- Regulation of the bound-state pole via $\epsilon$-prescription

Systematics

- **Unitarity test:** $\text{Im} A_2 = \frac{1}{2} \sigma q/m^2$ 
  
- **Convergence test:** $\Delta_{N} A_2 = 2 \times \frac{|A_2(s; N + 1, \epsilon) - A_2(s; N, \epsilon)|}{|A_2(s; N + 1, \epsilon) + A_2(s; N, \epsilon)|}$

Example methods

- "Brute force"
- Explicit pole removal
- Spline-based quadratures
Numerical procedure

- Discretization of the integral equation → N linear equations (Matrix equation)
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$$ A_2(s) = \lim_{\epsilon \to 0^+} \lim_{N \to \infty} A_2(s; N, \epsilon) $$

Systematics

- **Unitarity test:** $\text{Im} A_2(s) = \rho_2(s) |A_2(s)|^2$ → $\Delta \rho_2 = 100 \times \left| \frac{\text{Im} A_2^{-1}(s; N, \epsilon) + \rho_2(s)}{\rho_2(s)} \right|$

- **Convergence test:** $\Delta_N A_2 = 2 \times \left| \frac{A_2(s; N + 1, \epsilon) - A_2(s; N, \epsilon)}{A_2(s; N + 1, \epsilon) + A_2(s; N, \epsilon)} \right|$

Example methods

- “Brute force”
- Explicit pole removal
- Spline-based quadratures
Splines-based method

\[
d_{S}^{(u,u)}(p, k) = -G_{S}(p, k) - \int_{0}^{\infty} \frac{dk' k'^{2}}{(2\pi)^{2} \omega k'} G_{S}(p, k') M_{2}(k') d_{S}^{(u,u)}(k', k)
\]
Splines-based method

\[ d_S^{(u,u)}(p, k) = -G_S(p, k) - \int_0^\infty \frac{dk'k'^2}{(2\pi)^2\omega_{k'}} G_S(p, k')M_2(k')d_S^{(u,u)}(k', k) \]

\[ d_S^{(u,u)}(\sigma_k, \sigma_p) = -G_S(\sigma_k, \sigma_p) - \sum_{n=0}^{N_s} \omega_n(\sigma_k, s)d_S^{(u,u)}(\sigma_q, s) \]

\[ d_S^{(u,u)}(\sigma_k, \sigma_p) \approx \sum_{n=0}^{N_s} S_n(\sigma_k) d_S^{(u,u)}(\sigma_q, \sigma_p) \]

Three-body scattering in the presence of bound states

Numerical Treatment of Few Body Equations in Momentum Space by the Spline Method
Splines-based method

\[
\begin{align*}
    d_{S}^{(u,u)}(p,k) &= -G_{S}(p,k) - \int_{0}^{\infty} \frac{dk'k'^{2}}{(2\pi)^{2}\omega_{k'}} G_{S}(p,k') M_{2}(k') d_{S}^{(u,u)}(k',k) \\
    d_{S}^{(u,u)}(\sigma_{k},\sigma_{p}) &= -G_{S}(\sigma_{k},\sigma_{p}) - \sum_{n=0}^{N_{s}} \omega_{n}(\sigma_{k},s) d_{S}^{(u,u)}(\sigma_{q},s) \\
    d_{S}^{(u,u)}(\sigma_{k},\sigma_{p}) &\approx \sum_{n=0}^{N_{s}} S_{n}(\sigma_{k}) d_{S}^{(u,u)}(\sigma_{q},\sigma_{p})
\end{align*}
\]

Quadratures

Possible improvement
analytic removal of \( \epsilon \)

\[
\int_{0}^{(\sqrt{s}-m)^{2}} \frac{d\sigma_{q}}{\pi} G_{S}(\sigma_{k},\sigma_{q}) \frac{\lambda^{1/2}(s,\sigma_{q},m^{2})}{16\pi s} M_{2}(\sigma_{q}) S_{i}(\sigma_{q}) = \sum_{n=0}^{N_{s}} \omega_{n}(\sigma_{k}) S_{i}(\sigma_{q,n})
\]
Extrapolations of the results

- Expansion in 1/N
- Epsilon regulator fixed to $\epsilon \propto \eta/N$
Extrapolations of the results

- Expansion in $1/N$
- Epsilon regulator fixed to $\epsilon \propto \eta/N$
Example results, $M^2 = 3.89m^2$, i.e. $a=6$
Generalization of the B–matrix formalism

- Satisfies unitarity above the three–particle threshold
- New kernels include multi–channel couplings, for example

\[
\tilde{A}_{33} = \mathcal{H}_{33} + \int \mathcal{H}_{33} M_2 \tilde{A}_{33} \\
\mathcal{H}_{33} = \mathcal{B}_{33} + \frac{\mathcal{B}_{32} i \rho_2 \mathcal{B}_{23}}{1 - i \rho_2 \mathcal{B}_{22}}
\]

- Formal solutions can be found

\[
\tilde{A}_{33} = \frac{1}{1 - \mathcal{H}_{33} M_2} \mathcal{H}_{33} \\
\tilde{A}_{32} = \frac{1}{1 - \mathcal{H}_{33} M_2} \mathcal{H}_{32} \\
\tilde{A}_{23} = \frac{1}{1 - \mathcal{H}_{22} i \rho_2} \mathcal{H}_{23} \\
\tilde{A}_{22} = \frac{1}{1 - \mathcal{H}_{22} i \rho_2} \mathcal{H}_{22} \\
\mathcal{H}_{32} = \mathcal{B}_{32} + \frac{\mathcal{B}_{32} i \rho_2 \mathcal{B}_{22}}{1 - i \rho_2 \mathcal{B}_{22}} \\
\mathcal{H}_{23} = \mathcal{B}_{23} + \mathcal{B}_{23} M_2 \frac{1}{1 - \mathcal{B}_{33} M_2} \mathcal{B}_{33} \\
\mathcal{H}_{22} = \mathcal{B}_{22} + \mathcal{B}_{23} M_2 \frac{1}{1 - \mathcal{B}_{33} M_2} \mathcal{B}_{32}
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\[ \mathcal{H}_{23} = \mathcal{B}_{23} + \mathcal{B}_{23} \mathcal{M}_2 \frac{1}{1 - \mathcal{B}_{33} \mathcal{M}_2} \mathcal{B}_{33} \]
\[ \mathcal{H}_{22} = \mathcal{B}_{22} + \mathcal{B}_{23} \mathcal{M}_2 \frac{1}{1 - \mathcal{B}_{33} \mathcal{M}_2} \mathcal{B}_{32} \]
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\[ \tilde{A}_{23} = \frac{1}{1 - \mathcal{H}_{22} i \rho_2} \mathcal{H}_{23} \]

\[ \tilde{A}_{22} = \frac{1}{1 - \mathcal{H}_{22} i \rho_2} \mathcal{H}_{22} \]

\[ \mathcal{H}_{32} = \mathcal{B}_{32} + \frac{\mathcal{B}_{32} i \rho_2 \mathcal{B}_{22}}{1 - i \rho_2 \mathcal{B}_{22}} \]

\[ \mathcal{H}_{23} = \mathcal{B}_{23} + \mathcal{B}_{23} M_2 \frac{1}{1 - \mathcal{B}_{33} M_2} \mathcal{B}_{33} \]

\[ \mathcal{H}_{22} = \mathcal{B}_{22} + \mathcal{B}_{23} M_2 \frac{1}{1 - \mathcal{B}_{33} M_2} \mathcal{B}_{32} \]
Generalization of the B-matrix formalism

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\[ \tilde{A}_{33} = \frac{1}{1 - \mathcal{H}_{33} M_2} \mathcal{H}_{33} \]

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\[ \tilde{A}_{22} = \frac{1}{1 - \mathcal{H}_{22} i \rho_2} \mathcal{H}_{22} \]

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\[ \mathcal{H}_{23} = \mathcal{B}_{23} + \mathcal{B}_{23} M_2 \frac{1}{1 - \mathcal{B}_{33} M_2} \mathcal{B}_{33} \]

\[ \mathcal{H}_{22} = \mathcal{B}_{22} + \mathcal{B}_{23} M_2 \frac{1}{1 - \mathcal{B}_{33} M_2} \mathcal{B}_{32} \]
Generalization of the $B$-matrix formalism

- Satisfies unitarity above the three-particle threshold
- New kernels include multi-channel couplings, for example

\[
\tilde{A}_{33} = \mathcal{H}_{33} + \int \mathcal{H}_{33} M_2 \tilde{A}_{33}
\]

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\]

- Formal solutions can be found

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\tilde{A}_{33} = \frac{1}{1 - \mathcal{H}_{33} M_2} \mathcal{H}_{33}
\]

\[
\tilde{A}_{32} = \frac{1}{1 - \mathcal{H}_{32} M_2} \mathcal{H}_{32}
\]

\[
\tilde{A}_{23} = \frac{1}{1 - \mathcal{H}_{23} i\rho_2} \mathcal{H}_{23}
\]

\[
\tilde{A}_{22} = \frac{1}{1 - \mathcal{H}_{22} i\rho_2} \mathcal{H}_{22}
\]

\[
\mathcal{H}_{32} = \mathcal{B}_{32} + \frac{\mathcal{B}_{32} i\rho_2 \mathcal{B}_{22}}{1 - i\rho_2 \mathcal{B}_{22}}
\]

\[
\mathcal{H}_{23} = \mathcal{B}_{23} + \mathcal{B}_{23} M_2 \frac{1}{1 - \mathcal{B}_{33} M_2} \mathcal{B}_{33}
\]

\[
\mathcal{H}_{22} = \mathcal{B}_{22} + \mathcal{B}_{23} M_2 \frac{1}{1 - \mathcal{B}_{33} M_2} \mathcal{B}_{32}
\]
Including OPE

How does this generalized B-matrix approach compare with EFT?

- REFT ladder formal solution:

\[
d = \frac{1}{\mathbb{1} - \mathcal{K} i \rho_2} \mathcal{K}
\]

where the \( K \) matrix is

\[
\mathcal{K} = g^2 \frac{1}{\mathbb{1} + G_S \text{P.V.}[\mathcal{M}_2]} (-G_S)
\]
Including OPE

How does this generalized B–matrix approach compare with EFT?

- **B–matrix formal solution:**
  \[ \tilde{A}_{22} = \frac{1}{I - \mathcal{H}_{22}i\rho_2} \mathcal{H}_{22} \]

- **REFT ladder formal solution:**
  \[ d = \frac{1}{I - \mathcal{K}i\rho_2} \mathcal{K} \]

where the \( K \) matrix is

\[ \mathcal{K} = g^2 \frac{1}{I + G_S \text{P.V.}[\mathcal{M}_2]} (-G_S) \]
Including OPE

How does this generalized B–matrix approach compare with EFT?

- **B–matrix formal solution:**

  $$\tilde{A}_{22} = \frac{1}{1 - H_{22}i\rho_2} H_{22}$$

  where the H matrix is

  $$H_{22} = B_{22} + B_{23}M_2 \frac{1}{1 - B_{33}M_2} B_{32}$$

- **REFT ladder formal solution:**

  $$d = \frac{1}{1 - K_i\rho_2} K$$

  where the K matrix is

  $$K = g^2 \frac{1}{1 + G_S \text{P.V.}[M_2]} (-G_S)$$
Including OPE

How does this generalized B–matrix approach compare with EFT?

**B–matrix formal solution:**

\[
\tilde{A}_{22} = \frac{1}{1 - \mathcal{H}_{22} \rho_2} \mathcal{H}_{22}
\]

where the $\mathcal{H}$ matrix is

\[
\mathcal{H}_{22} = \mathcal{B}_{22} + \mathcal{B}_{23} \mathcal{M}_2 \frac{1}{1 - \mathcal{B}_{33} \mathcal{M}_2} \mathcal{B}_{32} g_B - G_S g_B
\]

**REFT ladder formal solution:**

\[
d = \frac{1}{1 - \mathcal{K} \rho_2} \mathcal{K}
\]

where the $\mathcal{K}$ matrix is

\[
\mathcal{K} = g^2 \frac{1}{1 + G_S \text{P.V.}[\mathcal{M}_2]} (-G_S)
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Including OPE

How does this generalized B–matrix approach compare with EFT?

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  \tilde{A}_{22} = \frac{1}{1 - \mathcal{H}_{22} i \rho_2} \mathcal{H}_{22}
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  where the H matrix is
  \[
  \mathcal{H}_{22} = \mathcal{B}_{22} + \mathcal{B}_{23} \mathcal{M}_2 \frac{1}{1 - \mathcal{B}_{33} \mathcal{M}} \mathcal{B}_{32}
  \]

- **REFT ladder formal solution:**
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  \[
  \mathcal{K} = g^2 \frac{1}{1 + G_S \text{P.V.} [\mathcal{M}_2]} (-G_S)
  \]

\[\]

Three-body scattering in the presence of bound states
Including OPE

How does this generalized B-matrix approach compare with EFT?

**B-matrix formal solution:**

\[
\tilde{A}_{22} = \frac{1}{1 - \mathcal{H}_{22}i\rho_2} \mathcal{H}_{22}
\]

where the \( H \) matrix is

\[
\mathcal{H}_{22} = B_{22} + B_{23} M_2 \left( 1 - B_{33} M \right) B_{32}
\]

\[
\mathcal{H}_{22} = g_B^2 \frac{1}{1 + G_S M} M_2
\]

**REFT ladder formal solution:**

\[
d = \frac{1}{1 - \mathcal{K}i\rho_2} \mathcal{K}
\]

where the \( K \) matrix is

\[
\mathcal{K} = g^2 \frac{1}{1 + G_S P.V.[M_2]} (-G_S)
\]

Those two are matrices defined in different momentum spaces!
The unitarity-based approach

Formalism

Relativistic three-body theory with applications to π-N scattering
Aaron, Amado, Young, Phys. Rev. 174, 2022 (1968)

Three-body unitarity in finite volume

Three-body unitarity with Isobars revisited

Three-body spectrum in a finite volume: the role of cubic symmetry

Phenomenology of relativistic 3 → 3 reaction amplitudes within the isobar approximation

Three-body scattering: ladders and resonances
Mikhasenko et al. (JPAC), JHEP 08 (2019) 1, 080

Bound states in the B-matrix formalism for the three-body scattering
Dawid, Szczepaniak, Phys. Rev. D 103 (2021) 1, 014009

Lattice spectrum and applications

Finite-Volume Spectrum of π−π and π−π−π Systems

Three-body unitarity vs finite-volume π−π−π spectrum from lattice QCD

Dalitz plots and lineshape of a(1260) from relativistic three-body unitary approach

Three-pion spectrum in the I=3 channel from lattice QCD

Finite volume energy spectrum of the K-K- system

Three-body interactions from the finite-volume spectrum
Brett et al., Phys. Rev. D 104 (2021) 1, 014501

Three-body dynamics of the a(1260) resonance from lattice QCD
Mai et al. (GWQCD), arxiv:2107.03973 (2021)
REFT approach

Formalism

Relativistic, model-independent, three-particle quantization condition
Hansen, Sharpe, Phys. Rev. D 90 (2014) 11, 116003

Expressing the three-particle finite-volume spectrum in terms of the three-to-three scattering amplitude

Threshold expansion of the three-particle quantization condition
Hansen, Sharpe, Phys. Rev. D 93 (2016) 9, 096006

Relating the FV spectrum and the 2-and-3-particle S matrix for relativistic systems of identical scalar particles
Briceño, Hansen, Sharpe, Phys. Rev. D 95 (2017) 7, 074510

Three-particle systems with resonant subprocesses in a finite volume

Unitarity of the infinite-volume three-particle scattering amplitude arising from a finite-volume formalism

Generalizing the relativistic quantization condition to include all three-pion isospin channels
Hansen, Romero-López, Sharpe, JHEP 07 (2020) 047

Relativistic three-particle quantization condition for nondegenerate scalars
Blanton, Sharpe, Phys. Rev. D 103 (2021) 5, 054503

Three-particle finite-volume formalism for π+π+K+ and related systems
Blanton, Sharpe, Phys. Rev. D 104 (2021) 3, 034509

Lattice spectrum and applications

Numerical study of the relativistic three-body quantization condition in the isotropic approximation
Briceño, Hansen, Sharpe, Phys. Rev. D 98 (2018) 1, 014506

Implementing the three-particle quantization condition including higher partial waves
Blanton, Romero-López, Sharpe, JHEP 03 (2019) 106

I=3 Three-Pion Scattering Amplitude from Lattice QCD

Interactions of two and three mesons including higher partial waves from lattice QCD
Blanton et al., arxiv:2106.05690 (2021)

Energy dependent π−π−π− scattering amplitude from lattice QCD
Hansen et al. (HadSpec), Phys. Rev. Lett. 126 (2021) 012001