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Homework Exercises

Exercise 2 - 1

The expected mean square error of an estimator from the true value θ_0 of an underlying distribution is defined as $\mathbb{E}_x[(\hat{\theta}(x) - \theta_0)^2]$.

a) In class we discussed bias and variance of estimators as key quantities. Derive the decomposition of the MSE into these two components.

b) Consider the exponential distribution $p(x|\lambda) = \frac{1}{\lambda}\exp(-x/\lambda)$. Derive an expression for the MLE estimator $\hat{\lambda}$ given n observations.

c) For $n = 1$, compute the bias, variance and MSE of two estimators: 1) the constant estimator $\hat{\lambda} = 0$ and 2) the MLE estimator. Does the variance of the constant estimator disprove the Cramér-Rao inequality?

d) Consider the family of estimators $\hat{\lambda} = (1 - m)\hat{\lambda}_{\text{MLE}}$ parametrized by $m \in [0, 1]$. Which value of m achieves the lowest possible MSE? Discuss the result qualitatively in terms of bias and variance.

Exercise 2 - 2

Consider a counting experiment where, we expect μ events from a signal processs, and b events from a background source. The experiment can be modeled with $\text{Pois}(n|\mu + b)$.

a) For a given μ implement a program that yields a rejection region ω in data space such that all rejected events have LRT test statistic $t > t_0$, with t_0 chosen such that the test size is $\alpha =\leq 0.1$

b) For $\mu = 0.5$ and $b = 3$ what is the rejection region, t_0, and the size of the test?

c) For an observed count of $n = 0$ and $b = 3.0$ what is thee range of μ values that is not rejected by a test? What is the corresponding interval for $b = 0.2$?

Exercise 2 - 3

Implement the on/off problem $\text{Pois}(n_1|\mu s + \gamma b)\,\text{Pois}(n_2|\gamma\tau b)$ in code with $\tau = 5, s = 20, b = 50$

a) Given $n_1 = 105, n_2 = 265$, visualize the 2-D likelihood contour as a function of the parameters $(\mu,\, \gamma)$

b) Perform a maximum likelihood fit to find the estimates $\hat{\mu}, \hat{\gamma}$

c) Perform a maximum likelihood likelihood fit but for μ fixed at 1.5

d) Numerically compute the restricted MLE of the nuisance parameter as a function of the parameter of interest and add it to the 2-D likelihood plot .

e) Sample toy data from a model with $\mu = 1.5, \gamma = 1.2$, perform maximum likelihood fits and observe the MLE distribution .