Scattering amplitudes at the conformal fixed point

— **PART 2** —

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Conformal symmetry on shell

- Correlators are conformal invariant at the conformal fixed point $\beta(u^*) = 0$

$$K^\mu \, C^{(n)} = 0$$

- We define amplitudes $M^{(n)}$ as the on-shell limit $p_i^2 \to 0$ of the correlators
- Is the on-shell limit compatible with the conformal symmetry?
- Conformal boost symmetry becomes anomalous on shell

$$K^\mu M^{(n)} = \mathrm{anomaly}$$

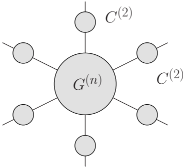
- Calculate the anomaly with the help of expansion by regions method

From correlators to scattering amplitudes

Lehmann–Symanzik–Zimmermann reduction in a CFT

- From correlator $C^{(n)}$ to amputated correlator $G^{(n)}$,

$$C^{(n)}(p_1,\dots,p_n) \equiv \left[\prod_{j=1}^{n} C^{(2)}(p_j,-p_j)\right] G^{(n)}(p_1,\dots,p_n) =$$



where the two-point correlator

$$C^{(2)}(p,-p) = \mathrm{const}\,\left(-p^2\right)^{-1+\gamma}$$

- Put all legs (or a subset of legs) on-shell

$$M^{(n)}(p_1,\dots,p_n) = \lim_{p_1^2\to 0}\dots\lim_{p_n^2\to 0} G^{(n)}(p_1,\dots,p_n)$$

Is LSZ reduction compatible with the conformal symmetry?

Two-point correlator is the intertwining operator

Conformal boost generator

$$K^\mu_\Delta = -p^\mu \frac{\partial}{\partial p_\nu}\frac{\partial}{\partial p^\nu} + 2p^\nu \frac{\partial}{\partial p^\nu}\frac{\partial}{\partial p_\mu} + 2(d-\Delta)\frac{\partial}{\partial p_\mu}$$

Representations of the conformal group of weights Δ and $d-\Delta$ are equivalent

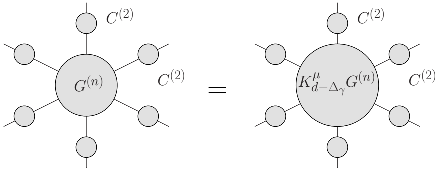
$$K^\mu_\Delta \left(p^2\right)^{\Delta-\frac{d}{2}} = \left(p^2\right)^{\Delta-\frac{d}{2}} K^\mu_{d-\Delta}$$

At $\Delta = \Delta_\gamma$, the intertwining operator is the two-point correlator

$$C^{(2)}(p,-p) \sim (p^2)^{-1+\gamma}$$

Conformal symmetry of the non-amputated/amputated off-shell correlator

$0 = K^\mu_{\Delta_\gamma}$



How to put legs of the correlator on-shell?

On-shell limit and the conformal boost do **NOT** commute

$$\left[K^{\mu}_{d-\Delta_\gamma}, \lim_{p^2\to 0}\right] G\left(p^2\right) = -4 \lim_{p^2\to 0} p^{\mu}\left[p^2 G''\left(p^2\right) + \gamma\, G'\left(p^2\right)\right]$$

two terms in the commutator = two sources of the on-shell conformal anomaly

- Conformal anomaly in integer number of dimensions $d = 6$. Not related to UV

  Example
  $G(p^2)$ is conformal and finite at $p^2 \to 0$, but its derivatives are singular

  $$G(p^2) \simeq M + p^2 \log(-p^2)\, A + p^2\, B + \dots$$

  $$K^{\mu} M = -4 p^{\mu} A$$

- Conformal anomaly because of the UV divergences, i.e. $\sim \gamma$

Toy example: conformal symmetry of the three-point correlator

Choose independent kinematic variables

$$s_1,\ s_2,\ s_3 \qquad \text{where} \qquad s_i \equiv (p_i)^2$$

Conformal symmetry off shell

$$K^{\mu}_{d-\Delta_\gamma} G\left(s_1, s_2, s_3\right) = 0$$

Conformal boost generator in the Mandelstam variables

$$K^{\mu}_{d-\Delta_\gamma} = \sum_{i=1}^{3} p_i^{\mu} \hat{K}^{(i)}_{d-\Delta_\gamma}$$

where scalar differential operators

$$\hat{K}^{(i)}_{d-\Delta_\gamma} \equiv 4 s_i \frac{\partial^2}{\partial s_i^2} + 4\gamma \frac{\partial}{\partial s_i}$$

Conformal anomaly of the three-point 'amplitude'

Conformal symmetry of the three-point correlator

$$\left(\hat{K}^{(2)}_{d-\Delta_\gamma} - \hat{K}^{(3)}_{d-\Delta_\gamma}\right) G = 0$$

$$\left(\hat{K}^{(2)}_{d-\Delta_\gamma} - \hat{K}^{(1)}_{d-\Delta_\gamma}\right) G = 0$$

From the correlator to the 'amplitude' (one of the legs is on shell)

$$M\left(s_2, s_3\right) = \lim_{s_1 \to 0} G\left(s_1, s_2, s_3\right)$$

On-shell variables : s_2, s_3

Conformal symmetry of the 'amplitude'

$$\left(\hat{K}^{(2)}_{d-\Delta_\gamma} - \hat{K}^{(3)}_{d-\Delta_\gamma}\right) M\left(s_2, s_3\right) = 0$$

$$\hat{K}^{(2)}_{d-\Delta_\gamma} M\left(s_2, s_3\right) = \lim_{s_1 \to 0} \hat{K}^{(1)}_{d-\Delta_\gamma} G\left(s_1, s_2, s_3\right) = \text{anomaly}$$

Anomaly from the asymptotics of the correlator

Asymptotic expansion at $s_1 \to 0$ of the three-point correlator

$$G\left(s_1, s_2, s_3\right) = \sum_{m \geq 0} \left(-s_1\right)^m G_{[m]}^{\text{light-like}}\left(s_2, s_3\right) + \sum_{m \geq 0} \left(-s_1\right)^{1-\gamma+m} G_{[m]}^{\text{hard}}\left(s_2, s_3\right)$$

In perturbation theory γ is a Taylor series in ϵ and it is natural to deal with log's

$$G\left(s_1, s_2, s_3\right) = M\left(s_2, s_3\right) + \sum_{m \geq 1} \sum_{k \geq 0} s_1^m \log^k\left(-s_1\right) \left[G\right]_{s_1^m \log^k(s_1)}$$

Anomalous conformal Ward identity for the 'amplitude'

$$K^{\mu}_{d-\Delta_\gamma} M\left(s_2, s_3\right) = \text{anomaly}$$
$$= -4 p_1^{\mu} \left(\gamma \left[G\right]_{s_1} + \left[G\right]_{s_1 \log(s_1)}\right)$$

Anomalous conformal Ward identities can be easily solved in the three-point case

- One dimensionless variable s_2/s_3
- (Anomalous) conformal Ward identities is a pair of the second order ordinary DE for the 'amplitude' M and anomaly
- Solution: the hypergeometric function

$$M(s_2,s_3) = \frac{4\tilde{c}_{123}}{(\tilde{c}_{12})^3}\frac{\epsilon-\gamma-3}{\epsilon-\gamma-2}\frac{e^{\epsilon\gamma_E}\Gamma\left(1-\gamma\right)\Gamma^2\left(2-\frac{\epsilon}{2}+\frac{\gamma}{2}\right)}{\Gamma\left(4-\epsilon+\gamma\right)\Gamma^2\left(2-\frac{\epsilon}{2}-\frac{\gamma}{2}\right)}(-s_3)^{\frac{\epsilon-3\gamma}{2}}\times$$
$${}_2F_1\left(1-\frac{\epsilon}{2}+\frac{\gamma}{2},\frac{3\gamma}{2}-\frac{\epsilon}{2},2-\epsilon+\gamma;1-\frac{s_2}{s_3}\right)$$

Are the anomalous conformal Ward identities useful?

- How to extend the conformal Ward identities to multi-point amplitudes?

$$K^{\mu}_{d-\Delta_\gamma} M^{(n)}(p_1, \ldots, p_n) = A^{(n)\,\mu}(p_1, \ldots, p_n)$$

- How to calculate the anomaly?
- How to solve the anomalous Ward identities?
- What could we learn about amplitudes away from the conformal fixed point?

Multi-point 'amplitude' from the correlator

Put a subset $L \subseteq \{1, \ldots, n\}$ of legs of an n-point correlator $G^{(n)}$ on shell

$$s_i \equiv p_i^2 = 0 \,, \qquad i \in L$$

On-shell Mandelstam variables $\vec{v}$ of the 'amplitude' $M^{(n)}$,

$$M^{(n)}(\vec{v}) = \left(\prod_{i \in L} \lim_{s_i^2 \to 0} \right) G^{(n)}\left(\vec{v}, \{s_j\}_{j \in L}\right)$$

Conformal boost in the multi-point case

Conformal boost generator in the Mandelstam variables $\vec{v}$ and $\{s_i\}_{i\in L}$

$$K^{\mu}_{d-\Delta_\gamma} = \sum_{i=1}^{n} p_i^{\mu} \left[\delta_{i\in L} \hat{K}^{(i)}_{d-\Delta_\gamma} + \mathbb{K}^{(i)}_{d-\Delta_\gamma} + \sum_{j\in L} s_j \partial_{s_j} A^{(ij)} + \sum_{j\in L} s_j B^{(ij)} \right]$$

- Differential operators $\mathbb{K}, A, B$ act on on-shell variables $\vec{v}$ only
- $\hat{K}$ is responsible for the on-shell anomaly

$$\hat{K}^{(i)}_{d-\Delta_\gamma} \equiv 4 s_i \frac{\partial^2}{\partial s_i^2} + 4\gamma \frac{\partial}{\partial s_i}$$

Anomalous Ward identity of multi-point 'amplitude'

Asymptotic expansion of the correlator at $s_i \to 0$ with $i \in L$,

$$G^{(n)}(\vec{v}, s_L) = M^{(n)}(\vec{v}) + \sum_{i \in L} \left(s_i \left[G^{(n)} \right]_{s_i} + s_i \log(-s_i) \left[G^{(n)} \right]_{s_i \log(s_i)} \right) + \dots$$

Anomalous conformal Ward identity for the 'amplitude'

$$\left(\sum_{i=1}^{n} p_i^{\mu} \, \mathbb{K}^{(i)}_{d-\Delta_\gamma} \right) M^{(n)}(\vec{v}) = -4 \sum_{i \in L} p_i^{\mu} \left(\gamma \left[G^{(n)} \right]_{s_i} + \left[G^{(n)} \right]_{s_i \log(s_i)} \right)$$

- No anomaly at the classical level
- The anomaly is additive with respect to on-shell legs
- Anomaly contributions of on-shell legs $\sim p_i^{\mu}$
- The anomaly is simpler to calculate than the 'amplitude' $M^{(n)}$
- There is a simple receipt to calculate $\left[G^{(n)} \right]_{s_i \log(s_i)}$ part of the anomaly

Collinear anomaly of six-dimensional Feynman integrals with an on-shell leg

- Perturbative calculation of the correlator
- Study asymptotics $p^2 \to 0$ of individual Feynman diagrams
- Consider a UV-finite $(n+1)$-point Feynman diagram $\mathcal{F}^{(n+1)}_{d=6}$ and put $p^2 \to 0$

$$K^{\mu}_{\Delta=4}\left(\lim_{p^2\to 0} \mathcal{F}^{(n+1)}_{d=6}\right) = -4p^{\mu}\left[\mathcal{F}^{(n+1)}_{d=6}\right]_{p^2\log(p^2)}$$

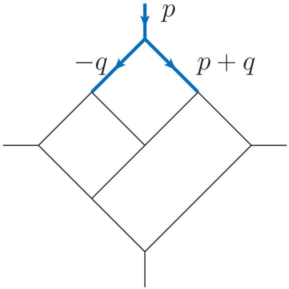
- There is a simple receipt how to evaluate $p^2\log(p^2)$ term in the asymptotics of the Feynman diagram $\mathcal{F}^{(n+1)}_{d=6}$

Collinear anomaly of six-dimensional Feynman integrals with an on-shell leg

Split out $(n+2)$-point $(\ell-1)$-loop subdiagram $\mathcal{G}$ of an $(n+1)$-point ℓ-loop UV-finite Feynman diagram $\mathcal{F}$,

$$\mathcal{F}^{(n+1)}_{d=6}(p^a, p_1^{a_1}, \dots, p_n^{a_n})$$
$$= g d^{abc} \int \frac{d^6 q}{(2\pi)^6} \frac{1}{q^2(p+q)^2} \mathcal{G}^{(n+2)}_{d=6}(q^b, (p-q)^c, p_1^{a_1}, \dots, p_n^{a_n})$$

Expansion by regions analysis of the Feynman integral asymptotics



$$\left[\mathcal{F}^{(n+1)}_{d=6}(p^a, p_1^{a_1}, \dots, p_n^{a_n})\right]_{p^2 \log(p^2)}$$
$$= \frac{g d^{abc}}{(4\pi)^3} \int_0^1 d\xi \, \xi \bar{\xi} \, \mathcal{G}^{(n+2)}_{d=6}(\xi p^b, \bar{\xi} p^c, p_1^{a_1}, \dots, p_n^{a_n})$$

where $\bar{\xi} \equiv 1 - \xi$

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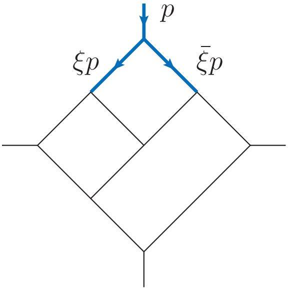
Expansion by regions analysis of the Feynman integral asymptotics



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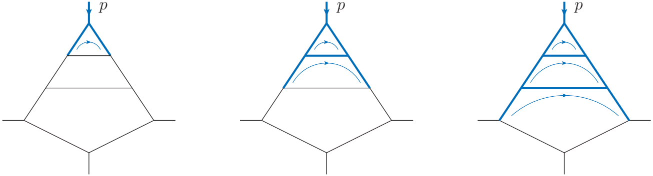
where $\bar{\xi} \equiv 1 - \xi$

Collinear anomaly of UV-divergent Feynman integrals with an on-shell leg

Split $(n+1)$-point ℓ-loop Feynman diagram $\mathcal{F}$ into $(n+2)$-point ℓ'-loop $(0 \le \ell' < \ell)$ subdiagram $\mathcal{G}$ and the three-point $(\mathcal{F}\backslash\mathcal{G})$ with ℓ' loop momenta



Only the collinear region of loop integrations contributes

$$\left[\mathcal{F}^{(n+1)}(p^a, p_1^{a_1}, \ldots, p_n^{a_n})\right]_{p^2 \log(p^2)}$$

$$= \frac{g d^{abc}}{(4\pi)^3} \sum_{\mathcal{G}\subset\mathcal{F}} \int_0^1 d\xi\, \Omega_{\mathcal{F}\backslash\mathcal{G}}(\xi, \epsilon, g)\, \mathcal{G}^{(n+2)}(\xi p^b, \bar{\xi} p^c, p_1^{a_1}, \ldots, p_n^{a_n})$$

Collinear anomaly of UV-divergent Feynman integrals with an on-shell leg

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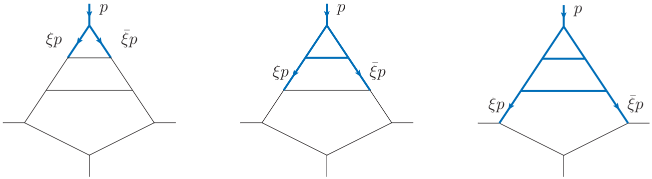

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$$\left[\mathcal{F}^{(n+1)}(p^a, p_1^{a_1}, \ldots, p_n^{a_n})\right]_{p^2 \log(p^2)}$$

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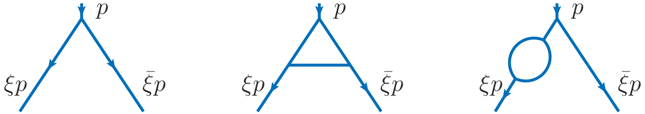
Collinear anomaly of the 'amplitude'

Sum over all Feynman diagrams and tune into the conformal fixed point $g = g^*$
Collinear part of the on-shell conformal anomaly

$$\left[G^{(n+1)}(p^a, p_1^{a_1}, \dots, p_n^{a_n})\right]_{p^2 \log(p^2)}$$
$$= \frac{g^* d^{abc}}{(4\pi)^3} \int_0^1 d\xi \, \Omega(\xi, \epsilon) \, G_{\mathrm{reg}}^{(n+2)}(\xi p^b, \bar{\xi} p^c, p_1^{a_1}, \dots, p_n^{a_n})$$

where Ω is the three-point collinear function



$$\Omega(\xi, \epsilon) = \Omega^{\text{1-loop}}(\xi, \epsilon) + g^{*2} \, \Omega^{\text{2-loop}}(\xi, \epsilon) + \mathcal{O}(g^{*4})$$

For example, the one-loop approximation reproduces the collinear anomaly of the six-dimensional Feynman integrals

$$\Omega^{\text{1-loop}}(\xi, \epsilon) = \xi \bar{\xi} + O(\epsilon)$$

Conclusions

- Anomalous conformal WI for the n-point 'amplitude' $M^{(n)}$ with $|L|$ on-shell legs,

$$\left(\sum_{i=1}^{n} p_i^{\mu} \mathbb{K}_{d-\Delta_\gamma}^{(i)}\right) M^{(n)}(\vec{v}) = -4 \sum_{i \in L} p_i^{\mu} \left(\gamma \left[G^{(n)}\right]_{s_i} + \left[G^{(n)}\right]_{s_i \log(s_i)}\right)$$

- $n = 3$: WI completely fixes the amplitude (and its anomaly)
- $n > 3$: calculate the anomaly in perturbation theory

  For example, conformal WI for one-loop $M^{(n)}$ requires
  - $\left[G^{(n)}\right]_{s_i}$ at tree level
  - $\left[G^{(n)}\right]_{s_i \log(s_i)}$: one-fold integral over the tree-level $(n+1)$-point 'amplitude'

Open questions

- Applications to Feynman integrals with non-integer propagator powers?
- Stability of the alphabet for the three-point functions?
- How to solve the Ward identities (e.g. via bootstrap)?
- Geometric understanding of the collinear anomaly?
- How to deal with gauge theories?
- How to deal with theories with IR divergences?
- Relation to Lorenzian OPE?

BACKUP SLIDES

Asymptotics of the correlator is determined by the confomal symmetry

$$G(s_1, s_2, s_3) = M(s_2, s_3) + s_1 \, G_{1;0}(s_2, s_3) + s_1 \log(-s_1) G_{1;1}(s_2, s_3) + \dots$$

Solution of the conformal Ward identities for the correlator

$$G(s_1, s_2, s_3) = {}_0F_1\left(\gamma; s_1\mathcal{K}\right) M(s_2, s_3) - \frac{1}{\gamma}\left(-s_1\right)^{1-\gamma} {}_0F_1\left(2-\gamma; s_1\mathcal{K}\right) G_{1;1}(s_2, s_3)$$

as operator series in $\mathcal{K} \equiv \frac{1}{4}\hat{K}^{(2)}_{d-\Delta_\gamma}$ acting on the 'amplitude' M and $G_{1;1}$