The Three-Proton System: A Preliminary Approach

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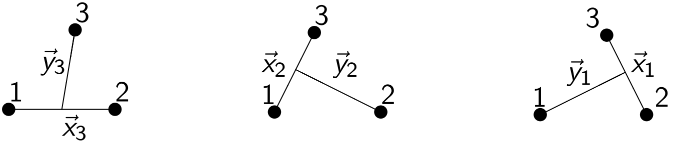
Introduction

- Recently the nnn and ppp systems have attracted a particular interest
- In the two-body case, the nn system is much less known than the pp or np systems
- In s-wave they interact in the $S = 0$, $T = 1$ channel.
- The associate scattering lengths are negative and large indicating the presence of a virtual state:
  $a_{nn} \approx 18.5\,\text{fm}$, $a_{pp}^{sr} \approx 17.5\,\text{fm}$, $a_{np} \approx 23.7\,\text{fm}$
- The effective ranges are all similar:
  $r_{nn} \approx r_{pp} \approx r_{np} \approx 2.8\,\text{fm}$
- The nnn and nnnn have been subject of intense investigations to determine if they have a low energy resonance state
- Here we study the ppp system

The Three-Nucleon System

The Jacobi coordinates ($\vec{x}_i$, $\vec{y}_i$)



The Jacobi coordinates allow to separate the center of mass motion

$$H = T + V = T_{CM} - \frac{\hbar^2}{m}\left(\nabla^2_{x_1} + \nabla^2_{y_1}\right) + \sum_{i<j} V(i,j) + \sum_{i<j<k} W(i,j,k)$$

The Three-Nucleon System

The three-nucleon wave function:

$$\Psi(\vec{x},\vec{y}) = \psi(\vec{x}_1,\vec{y}_1) + \psi(\vec{x}_2,\vec{y}_2) + \psi(\vec{x}_3,\vec{y}_3)$$

The amplitudes $\psi(\vec{x}_i,\vec{y}_i)$ are called Faddeev amplitudes. They can be decomposed in angular-spin-isospin channels

$$\psi(\vec{x}_i,\vec{y}_i) = \sum_{\alpha} \phi(x_i,y_i) \left[[Y_{l_1}(\hat{x}_i) Y_{l_2}(\hat{y}_i)]_L \otimes \chi_S^{S_{jk}} \right]_{JJ_z} \xi_{TT_z}^{T_{jk}}$$

$\chi_S^{S_{jk}}$ and $\xi_{TT_z}^{T_{jk}}$ are the spin and isospin functions of the three nucleons.

- Each channel $\alpha = [l_1, l_2, L, S_{jk}, S, T_{jk}, T]$ is compatible with J and parity.
- $l_1 + S_{jk} + T_{jk} =$ odd for antisymmetrization.
- The two-dimensional amplitudes $\phi(x_i,y_i)$ can be obtained solving the Faddeev equations or by a variational description.
- The number of channels is not limited and some truncation criteria is needed (convergence in some observables).

The Hyperspherical Harmonic Functions

The kinetic term of the center of mass Hamiltonian

$$T = -\frac{\hbar^2}{m}\left(\nabla^2_{x_1} + \nabla^2_{y_1}\right)$$

can be written as

$$T = -\frac{\hbar^2}{m}\left(\frac{\partial^2}{\partial\rho^2} + \frac{5}{\rho}\frac{\partial}{\partial\rho} + \frac{\Lambda^2(\Omega)}{\rho^2}\right)$$

We have introduced the hyperradius and the hyperspherical coordinates $[\rho, \Omega] = [\rho, \phi, \hat{x}_i, \hat{y}_i]$

$$\begin{cases} x_i = \rho\cos\phi_i \\ y_i = \rho\sin\phi_i \end{cases}$$

and the grand angular operator $\Lambda^2(\Omega)$.

The Hyperspherical Harmonic Functions

The eigenvectors of $\Lambda^2(\Omega)$ are the hyperspherical harmonic (HH) functions

$$\left(\Lambda^2(\Omega) + K(K+4)\right) Y_{[K]}(\Omega) = 0$$

The HH functions, having well defined angular momentum, are

$$Y_{[K]}^{LM}(\Omega_i) = \mathcal{N}_{n,l_1,l_2} \, {}^{(2)}P_n^{l_1,l_2}(\phi_i) \left[Y_{l_1}(\hat{x}_i) Y_{l_2}(\hat{y}_i)\right]_{LM}$$

The set of quantum numbers is $[K] = [n, l_1, l_2, L, M]$. The HH functions form a complete basis useful to expand the three-nucleon wave function.

$$\psi(\vec{x}_i, \vec{y}_i) = \rho^{-5/2} \sum_{[K]} u_{[K]}(\rho) \mathcal{Y}_{[K]}^{J\pi}(\Omega_i)$$

The functions $\mathcal{Y}_{[K]}^{J\pi}(\Omega_i)$ are hyperangular-spin-isospin function

$$\mathcal{Y}_{[K]}^{J\pi}(\Omega_i) = \left[Y_{[K]}^{LM}(\Omega_i) \otimes \chi_S^{S_{jk}} \right]_{JJ_z} \xi_{TT_z}^{T_{jk}}$$

with quantum numbers $[K] = [n, l_1, l_2, L, M, S_{jk}, S, T_{jk}, T]$.

The ppp Wave Function

The total wave function is

$$\Psi(\vec{x},\vec{y}) = \sum_i \psi(\vec{x}_i,\vec{y}_i) = \rho^{-5/2} \sum_{[K]} u_{[K]}(\rho)\mathcal{B}^{J\pi}_{[K]}(\Omega)$$

with $\mathcal{B}^{J\pi}_{[K]}$ antisymmetric HH-spin functions

The ppp wave is completely determined from the hyperradial functions $u_{[K]}(\rho)$. And they are determined from the boundary conditions as $\rho \to \infty$.

For a given energy, $E = \hbar^2 Q^2/m$, and in the nnn case

$$u_{[K]}(\rho \to \infty) \to \sqrt{Q\rho}\,[J_{K+2}(Q\rho) + \tan\delta_K\, Y_{K+2}(Q\rho)]$$

In the ppp case the asymptotic equations are coupled not allowing this simple picture

ppp Correlation Analysis

Using the property of the HH functions

$$\Psi_s^0 = e^{i\vec{Q}\cdot\vec{\rho}} = \frac{(2\pi)^3}{(Q\rho)^2}\sum_{[K]} i^K J_{K+2}(Q\rho)\mathcal{Y}_{[K]}(\Omega)\mathcal{Y}^*_{[K]}(\hat{Q})$$

where $\vec{Q}\cdot\vec{\rho} = \vec{k}_1\cdot\vec{x} + \vec{k}_2\cdot\vec{y}$ and J_{K+2} a Bessel function.

- For the case of three nucleons we have to include the correct symmetrization.
- For the case of three protons we have to include the correct asymptotics

The nnn (or ppp) case:

$$\Psi_s^0 = \frac{(2\pi)^3}{(Q\rho)^2}\sum_{[K]} i^K J_{K+2}(Q\rho)\mathcal{B}_{[K]}(\Omega)\mathcal{B}^*_{[K]}(\hat{Q})$$

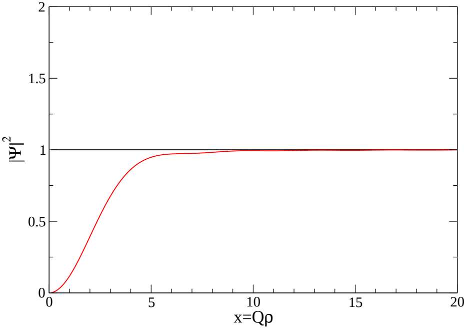
with $\mathcal{B}_{[K]}(\Omega)$ antisymmetric in the hyperangle-spin space.

ppp Correlation Analysis

Taken spin traces and performing the hyperangular integration, the norm results

$$|\Psi_s^0|^2 = \frac{(2\pi)^6}{(Q\rho)^6} \sum_{[K]} J_{K+2}^2(Q\rho) N_{ST}(K)$$

where $N_{ST}(K)$ is the number of states.



ppp Correlation Analysis

For three protons the asymptotic form changes (and it is not known in a close form)

The Coulomb interaction coupled the asymptotic equations through the term

$$\sum_{ij} \frac{e^2}{r_{ij}}$$

In this preliminary study we perform an average of the Coulomb interaction on the hyperangles

$$V_c(\rho) = \int d\Omega \sum_{ij} \frac{e^2}{r_{ij}} |\mathcal{Y}_0(\Omega)|^2 = \frac{16}{\pi} \frac{e^2}{\rho}$$

and the plane wave takes the form

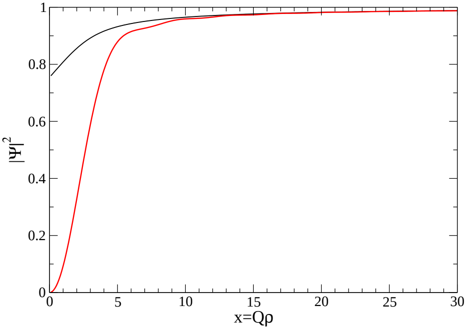
$$e^{i\vec{Q}\cdot\vec{\rho}} \to \Psi_s^0 = \frac{1}{C_{3/2}(0)} \frac{(\pi)^3}{(Q\rho)^{5/2}} \sum_{[K]} i^K F_{K+3/2}(\eta, Q\rho) \mathcal{B}_{[K]}(\Omega) \mathcal{B}^*_{[K]}(\hat{Q})$$

ppp Correlation Analysis

Taken spin traces and performing the hyperangular integration, the norm results

$$|\Psi_s^0|^2 = \frac{1}{C_{3/2}^2} \frac{1}{(Q\rho)^5} \sum_K F_{K+3/2}^2(\eta, Q\rho) N_{ST}(K)$$

where $N_{ST}(K)$ is the number of states.



Integrating on a Hyperradial source

The hyperradial source in the case of three particlers is defined as

$$S_{123} = \frac{1}{\pi^3 \rho_0^6} e^{-(\rho/\rho_0)^2} \tag{1}$$

with the condition

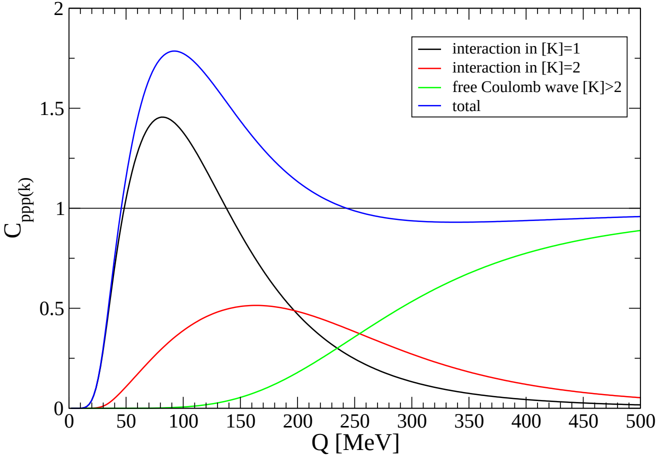
$$\int S_{123} \rho^5 d\rho \, d\Omega = 1 \tag{2}$$

The correlation function is defined now as

$$C_{123}(Q) = \int \rho^5 d\rho \, d\Omega \, S_{123} |\Psi_s|^2$$

Integrating on a Hyperradial source

Preliminary results with size source of 2 fm



Summary

- Although its apparent simplicity, the three-nucleon problem is of great complexity
- The antisymmetrization of the wave function is perfomed using the HH basis
- In the ppp case the Coulomb interaction couples the asymptotic dynamics increaing the difficulties of the numerical treatment.
- In this preliminary description the Coulomb interaction was averaged through the hyperangles
- Moreover only the $K = 1$ and $K = 2$ hyperangular channels were included

- The next work is to inlcude more channels and to relax the average of the Coulomb force