Feynman integrals, geometries and differential equations

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1. Introduction: Curves of genus 0 and 1
2. Higher genus
3. Higher dimensions

Section 1

Introduction

Setting the stage

- We are interested in **Feynman integrals**.
- These depend on kinematic variables $x_1, x_2, \dots$.
- They are regulated with the dimensional regularisation parameter ε.
- Integration-by-parts allows us to reduce Feynman integrals to master integrals $I_1, I_2, \dots$.
- Integration-by-parts allows us to derive a differential equation for the master integrals.

Notation

$\mathbf{I}$	$= (I_1, ..., I_{N_F})$	Master integrals
$\mathbf{N_F}$	$= N_{\mathrm{Fibre}}$	Number of master integrals
$\mathbf{x}$	$= (x_1, ..., x_{N_B})$	Kinematic variables
$\mathbf{N_B}$	$= N_{\mathrm{Base}}$	Number of kinematic variables
ω	$= (\omega_1, ..., \omega_{N_L})$	Differential one-forms/letters
$\mathbf{N_L}$	$= N_{\mathrm{Letters}}$	Number of letters

The method of differential equations

We want to calculate

$$I = (I_1, \dots, I_{N_F})$$

1. **Find** *a differential equation with respect to the kinematic variables for the Feynman integrals:*

$$\left[d + A(\varepsilon, x) \right] I = 0.$$

2. **Transform** *the differential equation into an* ε*-factorised form:*

$$\left[d + \varepsilon A(x) \right] I = 0, \qquad A(x) = \sum_{j=1}^{N_L} M_j \, \omega_j(x).$$

Henn '13

3. **Solve** *the latter differential equation with appropriate boundary conditions.*

Fibre bundles

We have a **vector bundle**:

- Fibre spanned by the master integrals $I_1, ..., I_{N_F}$.

  (The master integrals $I_1(x), \dots, I_{N_F}(x)$ can be viewed as local sections, and for each x they define a basis of the vector space in the fibre. In other words, they define a local frame.)
- Base space with coordinates $x = (x_1, ..., x_{N_B})$ corresponding to kinematic variables.
- Connection defined by the matrix A appearing in the differential equation.

Allowed **transformations**:

- a change of basis in the fibre,
- a coordinate transformation on the base manifold.

Section 2

Geometry

The base space

Question:

After a suitable coordinate transformation, can we relate the base space to a space known from mathematics?

The base space

- Assume we have $(n-3)$ variables $z_1, \dots, z_{n-3}$ and differential one-forms

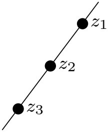
$$
\begin{aligned}
\omega_k \quad \in \quad & \{ d\ln(z_1), d\ln(z_2), \dots, d\ln(z_{n-3}), \\
& d\ln(z_1 - 1), \dots, d\ln(z_{n-3} - 1), \\
& d\ln(z_1 - z_2), \dots, d\ln(z_i - z_j), \dots, d\ln(z_{n-4} - z_{n-3}) \}
\end{aligned}
$$

- The iterated integrals $I_\gamma(\omega_1, \dots, \omega_r; \lambda)$ are multiple polylogarithms.
- We require $z_i \notin \{0, 1, \infty\}$ and $z_i \neq z_j$:
  This defines the **moduli space** $\mathcal{M}_{0,n}$: The space of configurations of n points on a Riemann sphere modulo Möbius transformations.
- Usually the z_i are functions of the kinematic variables x and the arguments of the dlog-forms define the Landau singularities.

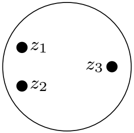
Moduli spaces

$\mathcal{M}_{g,n}$: Space of **isomorphism classes of** smooth (complex, algebraic) **curves of genus g with n marked points**.

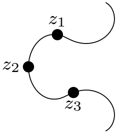
complex curve



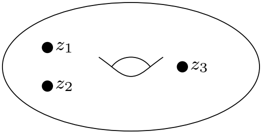
$\Leftrightarrow$



real surface



$\Leftrightarrow$



Coordinates

Genus 0: $\dim \mathcal{M}_{0,n} = n - 3$.

Sphere has a unique shape

Use Möbius transformation to fix $z_{n-2} = 1$, $z_{n-1} = \infty$, $z_n = 0$

Coordinates are $(\mathbf{z_1}, \dots, \mathbf{z_{n-3}})$

Genus 1: $\dim \mathcal{M}_{1,n} = n$.

One coordinate describes the shape of the torus

Use translation to fix $z_n = 0$

Coordinates are $(\tau, \mathbf{z_1}, \dots, \mathbf{z_{n-1}})$

Differential one-forms on $\mathcal{M}_{0,n}$

Multiple polylogarithms:

$$\omega^{\mathrm{MPL}} = \frac{dz}{z-c}.$$

Differential one-forms on $\mathcal{M}_{1,n}$

1. *From modular forms ($f_k(\tau)$ modular form):*

$$\omega_k^{\mathrm{modular}} = 2\pi i \, f_k(\tau) d\tau$$

Adams, S.W. '17

2. *From the Kronecker function:*

$$\omega_k^{\mathrm{Kronecker}} = (2\pi i)^{2-k} \left[g^{(k-1)}(z,\tau)\, dz + (k-1)\, g^{(k)}(z,\tau) \frac{d\tau}{2\pi i} \right]$$

Broedel, Duhr, Dulat, Tancredi, '17

The Kronecker function

Define the **first Jacobi theta function** $\theta_1(z,q)$ by (with $q = e^{2\pi i \tau}$)

$$\theta_1(z,q) = -i \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} e^{i\pi(2n+1)z}.$$

The **Kronecker function** $F(z,\alpha,\tau)$:

$$F(z,\alpha,\tau) = \theta_1'(0,q) \frac{\theta_1(z+\alpha,q)}{\theta_1(z,q)\,\theta_1(\alpha,q)} = \frac{1}{\alpha} \sum_{k=0}^{\infty} \mathbf{g}^{(\mathbf{k})}(\mathbf{z},\tau) \alpha^k$$

We are interested in the coefficients $g^{(k)}(z,\tau)$ of the Kronecker function.

Numerics

Physics is about numbers:

- Iterated integrals of modular forms and elliptic multiple polylogarithms can be evaluated numerically with arbitrary precision.
- Implemented in `GiNaC`.

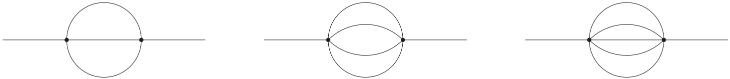
  Walden, S.W, '20

```
ginsh - GiNaC Interactive Shell (GiNaC V1.8.1)
  __,  _______  Copyright (C) 1999-2021 Johannes Gutenberg University Mainz,
 (__) *       | Germany.  This is free software with ABSOLUTELY NO WARRANTY.
  ._) i N a C | You are welcome to redistribute it under certain conditions.
<-------------' For details type `warranty;'.

Type ?? for a list of help topics.
> Digits=50;
50
> iterated_integral({Eisenstein_kernel(3,6,-3,1,1,2)},0.1);
0.23675657575197179243274817775862177623438999192840338805367
```

Generalisations

- We understand by now very well Feynman integrals related to algebraic curves of genus 0 and 1. These correspond to iterated integrals on the moduli spaces $\mathcal{M}_{0,n}$ and $\mathcal{M}_{1,n}$.
- The obvious generalisation is the generalisation to algebraic curves of **higher genus** g, i.e. iterated integrals on the moduli spaces $\mathcal{M}_{g,n}$.
- However, we also need the generalisation from curves to surfaces and **higher dimensional objects**: The geometry of the banana graphs with equal non-vanishing internal masses



are Calabi-Yau manifolds.

Section 3

Higher genus curves

Hyperelliptic curves

Definition

A hyperelliptic curve is an algebraic curve of genus $g \geq 2$ whose defining equation takes the form

$$y^2 = P(z),$$

for some polynomial $P(z)$ of degree $(2g+1)$ or $(2g+2)$.

They generalise elliptic curves, whose defining equation takes the same form when $g = 1$.

We are interested in **Feynman integrals**, where the maximal cut takes the form

$$\int dz \frac{N(z)}{\sqrt{P(z)}}$$

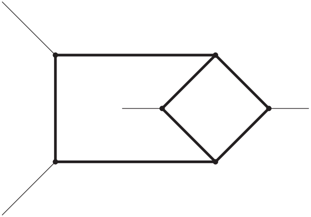
Non-planar double boxes

Non-planar double boxes (with sufficient internal/external masses) provide examples of higher-genus Feynman integrals.

- In the loop momentum representation one obtains a genus 3 curve.

  Georgoudis, Zhang, '15

- In the Baikov representation one obtains a genus 2 curve.



Can we understand this?

Yes we can!

R. Marzucca, A. McLeod, B. Page, S.Pögel, S.W., '23

Extra involutions

- Any hyperelliptic curve $H : y^2 = P(z)$ has an involution symmetry $e_0 : y \to -y$.
- **The solution to this riddle**: The higher genus curve has an extra involution. In the simplest case, if $P(z)$ is of the form

$$P(z) = Q\left(z^2\right) = \left(z^2 - \alpha_1^2\right) \dots \left(z^2 - \alpha_{g+1}^2\right)$$

  the extra involution is given by $e_1 : z \to -z$.
- There is an algorithm to detect the extra involution.
- To a hyperelliptic curve with an extra involution we can associate two curves

$$\begin{aligned} H_1 &: \quad y_1^2 = Q(w) \qquad && \text{(corresponding to } e_1\text{)} \\ H_2 &: \quad y_2^2 = wQ(w) \qquad && \text{(corresponding to } e_1 \circ e_0\text{)} \end{aligned}$$

  of genus $\lfloor \frac{g}{2} \rfloor$ and $\lceil \frac{g}{2} \rceil$, respectively.

Why two curves?

Start from $H : y^2 = P_{2g+2}(z) = Q_{g+1}(z^2)$, where P_{2g+2} is of degree $2g+2$:

- **Curve 1**: The substitution $w = z^2$ gives

$$H_1 \quad : \quad y^2 = Q_{g+1}(w).$$

  This has genus $\lfloor \frac{g}{2} \rfloor$.

- **Curve 2**: The differential transforms as

$$\frac{dz}{\sqrt{P_{2g+2}(z)}} \quad = \quad \frac{1}{2} \frac{dw}{\sqrt{w Q_{g+1}(w)}}$$

  This gives

$$H_2 \quad : \quad y_2^2 = w Q(w).$$

  This curve has genus $\lceil \frac{g}{2} \rceil$.

Period relations

- A hyperelliptic curve H of genus g has g holomorphic differentials and $(2g)$ cycles. This defines a **$g \times (2g)$-period matrix $\mathcal{P}$**.
- If H has an extra involution, there will be relations among the periods. One can find matrices $M_\omega \in \mathbb{C}^{g \times g}$ and $M_\Gamma \in \mathbb{Z}^{2g \times 2g}$ such that

$$M_\omega^T \mathcal{P} M_\Gamma = \begin{pmatrix} \mathcal{P}_1 & 0 \\ 0 & \mathcal{P}_2 \end{pmatrix},$$

where $\mathcal{P}_1$ and $\mathcal{P}_2$ are period matrices of curves isogenous to H_1 and H_2.

Why is there an extra involution?

For our example we can trace it back to discrete Lorentz transformations (parity, time reversal):

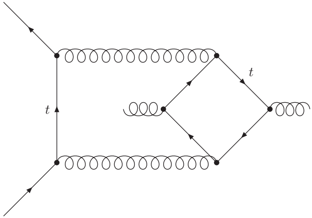
- In the Baikov representation everything is manifestly Lorentz invariant, the Baikov variables are Lorentz invariants:

$$z = k^2 - m^2.$$

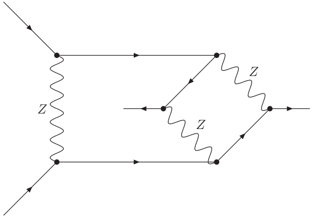
- In the loop momentum representation we choose a frame, we choose a parametrisation of the loop momenta, we choose an elimination order: The full Lorentz symmetry is not required to be trivially realised, but may manifest itself through extra symmetries of the curve.

Examples

- Top pair production at NNLO
  (genus drop from 3 to 2)



- Møller scattering at NNLO
  (genus drop from 3 to 2)



Section 4

Calabi-Yau manifolds

Calabi-Yau manifolds

Definition

A Calabi-Yau manifold of complex dimension n is a compact Kähler manifold M with vanishing first Chern class.

Theorem (conjectured by Calabi, proven by Yau)

An equivalent condition is that M has a Kähler metric with vanishing Ricci curvature.

Mirror symmetry

The **mirror map** relates a Calabi-Yau manifold A to another Calabi-Yau manifold B with Hodge numbers $h_B^{p,q} = h_A^{n-p,q}$.

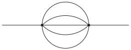
Candelas, De La Ossa, Green, Parkes '91

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\qquad\qquad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{2,1} & & 0 & \\
1 & & h^{1,1} & & h^{1,1} & & 1 \\
 & 0 & & h^{2,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

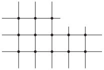
Calabi-Yau manifold A | mirror image B

Fantastic Beasts and Where to Find Them

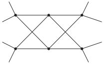
- Bananas



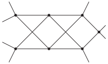
- Fishnets



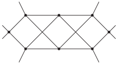
- Amoebas



- Tardigrades



- Paramecia



Aluffi, Marcolli, '09, Bloch, Kerr, Vanhove, '14
Bourjaily, McLeod, von Hippel, Wilhelm, '18
Duhr, Klemm, Loebbert, Nega, Porkert, '22

Bananas

- The l-loop banana integral with (equal) non-zero masses is related to a **Calabi-Yau $(l-1)$-fold**.
- An elliptic curve is a Calabi-Yau 1-fold, this is the geometry at two-loops.
- The system of differential equations for the equal mass l-loop banana integral can be transformed to an **ε-factorised form**.
  - Change of variables from $x = p^2/m^2$ to τ given by mirror map.
  - Transformation constructed from special local normal form of a Calabi-Yau operator.

    M. Bogner '13, D. van Straten '17
- Strong support for the conjecture that a transformation to an ε-factorised differential equation exists for all Feynman integrals.

Section 5

The mirror map

The mirror map

- The point $x = \infty$ is a point of maximal unipotent monodromy, the Frobenius method gives solutions ordered by powers of logarithms.
- The holomorphic solution ψ_0 and the single-logarithmic solution ψ_1 are used to define a change of variables from x to τ (or q):

$$\tau = \frac{\psi_1}{\psi_0}, \qquad q = e^{2\pi i \tau}.$$

- In the context of Calabi-Yau manifolds the map from x to τ is called the **mirror map**.

Candelas, De La Ossa, Green, Parkes, '91

- In the special case of $l = 2$ the map corresponds to the transformation from x to the modular parameter τ of an elliptic curve.

Section 6

The special local normal form of a Calabi-Yau operator

Special local normal form

Consider a sequence which starts as

$$\begin{aligned} l &= 0: & 1 \\ l &= 1: & \theta \\ l &= 2: & \theta \cdot \theta \\ l &= 3: & \theta \cdot \theta \cdot \theta \end{aligned}$$

We would like to understand the general term at l loops.

Special local normal form

We first compute the $(l = 4)$-term:

$$\begin{aligned} l &= 0: & 1 \\ l &= 1: & \theta \\ l &= 2: & \theta \cdot \theta \\ l &= 3: & \theta \cdot \theta \cdot \theta \\ l &= 4: & \theta \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \theta \end{aligned}$$

Special local normal form

The general term at l loops is given by

$$\theta \cdot \frac{1}{Y_{l-1}} \cdot \theta \cdot \frac{1}{Y_{l-2}} \cdot \theta \cdot \frac{1}{Y_{l-3}} \cdot \ldots \cdot \frac{1}{Y_3} \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \frac{1}{Y_1} \cdot \theta$$

and we have

$$Y_1 = 1$$

and the duality

$$Y_j = Y_{l-j}.$$

Special local normal form

Up to seven loops we therefore have

$$\begin{aligned}
l &= 0: & & 1 \\
l &= 1: & & \theta \\
l &= 2: & & \theta \cdot \theta \\
l &= 3: & & \theta \cdot \theta \cdot \theta \\
l &= 4: & & \theta \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \theta \\
l &= 5: & & \theta \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \theta \\
l &= 6: & & \theta \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \frac{1}{Y_3} \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \theta \\
l &= 7: & & \theta \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \frac{1}{Y_3} \cdot \theta \cdot \frac{1}{Y_3} \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \theta
\end{aligned}$$

Special local normal form

- θ is the **Euler operator** $\theta = q \frac{d}{dq}$ in the variable q, the functions Y_j are called Y-invariants.
- $N = \theta^2 \frac{1}{Y_2} \theta \frac{1}{Y_3} \dots \frac{1}{Y_3} \theta \frac{1}{Y_2} \theta^2$ is the **special local normal form** of a Calabi-Yau operator.
- Operators like N are related to **Picard-Fuchs operators** of Calabi-Yau Feynman integrals.
- From the factorisation of N we may construct the ε-factorised differential equation.

Section 7

The ansatz

The ansatz

- We set $D = 2 - 2\varepsilon$.
- Instead of $x = p^2 / m^2$ we work with the variable τ (or q).
- We now **construct master integrals**

$$M = (M_0, M_1, \dots, M_l)^T,$$

which put the differential equation into an ε-factorised form.

- M_0 is proportional to the l-loop tadpole integral:

$$M_0 = \varepsilon^l I_{1\dots10}.$$

The ansatz

- $I_{1...11}$ has Picard-Fuchs operator $L^{(l)}$, the ε^0-part $L^{(l,0)}$ is of the form

$$L^{(l,0)} = \beta \theta^2 \frac{1}{Y_{l-2}} \theta \frac{1}{Y_{l-3}} \dots \frac{1}{Y_3} \theta \frac{1}{Y_2} \theta^2 \frac{1}{\psi_0}$$

- M_1 should start at order ε^l.
- $L^{(l,0)}$ **annihilates** $I_{1...11}$ modulo ε and modulo tadpoles.
- This suggests

$$M_1 = \frac{\varepsilon^l}{\psi_0} I_{1...11}.$$

The ansatz

- We construct a derivative basis. The **factorisation** of $L^{(l,0)}$ in the variable q suggests for the master integrals $M_2 - M_l$

$$M_j = \frac{\mathbf{1}}{\mathbf{Y_{j-1}}} \left[\frac{1}{2\pi i \varepsilon} \frac{d}{d\tau} M_{j-1} + \text{junk} \right],$$

- **Griffiths transversality**:

$$M_j = \frac{1}{Y_{j-1}} \left[\frac{1}{2\pi i \varepsilon} \frac{d}{d\tau} M_{j-1} - \sum_{k=1}^{j-1} \mathbf{F_{(j-1)k} M_k} \right],$$

with a priori unkown but ε-independent functions $F_{ij}(\tau)$.

Summary of the ansatz

$$
\begin{aligned}
M_0 &= \varepsilon^l I_{1\dots10} \\
M_1 &= \frac{\varepsilon^l}{\psi_0} I_{1\dots11} \\
M_j &= \frac{1}{Y_{j-1}} \left[\frac{1}{2\pi i \varepsilon} \frac{d}{d\tau} M_{j-1} - \sum_{k=1}^{j-1} F_{(j-1)k} M_k \right] \qquad \text{for } j \geq 2
\end{aligned}
$$

The differential equation

The ansatz leads to the differential equation

$$\frac{1}{2\pi i}\frac{d}{d\tau}M = \varepsilon \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & F_{11} & 1 & 0 & 0 & & 0 & 0 \\ 0 & F_{21} & F_{22} & Y_2 & 0 & & 0 & 0 \\ 0 & F_{31} & F_{32} & F_{33} & Y_3 & & 0 & 0 \\ \vdots & & & & & \ddots & & \vdots \\ 0 & F_{(l-2)1} & F_{(l-2)2} & F_{(l-2)3} & F_{(l-2)4} & \dots & Y_{l-2} & 0 \\ 0 & F_{(l-1)1} & F_{(l-1)2} & F_{(l-1)3} & F_{(l-1)4} & \dots & F_{(l-1)(l-1)} & 1 \\ * & * & * & * & * & \dots & * & * \end{pmatrix} M.$$

- The first l rows are in an ε-factorised form.
- Determine the functions F_{ij} such that the $(l+1)$-th row is in ε-factorised form.

The differential equation

The condition that in the $(l+1)$-th row only terms of order ε^1 are present leads to

- differential equations
- **algebraic equations** from self-duality

$$\frac{1}{2\pi i}\frac{d}{d\tau}M = \varepsilon \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & F_{11} & 1 & 0 & 0 & & 0 & 0 \\ 0 & F_{21} & F_{22} & Y_2 & 0 & & 0 & 0 \\ 0 & F_{31} & F_{32} & F_{33} & Y_3 & & 0 & 0 \\ \vdots & & & & & \ddots & & \vdots \\ 0 & F_{(l-2)1} & F_{(l-2)2} & F_{(l-2)3} & F_{(l-2)4} & \dots & Y_{l-2} & 0 \\ 0 & F_{(l-1)1} & F_{(l-1)2} & F_{(l-1)3} & F_{(l-1)4} & \dots & F_{(l-1)(l-1)} & 1 \\ * & * & * & * & * & \dots & * & * \end{pmatrix} M$$

The differential equation

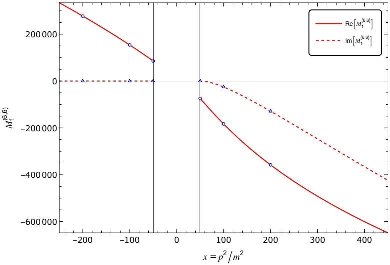
- The equations for F_{ij}'s have a natural triangular structure and can be solved systematically.
- We arrive at the **differential equation in ε-factorised form**:

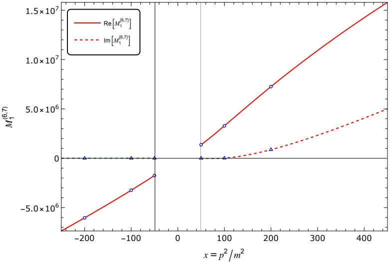
$$dM = \varepsilon A M$$

Section 8

Results and potential applications

Results: Six loops





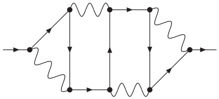
Expansion around $y = 0$ converges at six loops for $|p^2| > 49m^2$.
Agrees with results from `pySecDec`.

The geometry of this Feynman integral is a **Calabi-Yau five-fold**.

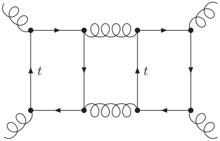
Pögel, Wang, S.W. '22

Examples

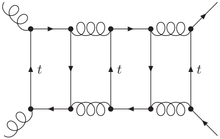
- Electron self-energy in QED (related to a Calabi-Yau 3-fold).



- Dijet production at N^{3}LO (related to a Calabi-Yau 2-fold).



- Top pair production at N^{4}LO (related to a Calabi-Yau 3-fold)



Conclusions

- Feynman integrals are needed for precision calculations in perturbative quantum field theory.
- Method of differential equations is a powerfull tool for computing Feynman integrals.
- It is helpful to relate a Feynman integral to a geometric object (spheres, elliptic curves, curves of higher genus, Calabi-Yau n-folds, ...).
  Algebraic geometry gives us information on the original Feynman integral.