On topological quantum field theories

Ilka Brunner, LMU München

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Introduction

- Topological quantum field theories are very simple models of TQFT
- ... so simple, that mathematicians can define them
- ... offer rigorous framework
- **part 1**: Explain in which physical context these models may play a role
- **part 2**: topological TQFT
- **part 3**: extended topological TQFT

Recap: quantum field theories

- General quantum field theories:
  E.g. defined by some action functional

$$S[\Phi] = \int_M \omega L(\Phi, \partial_\mu \Phi)$$

  ω volume element, L Lagrangian, Φ: fields
  M: space time manifold
- Computation of path integral $Z = \int D\Phi e^{-S}$
- Computation of correlation functions

$$\langle O_1 \dots O_n \rangle = \int D\Phi \; O_1 \dots O_n \; e^{-S}$$

  O_i (local) observables

Extended operators

- "Often" it is interesting to include defects
- Localized on $N \subset M$ with e.g. extra degrees of freedom confined to the defect.
- E.g. described on the level of an action:

$$S = S^{Bulk} + S^{defect}$$

- Boundaries are special cases.

What are topological quantum field theories

- They do not depend on a choice of metric
- Correlation functions do not depend on insertion points
- No divergence when operators approach each other
- The path integral only depends on the topology of the space-time manifold M

Examples

- Sometimes, the action does not depend on the metric
- Famous example: 3-dimensional Chern-Simons theory

$$S[A] = \frac{k}{4\pi} \int_{M_3} Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

- A: gauge connection
- Expectation values of Wilson loop operators $\leftrightarrow$ knot invariants
- Famous connection between physics, knot invariants and (rational) conformal field theory Witten

Examples: Twist of SUSY theories

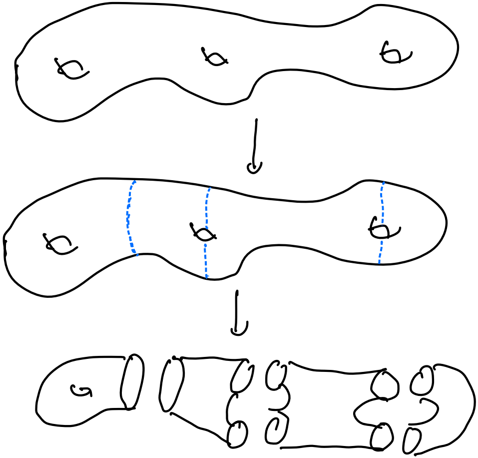
- SUSY algebras contain fermionic operators that anticommute.
- They extend the Poincare algebra
- Very schematically

$$\{Q, G\} \sim P, \quad Q^2 = G^2 = 0$$

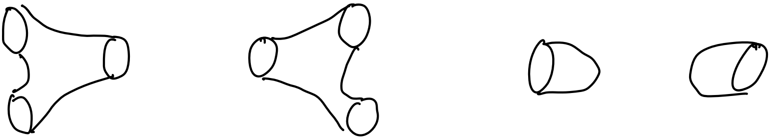
- Restricting to the cohomology of a SUSY generator provides examples for topological QFT, describing a subsector of the full SUSY QFT
- Geometric version: Instead of ordinary space, consider an extension by anticommuting directions. Twist is dimensional reduction along a fermionic direction.

Axiomatic formulation of TQFT

Computation of path integral on M



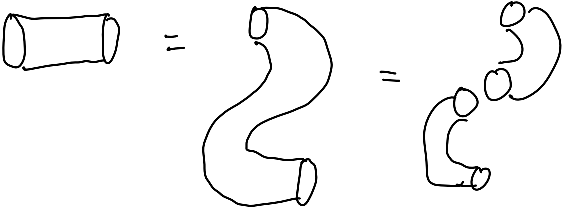
Basic building blocks for 2d TQFT



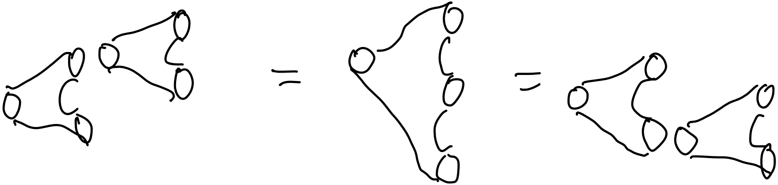
Build up any surface from these ingredients.

"bordisms"

Gluing relations



snake relation



Evaluation of diagrams

- Associate to a circle a space of states V
- Associate to the empty set the space $\mathbb{C}$.
- Associate to a bunch of n circles the tensor product $V \otimes \cdots \otimes V$
- Associate to the bordisms that connect unions of circles to other circles linear maps between the respective state spaces.
- Closed surfaces are read as maps between the empty set and the empty set: Associate maps $\mathbb{C} \to \mathbb{C}$
- Closed surface $\Sigma \to Z(\Sigma)$
- In case $\partial\Sigma = S^1$: $\mathbb{C} \to V$: state $\in V$.
- Read diagrams on previous slide as maps

Mathematical definition:TQFT

- a TQFT is a functor between a category of bordisms and (for example) the category of vector spaces
- Category: Set of objects with morphisms between them
- Category of vector spaces: Objects are vector spaces, morphisms are linear maps between them
- Category of bordisms: Objects are $d-1$ manifolds, morphisms are d dimensional bordisms.

$$Z : Bord_{d,d-1} \to Vect$$

Atiyah, Segal

Extended TQFT

- So far: Manifolds of dimension $d-1$ and d
- In (fully) extended TQFT we want to consider manifolds of any codimension!
- Start with points
- can be connected by lines
- Collection of lines can in turn be connected by 2-dimensional surfaces
- (...)
- There are again basic building blocks as well as gluing relations between them
- We obtain a higher category, in 2 dimensions a 2 category

Lurie, Freed, Baez, Dolan...

Building blocks and their relations, 0+1 d

0-dimensions: points

emptyset

1-dimensions: 1-manifolds connecting 0-manifolds

co-evaluation: $\emptyset \to pt_+ \sqcup pt_-$

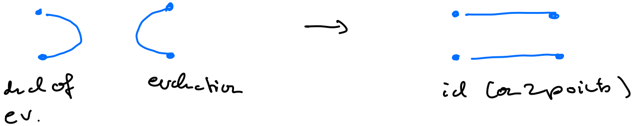
evaluation: $pt \sqcup pt \to \emptyset$

identity on a pt

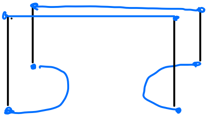
identity on $pt \sqcup pt$

snake relation

Building blocks, 2d

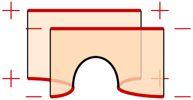


2-d interpolating surface

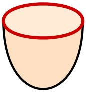


This is
"the evaluation on the evaluation"
ev_{ev}
(a 2-morphism)

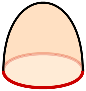
Building blocks, 2d: The adjunction 2-morphisms



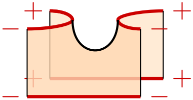
$$= \mathrm{ev}_{\widetilde{\mathrm{ev}}_+} \tag{1}$$



$$= \widetilde{\mathrm{coev}}_{\widetilde{\mathrm{coev}}_+} \, , \tag{2}$$

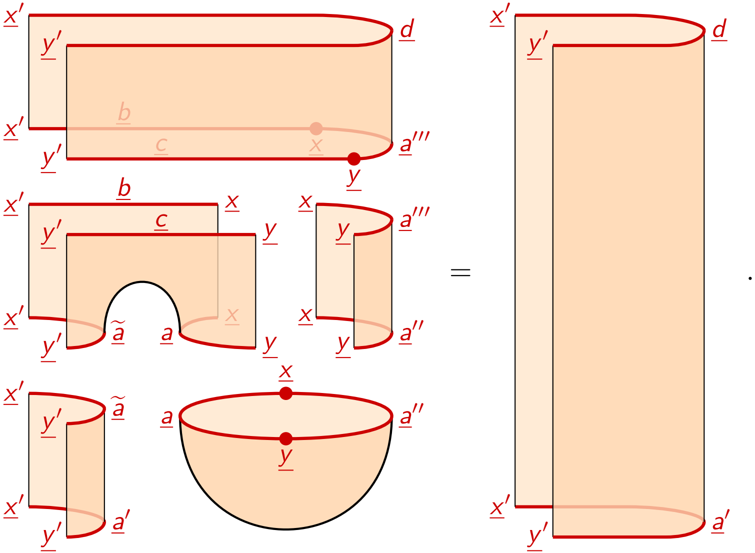


$$= \mathrm{ev}_{\widetilde{\mathrm{coev}}_+} \, , \tag{3}$$



$$= \widetilde{\mathrm{coev}}_{\widetilde{\mathrm{ev}}_+} \tag{4}$$

...and a snake-condition



(5)

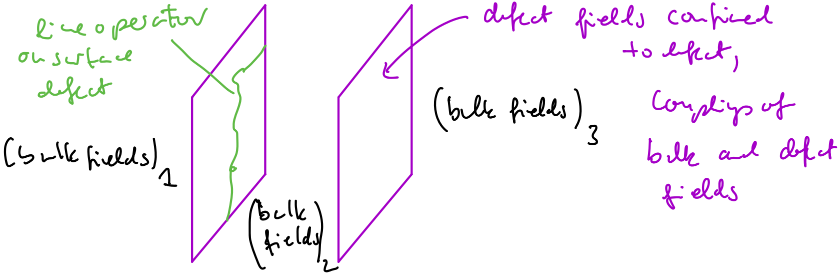
Extended TQFT

- We have introduced a higher category of bordisms.
- (i.e. start from points, look at lines between them, consider surfaces between the lines ...)
- An extended TQFT is a functor between this category and a target category that needs to have the same structure

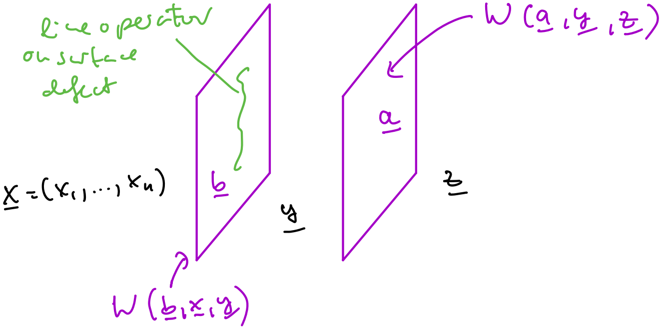
$$Z : Bord_{(2,1,0)} \to \mathcal{C}$$

- Suitable target category come for example from defects

Defects


line operator
on surface
defect
defect fields confined
to defect,
couplings of
bulk and defect
fields
(bulk fields)1
(bulk fields)2
(bulk fields)3

Example: Affine Rozansky Witten models



Example: Affine Rozansky Witten models

Theory of free $3d$ $N = 4$ hypermultiplets, in particular scalars $\Phi : M_3 \to T^*\mathbb{C}^n$

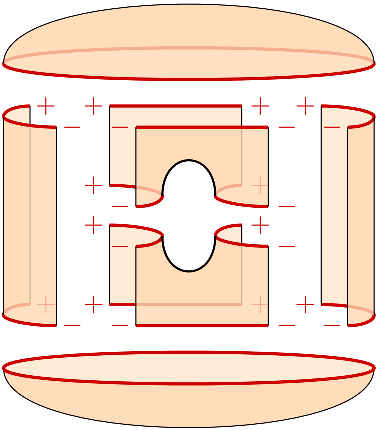
- objects: lists of variables $\underline{x} = (x_1, \dots, x_n)$
- surface defects: extra variables together with a superpotential $(\underline{a}; W(\underline{x}, \underline{a}))$
- invisible defect $(\underline{a}; \underline{a}(\underline{x} - \underline{y}))$
- Line defects: matrix factorizations
- One can show that this is a monoidal 2-category and that all objects are dualizable

Now we can compute anything!

IB, N. Carqueville, P. Fragkos, D. Roggenkamp

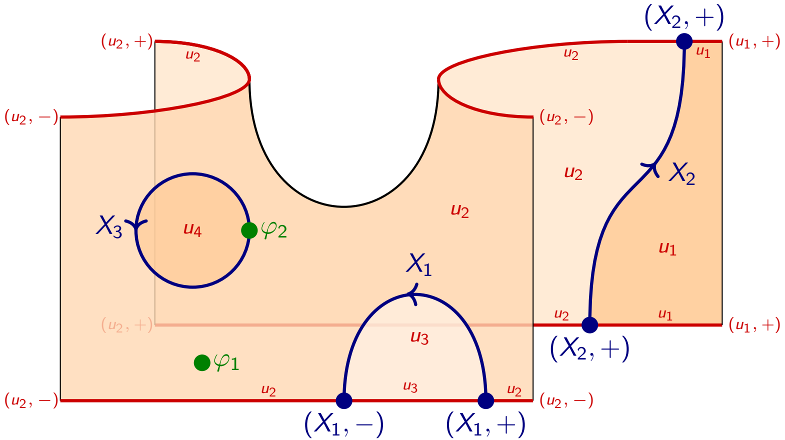
at least in the topological subsector, for affine RW or models under equally good control

$$Z(T^2) = Z\left(\vphantom{\int} \right.$$



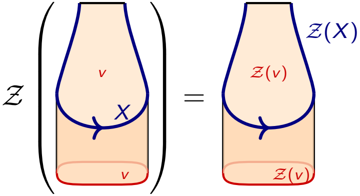
$$\left. \vphantom{\int} \right) = \widetilde{\mathrm{ev}}_{\widetilde{\mathrm{ev}}_+} \cdot \Big[1_{\widetilde{\mathrm{ev}}_u} \circ \big(\mathrm{ev}_{\widetilde{\mathrm{ev}}_u} \cdot \widetilde{\mathrm{coev}}_{\widetilde{\mathrm{ev}}_u} \big) \circ 1_{\mathrm{coev}_u} \Big] \cdot \mathrm{coev}_{\widetilde{\mathrm{ev}}_u} \; . \qquad (6)$$

or for example...



(7)

and more



(8)

This is the bulk-boundary map ...

Some results on affine Rozansky Witten models

- Applying the cobordism hypothesis systematically, we construct a unique extended TQFT for each number of variables.
- Compute for example state spaces on any orientable surface.
- use symmetry defects to model background gauge fields
- put any defect network on any surface (....)