

Symbol Alphabets from the Landau Singular Locus

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Scattering Amplitudes across Germany
Raitenhaslach, July 28, 2023

Motivation

For basis of polylogarithmic Feynman integrals (FI) \vec{g} , canonical differential equations (CDE) are gold: [Gehrmann,Remiddi'99] [Henn'13]

[Talks by Ben, Andreas, Ekta, Julian]

$$d\vec{g} = \epsilon d\widetilde{M} \vec{g}, \quad \widetilde{M} \equiv \sum_i \overbrace{\tilde{a}_i}^{\text{constant matrices}} \log \underbrace{W_i}_{\text{letters}},$$

Bottlenecks

1. Solving integration-by-parts identities to determine initial basis \vec{f}
2. Finding transformation $\vec{g} = T \cdot \vec{f}$

Could we predict kinematically dependent letters W_i beforehand?

Would reduce both steps to much easier, purely numeric problem!
In line with strategy of state of the art precision calculations such as

[Abreu, Ita, Moriello, Page, Tschernow, Zeng'20]

The Landau equations

Yield specific values of (kinematic) parameters of any (Feynman) integral, for which it may become singular. [Landau'59]

$$\mathcal{I} = \int \prod_{l=1}^L \frac{d^D k_l}{i\pi^{D/2}} \prod_{i=1}^E \frac{1}{D_i^{\nu_i}} = \int \prod_{l=1}^L \frac{d^D k_l}{i\pi^{D/2}} \int_0^\infty \prod_{i=1}^E dx_i \frac{\delta(\sum_j x_j - 1)}{(\sum_j x_j D_j)^{\sum_k \nu_k}}$$

where $D_i = -q_i^2 + m_i^2$.

$$x_i D_i = 0 \quad \forall i = 1, \dots, E$$

Landau equations:
$$\frac{\partial}{\partial k_l} \sum_{i=1}^E x_i D_i = 0, \quad \forall l = 1, \dots, L.$$

Formulated as conditions for the contour of integration to become trapped between two poles of integrand.

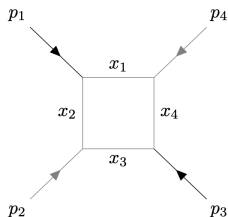
Believed for long to only provide information on where $W_i = 0$.

This work

Evidence through two loops: Rational letters of polylogarithmic FI captured by Landau equations, when recast as polynomial of the kinematic variables of integral, known as the *principal A-determinant* $E_A!$

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Example: 'Two-mass easy' box with $p_2^2 = p_4^2 = 0$, $p_1^2, p_3^2 \neq 0$:



E_A equipped with natural factorization, ($s = (p_1 + p_2)^2$, $t = (p_1 + p_4)^2$)

$$E_A = (p_1^2 p_3^2 - st) p_1^2 p_3^2 st (p_1^2 + p_3^2 - s - t) (p_3^2 - t) (p_3^2 - s) (p_1^2 - t) (p_1^2 - s).$$

where each factor is indeed a letter of the integral!

Outline

Introduction and Motivation

Feynman integrals, Landau singularities and GKZ systems

One-loop principal A -determinants and symbol letters

Further mathematical properties of Feynman integrals

Conclusions and Outlook

Feynman integrals in the Lee-Pomeransky representation:

$$\mathcal{I} = \frac{\Gamma(D/2)}{\Gamma((L+1)D/2 - \sum_i \nu_i)} \int_0^\infty \prod_{i=1}^E \left(\frac{x^{\nu_i-1} dx_i}{\Gamma(\nu_i)} \right) \frac{1}{\mathcal{G}^{D/2}}$$

where $\mathcal{G} = \mathcal{U} + \mathcal{F}$, and for graph G associated to integral \mathcal{I} ,

$$\mathcal{U} = \sum_{\substack{T \text{ a spanning} \\ \text{tree}^1 \text{ of } G}} \prod_{e \notin T} x_e,$$

$$\mathcal{F} = F_m + F_0 = \mathcal{U} \sum_{e \in E} m_e^2 x_e - \sum_{\substack{F \text{ a spanning} \\ \text{2-forest}^2 \text{ of } G}} p(F)^2 \prod_{e \notin F} x_e,$$

are the 1st and 2nd Symanzik polynomials, of degree $L, L+1$ in the x_i .

In this form, \mathcal{I} is special case³ of \mathcal{A} -hypergeometric function as defined by Gelfand, Graev, Kapranov and Zelevinsky (GKZ). [\[de la Cruz'19\]](#) [\[Klausen'19\]](#)

¹Connected subgraph of G containing all vertices but no loops.

²Defined similarly, but with 2 connected components.

³Generic case: All \mathcal{G} polynomial coefficients are variables, different from each other.

Singularities of GKZ-systems

Let $\mathcal{G} = \sum_{j=1}^m c_j \prod_{i=1}^E x_i^{a_{ij}}$, c_j all independent variables.

Values of c_i for which GKZ-system becomes singular are solutions to

$$E_A(\mathcal{G}) = 0$$

where $E_A(\mathcal{G})$ is the *principal A-determinant* of \mathcal{G} : Polynomial in c_j with integer coefficients, that vanishes whenever the system of equations

$$\mathcal{G} = x_1 \frac{\partial \mathcal{G}}{\partial x_1} = \dots = x_E \frac{\partial \mathcal{G}}{\partial x_E} = 0 \text{ has a solution for } x_i \in \mathbb{C}^*.$$

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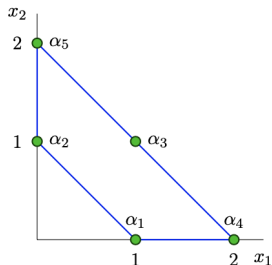
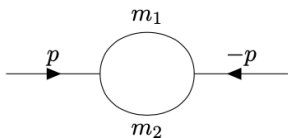
In practice, compute via theorem factorizing it into contributions from each face Γ of polytope with vertices (a_{1j}, \dots, a_{Ej}) , $j = 1, \dots, m$

$$E_A(\mathcal{G}) = \prod_{\Gamma} \Delta_{\Gamma}(\mathcal{G})$$

where the *A-discriminant* $\Delta_{\Gamma}(\mathcal{G})$ also polynomial in c_i , that vanishes when

$$\mathcal{G} = \frac{\partial \mathcal{G}}{\partial x_1} = \dots = \frac{\partial \mathcal{G}}{\partial x_E} = 0 \text{ has a solution for } x_i \in \mathbb{C}^*.$$

Example: Principal A -determinant of bubble



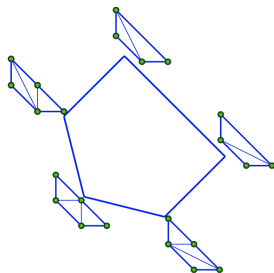
(Newton) polytope of \mathcal{G} polynomial exponents, $\text{Newt}(\mathcal{G})$

$$\mathcal{G} = x_1 + x_2 + (m_1^2 + m_2^2 - p^2)x_1x_2 + m_1^2x_1^2 + m_2^2x_2^2,$$

$$\begin{aligned} E_A(\mathcal{G}) &= \Delta_{\alpha_4} \Delta_{\alpha_5} \Delta_{\alpha_4\alpha_5} \Delta_{\alpha_1\alpha_2\alpha_4\alpha_5} \\ &= m_1^2 m_2^2 (p^4 + m_1^4 + m_2^4 - 2p^2 m_1^2 - 2p^2 m_2^2 - 2m_1^2 m_2^2) p^2, \end{aligned}$$

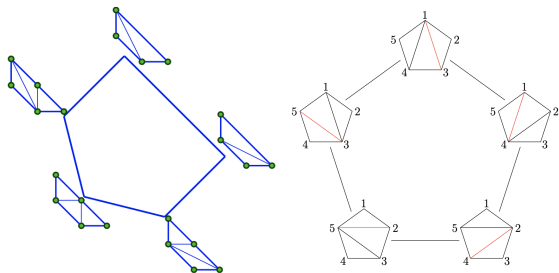
Interpretation of $E_A(\mathcal{G})$ polytope

$\text{Newt}(E_A(\mathcal{G}))$, built out of exponents of $E_A(\mathcal{G})$ polynomial: Keeps track of *triangulations* of $\text{Newt}(\mathcal{G})$.



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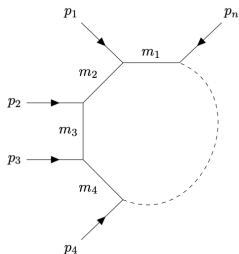
Cluster algebras also describe triangulations of geometric spaces

[Fomin, Zelevinsky'01] [Felikson, Shapiro, Tumarkin'11]

First-principle derivation of observed cluster-algebraic structure of Feynman integrals? [Chicherin, Henn, Papathanasiou'20] ... [He, Liu, Tang, Yang'22]

Generic 1-loop integrals

All $m_i, p_i^2 \neq 0$ and different from each other



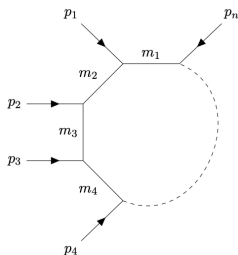
All Landau singularity information captured in *modified Cayley matrix* \mathcal{Y} , [Melrose'65]

$$\mathcal{Y} = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & Y_{11} & Y_{12} & \cdots & Y_{1n} \\ 1 & Y_{12} & Y_{22} & \cdots & Y_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & Y_{1n} & Y_{2n} & \cdots & Y_{nn} \end{pmatrix} \quad \begin{aligned} Y_{ii} &= 2m_i^2 \\ Y_{ij} &= m_i^2 + m_j^2 - s_{ij-1} \\ s_{ij} &= (p_i + \dots + p_j)^2 \end{aligned}$$

¹Where all $x_i \neq 0$

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because A -discriminants reduce to usual determinants:

- ▶ $\Delta_{\text{Newt}(\mathcal{F})}(\mathcal{F}) = \det Y$: Leading¹ Landau singularity of type I (finite k)
- ▶ $\Delta_{\text{Newt}(\mathcal{G})}(\mathcal{G}) = \det \mathcal{Y}$: Leading¹ Landau singularity of type II ($k \rightarrow \infty$)
- ▶ Subleading Landau singularity where $x_{i_1}, \dots, x_{i_m} = 0 \sim$ Leading singularity of subgraph where internal edges i_1, \dots, i_m removed

[Klausen'21]

¹Where all $x_i \neq 0$

Minors of modified Cayley matrix

For any matrix A with elements a_{mn} , let (j, k) -th minor of A be

$$A \begin{matrix} [j] \\ [k] \end{matrix} \equiv \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,k-1} & k & a_{1,k+1} & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,k-1} & & a_{2,k+1} & \cdots & a_{2,N} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ a_{j-1,1} & a_{j-1,2} & a_{j-1,3} & \cdots & a_{j-1,k-1} & & a_{j-1,k+1} & \cdots & a_{j-1,N} \\ j & & & & & & & & \\ a_{j+1,1} & a_{j+1,2} & a_{j+1,3} & \cdots & a_{j+1,k-1} & & a_{j+1,k+1} & \cdots & a_{j+1,N} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ a_{N,1} & a_{N,2} & a_{N,3} & \cdots & a_{N,k-1} & & a_{N,k+1} & \cdots & a_{N,N} \end{vmatrix},$$

where shading indicates removal of row and column. Similarly $A \begin{bmatrix} i_1 \cdots i_k \\ j_1 \cdots j_k \end{bmatrix}$,

$$A \begin{bmatrix} \cdot \\ \cdot \end{bmatrix} = \det A.$$

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$$\mathcal{Y} \begin{bmatrix} 3 \\ 3 \end{bmatrix} \left(\begin{array}{c} p_1 \quad p_n \\ \swarrow \quad \searrow \\ m_1 \\ \swarrow \quad \searrow \\ p_2 \quad p_3 \\ \swarrow \quad \searrow \\ m_2 \quad m_4 \\ \swarrow \quad \searrow \\ p_4 \end{array} \right) = \mathcal{Y} \begin{bmatrix} \cdot \\ \cdot \end{bmatrix} \left(\begin{array}{c} p_1 \quad p_n \\ \swarrow \quad \searrow \\ m_1 \\ \swarrow \quad \searrow \\ p_2 \quad p_3 \\ \swarrow \quad \searrow \\ m_3 \quad m_4 \\ \swarrow \quad \searrow \\ p_4 \end{array} \right)$$

Principal A -determinant of generic 1-loop graphs

Gathering previous bits of information, arrive at

$$E_A(\mathcal{G}) = \mathcal{Y} \left[\begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \right]_{i=1}^{n+1} \mathcal{Y} \left[\begin{array}{c} i \\ \cdot \\ i \end{array} \right] \cdots \prod_{i_{n-1} > \dots > i_1 = 1}^{n+1} \mathcal{Y} \left[\begin{array}{c} i_1 \dots i_{n-1} \\ \cdot \\ i_1 \dots i_{n-1} \end{array} \right] \prod_{i=2}^{n+1} \mathcal{Y}_{ii}.$$

product of all diagonal k -dimensional minors of \mathcal{Y} with $k = 1, \dots, n + 1$, except $\mathcal{Y}_{11} = 0$.

$2^{n+1} - n - 2$ factors, e.g. 1, 4, 11, 26, 57, 120 factors for $n = 1, \dots, 6$.

From 1-loop rational to square-root letters

Working assumption: Square-root letters produced by re-factorizing E_A using Jacobi determinant identities of the form

$$p \cdot q = f^2 - g = (f - \sqrt{g})(f + \sqrt{g}),$$

where

1. p, q factors of E_A , i.e. rational letters.
2. Square-root letters $f \pm \sqrt{g}$ obtained contain leading singularity of the Feynman integral considered in second term.

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Motivated by interpretation of 1-loop integrals as volumes of spherical simplices. [\[Davydychev, Delbourgo'99\]](#) Jacobi identities,

$$A \begin{bmatrix} \cdot \\ \cdot \end{bmatrix} A \begin{bmatrix} i & j \\ i & j \end{bmatrix} = A \begin{bmatrix} i \\ i \end{bmatrix} A \begin{bmatrix} j \\ j \end{bmatrix} - A \begin{bmatrix} i \\ j \end{bmatrix} A \begin{bmatrix} j \\ i \end{bmatrix} \stackrel{A=A^T}{=} A \begin{bmatrix} i \\ i \end{bmatrix} A \begin{bmatrix} j \\ j \end{bmatrix} - A \begin{bmatrix} i \\ j \end{bmatrix}^2$$

crucial for their computation. Point 2 adopts widely observed pattern in 1- and 2-loop computations.

All 1-loop letters I

Need only ratio $\frac{f-\sqrt{g}}{f+\sqrt{g}}$, as product already contained in rational alphabet.

Letting $D = D_0 - 2\epsilon$, obtain N letters of type,

$$W_{1,\dots,(i-1),\dots,n} = \begin{cases} \frac{\mathcal{Y} \begin{bmatrix} i \\ 1 \end{bmatrix} - \sqrt{-\mathcal{Y} \begin{bmatrix} \cdot \\ \cdot \end{bmatrix} \mathcal{Y} \begin{bmatrix} 1 & i \\ 1 & i \end{bmatrix}}{\mathcal{Y} \begin{bmatrix} i \\ 1 \end{bmatrix} + \sqrt{-\mathcal{Y} \begin{bmatrix} \cdot \\ \cdot \end{bmatrix} \mathcal{Y} \begin{bmatrix} 1 & i \\ 1 & i \end{bmatrix}}}, & D_0 + n \text{ odd,} \\ \frac{\mathcal{Y} \begin{bmatrix} i \\ 1 \end{bmatrix} - \sqrt{\mathcal{Y} \begin{bmatrix} i \\ i \end{bmatrix} \mathcal{Y} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}}{\mathcal{Y} \begin{bmatrix} i \\ 1 \end{bmatrix} + \sqrt{\mathcal{Y} \begin{bmatrix} i \\ i \end{bmatrix} \mathcal{Y} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}}, & D_0 + n \text{ even.} \end{cases}$$

All 1-loop letters II

In addition, $n(n-1)/2$ letters of type,

$$W_{1, \dots, (i-1), \dots, (j-1), \dots, n} = \begin{cases} \frac{\mathcal{Y} \begin{bmatrix} i \\ j \end{bmatrix} - \sqrt{-\mathcal{Y} \begin{bmatrix} \cdot \\ \cdot \end{bmatrix} \mathcal{Y} \begin{bmatrix} i & j \\ i & j \end{bmatrix}}}{\mathcal{Y} \begin{bmatrix} i \\ j \end{bmatrix} + \sqrt{-\mathcal{Y} \begin{bmatrix} \cdot \\ \cdot \end{bmatrix} \mathcal{Y} \begin{bmatrix} i & j \\ i & j \end{bmatrix}}}, & D_0 + n \text{ odd,} \\ \frac{\mathcal{Y} \begin{bmatrix} 1 & j \\ 1 & i \end{bmatrix} - \sqrt{-\mathcal{Y} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathcal{Y} \begin{bmatrix} 1 & i & j \\ 1 & i & j \end{bmatrix}}}{\mathcal{Y} \begin{bmatrix} 1 & j \\ 1 & i \end{bmatrix} + \sqrt{-\mathcal{Y} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathcal{Y} \begin{bmatrix} 1 & i & j \\ 1 & i & j \end{bmatrix}}}, & D_0 + n \text{ even,} \end{cases}$$

All 1-loop letters III

Our procedure also predicts $\mathcal{Y}[\cdot]$ and $\mathcal{Y}\left[\begin{smallmatrix} \cdot \\ 1 \end{smallmatrix}\right]$ as individual rational letters, but in fact only the ratio

$$W_{1,2,\dots,n} = \frac{\mathcal{Y}\left[\begin{smallmatrix} \cdot \\ \cdot \end{smallmatrix}\right]}{\mathcal{Y}\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]},$$

appears, as we'll get back to in next slide.

Finally, obtain remaining letters of n -point graph by applying above formulas to all of its subgraphs.

Assuming $n \leq d+1$ for external kinematics dimension d , total letter number

$$|W| = \begin{cases} 2^{n-3} (n^2 + 3n + 8) - \frac{1}{6} (n^3 + 5n + 6), & D_0 \text{ even,} \\ 2^{n-3} (n^2 + 3n + 8) - \frac{1}{2} (n^2 + n + 2), & D_0 \text{ odd,} \end{cases}$$

e.g. $|W| = 1, 5, 18, 57, 166$ (even), $|W| = 1, 5, 19, 61, 176$ (odd), $n = 1, \dots, 5$

Verification through differential equations & comparison with literature

From letter prediction, derived canonical differential equations through numeric IBP identities \Rightarrow confirmation.

By explicit computation up to $n = 10$, infer general form, e.g. $n + D_0$ even:

$$\begin{aligned} d\mathcal{J}_{1\dots n} &= \epsilon d \log W_{1\dots n} \mathcal{J}_{1\dots n} \\ &+ \epsilon \sum_{1 \leq i \leq n} (-1)^{i + \lfloor \frac{n}{2} \rfloor} d \log W_{1\dots(i)\dots n} \mathcal{J}_{1\dots\hat{i}\dots n} \\ &+ \epsilon \sum_{1 \leq i < j \leq n} (-1)^{i+j + \lfloor \frac{n}{2} \rfloor} d \log W_{1\dots(i)\dots(j)\dots n} \mathcal{J}_{1\dots\hat{i}\dots\hat{j}\dots n}. \end{aligned}$$

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Furthermore, compared to previous results for D_0 even based on

1. the diagrammatic coaction [Abreu,Britto,Duhr,Gardi'17]
2. the Baikov representation [Chen,Ma,Yang'22]

Agreement in form of CDE, where each line $i = 1, 2, 3$ corresponds (next-to-) i^{-1} maximal cut. Also in letters for orientations presented in 2.

Limits of generic to non-generic graphs

Proved that E_A has well-defined limit when any $m_i^2, p_j^2 \rightarrow 0$, namely it is unique regardless of the order with which we send them to zero.

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Limit of E_A when single parameter x takes value a may be defined as

$$\lim_{x \rightarrow a} E_A = \frac{\partial^l \widetilde{E}_A}{\partial x^l} \Big|_{x=a} \neq 0, \text{ with } \frac{\partial^{l'} E_A}{\partial x^{l'}} \Big|_{x=a} = 0 \text{ for } l' = 0, \dots, l-1,$$

While multivariate generalization straightforward, highly nontrivial that limit does not depend on order. E.g. triangle Cayley in limit $p_i^2 \rightarrow 0$:

$$\det Y = 0 + 2 \sum_{i=1}^3 p_i^2 (m_i^2 - m_{i-1}^2)(m_{i+1}^2 - m_{i-1}^2) + \mathcal{O}(p_j^2 p_k^2),$$

While limits of individual factors in E_A depend on limit order, E_A as a whole does not, since different orders produce factors it already contains.

Strong evidence that alphabet of non-generic FI correctly obtained as limit of generic one, in line with previous observations.

Mathematica Notebook

```

(* Symbol alphabets*)
(* Generic Box in even dimension *)
DB = 4;
EvaluateLetter[AllLettersList[4]] // Short
(* Two-mass easy box limit *)
Factor[PLimit[%, ms[1] -> 0, ms[2] -> 0, ms[3] -> 0, ms[4] -> 0, s[1, 3] -> 0, ps[2] -> 0]];
(* In the limit the letters become multiplicatively dependent. Since all of them are rational, a basis may be found as follows *)
d[Expand[d / %]];
d[Collect[#, DeleteCases[RowReduce[CoefficientArrays[%, Variables[%]]][{2}]] -> Variables[%, 0] / d[[_x_] - x
Length[%]
(* Product indeed yields corresponding limit of the principal A-determinant *)
LS2neBox - Times @@ %

(* Differential equations *)
(*Box basis*)
basis = Range[4]
(*Box canonical differential equations*)
CDEs[%] // MatrixForm

In[100]: ms[1] <<24>> (ms[3]^2 ps[1]^2 - 2 ms[3] - ms[4] ps[1]^2 + ms[4]^2 ps[1]^2 - 2 ms[1] - ms[3] - ps[1] - ps[2] +
2 ms[1] - ms[4] - ps[1] - ps[2] + <<172>> - ms[3]^2 s[2, 3]^2 - 2 ms[1] - s[1, 2] s[2, 3]^2 - 2 ms[3] - s[1, 2] s[2, 3]^2 + s[1, 2]^2 s[2, 3]^2)
Out[100]: ps[1] - ps[3] (ps[1] - s[1, 2]) (ps[3] - s[1, 2]) s[1, 2] (ps[1] - s[2, 3]) (ps[3] - s[2, 3]) (ps[1] + ps[3] - s[1, 2] - s[2, 3]) s[2, 3] (ps[1] - ps[3] - s[1, 2] - s[2, 3])
In[101]: {
1 / (2 ms[1]), 1 / (2 ms[2]), <<53>>, -ms[1] - ms[3] - ps[1] + <<73>>, -2 ms[1] - ms[2] - ps[1] + <<65>>
- ms[1] - ms[3] - ps[1] + <<71>> + <<130>>, -2 ms[1] - ms[2] - ps[1] + <<63>> + sqrt[-ms[1]^2 + 2 ms[1] - ms[2] - <<130>>^2 + <<130>> + 2 ms[2] - ps[1] - ps[1]^2] (<<130>>)
}
Out[101]: {ps[1], ps[3], ps[1] - s[1, 2], ps[3] - s[1, 2], s[1, 2], ps[1] - s[2, 3], ps[3] - s[2, 3], ps[1] + ps[3] - s[1, 2] - s[2, 3], s[2, 3], ps[1] - ps[3] - s[1, 2] - s[2, 3]}
Out[102]: 18
Out[103]: 0
Out[104]: {IG[2][3], IG[2][2], IG[2][3], IG[2][4], IG[2][1, 2], IG[2][1, 3], IG[2][1, 4], IG[2][2, 3], IG[2][2, 4], IG[2][3, 4], IG[4][1, 2, 3], IG[4][1, 2, 4], IG[4][1, 3, 4], IG[4][2, 3, 4], IG[4][1, 2, 3, 4]}
MatrixForm
w[1] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 w[2] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 w[3] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 w[4] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
-w[1, {2}] w[1, 2] 0 0 0 w[1, 2] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
-w[1, {3}] 0 0 w[1, 3] 0 0 w[1, 3] 0 0 0 w[1, 3] 0 0 0 0 0 0 0 0 0 0 0
-w[1, {4}] 0 0 0 w[1, 4] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 -w[2, {3}] w[2, 3] 0 0 0 0 0 0 0 w[2, 3] 0 0 0 0 0 0 0 0 0 0 0
0 -w[2, {4}] w[2, 4] 0 0 0 0 0 0 0 0 w[2, 4] 0 0 0 0 0 0 0 0 0 0 0
0 0 -w[3, {4}] w[3, 4] 0 0 0 0 0 0 0 0 w[3, 4] 0 0 0 0 0 0 0 0 0 0 0
-w[1, {2}, {3}] w[1, {2}, {3}] + w[1, {2}, {3}] -w[1, {2}, {3}] 0 -w[1, {2}, {3}] w[1, {2}, {3}] 0 0 -w[1, {2}, {3}] 0 0

```

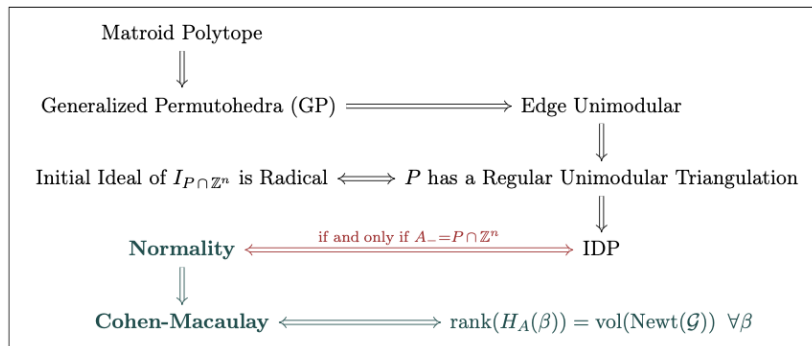
Cohen-Macaulay property

Guarantees that

master integrals = volume of $\text{Newt}(\mathcal{G})$

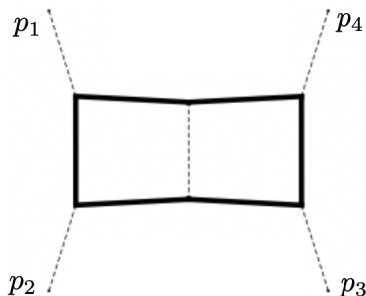
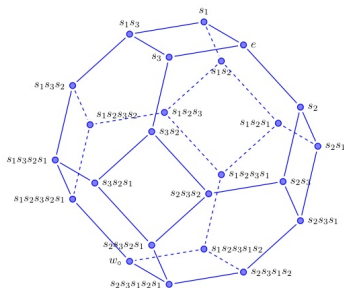
Proved it for currently largest known class of 1-loop integrals, including completely on-shell/massless. For earlier work, see [\[Tellander, Helmer'21\]](#) [\[Walther'22\]](#)

Relation to other properties:



Generalized permutohedron (GP) property

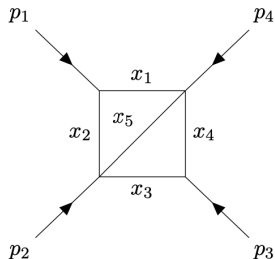
A polytope $P \subset \mathbb{R}^n$ is GP if and only if every edge is parallel to $\mathbf{e}_i - \mathbf{e}_j$, where \mathbf{e}_i is unit vector on coordinate axis, for some $i, j \in \{1, \dots, n\}$. E.g.



Practical utility: This property facilitates new methods for fast Monte Carlo evaluation of Feynman integrals. [Borinsky'20] [Borinsky, Munch, Tellander'23]

Previously proven for generic kinematics. [Schultka'18] Here: Generalized to any graph where all external vertices joined by massive path.

Two-loop example of principal A -determinant-alphabet relation



1-mass slashed box,

$$p_1^2 \neq 0, p_2^2 = p_3^2 = p_4^2 = 0$$

$$E_A(\mathcal{G}) = (p_1^2 - t)(p_1^2 - s)(p_1^2 - s - t)(s + t)stp_1^2.$$

Agrees precisely with (2dHPL) alphabet known to describe 2-loop master integrals with these kinematics! [\[Gehrmann, Remiddi'00\]](#)

Evidence that rational letters of polylogarithmic FI captured by polynomial form of Landau equations in terms of *principal A-determinant* E_A !

- ▶ Through 2 loops
- ▶ 1 loop: Also obtain square-root letters from Jacobi identities + CDE
- ▶ Strong evidence for well-defined limits to non-generic kinematics
- ▶ Easy-to-use Mathematica file with our results

Next Stage

1. More efficient evaluation of E_A + more 2-loop checks
2. New predictions for pheno, e.g. letters for $2 \rightarrow 3$ with 2 massive legs
3. Explore implications for beyond-polylogarithmic case