TENSOR NETWORK METHODS FOR LATTICE GAUGE THEORIES

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Conventional LGT techniques

- Discretization of both space and time
- Monte Carlo computations on a Wick-rotated, Euclidean lattice

$$\left\langle \hat{A}\left(\hat{\Phi}\right)\right\rangle = \frac{\int \mathcal{D}\phi A(\phi) e^{iS_M}}{\int \mathcal{D}\phi e^{iS_M}}$$

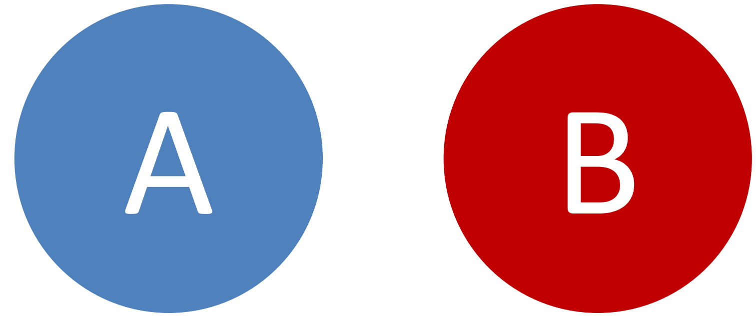
$$\xrightarrow[t\rightarrow -i\tau]{} \frac{\int \mathcal{D}\phi A(\phi) e^{-S_E}}{\int \mathcal{D}\phi e^{-S_E}} \equiv \int \mathcal{D}\phi A\left(\phi\right) p\left(\phi\right)$$

- **Very (very) successful for many applications, e.g. the hadronic spectrum**
- **Problems:**
  - **Real-Time evolution:**
    - Not available in Wick rotated, Euclidean spacetimes, used in conventional Monte-Carlo path integral LGT calculations
  - **Sign problem:**
    - Appears in several scenarios with fermions (finite density), represented by Grassman variables in a Wick-rotated, Euclidean spacetime

$\rightarrow$ **New approaches: quantum simulation and computation, tensor networks.**

Entanglement

$$\mathcal{H} = \mathcal{H}_A \times \mathcal{H}_B$$



- The most general state of this bipartite system may be written as

$$|\psi\rangle = \sum_{ij} a_{ij} \, |i\rangle_A \, |j\rangle_B$$

- However it may be brought to the **Schmidt form**

$$|\psi\rangle = \sum_{\mu} b_{\mu} \, |\mu\rangle_A \, |\mu\rangle_B$$

Entanglement

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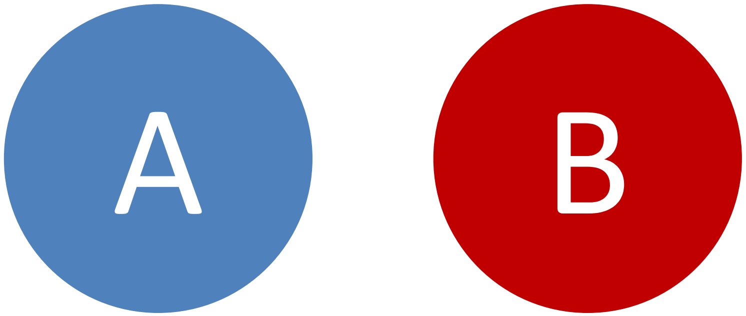

- If all the b_μ are zero but one, it is a **product state**,

$$|\psi\rangle = |\mu_0\rangle_A \, |\mu_0\rangle_B$$

- Otherwise we say that the state is **entangled**

Entanglement

$$\mathcal{H} = \mathcal{H}_A \times \mathcal{H}_B$$

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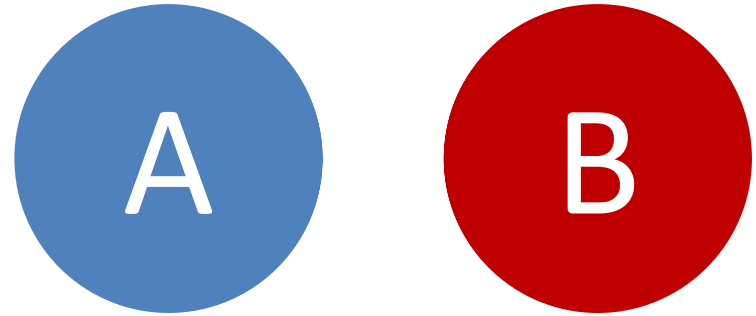


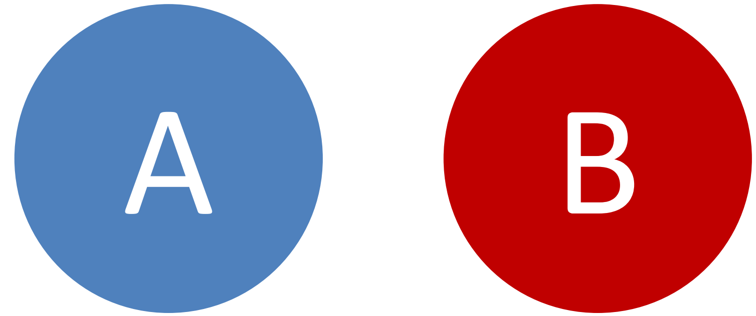
Entanglement is a nontrivial quantum correlation; if we measure the system A and find it in $|\mu_0\rangle_A$, we automatically know the state of B: $|\mu_0\rangle_B$

This has been used widely for quantum communication and cryptography protocols.

Entanglement Entropy

$$\mathcal{H} = \mathcal{H}_A \times \mathcal{H}_B$$

$$|\psi\rangle = \sum_{\mu} b_{\mu} \, |\mu\rangle_A \, |\mu\rangle_B$$



We define the probabilities, as usual, by

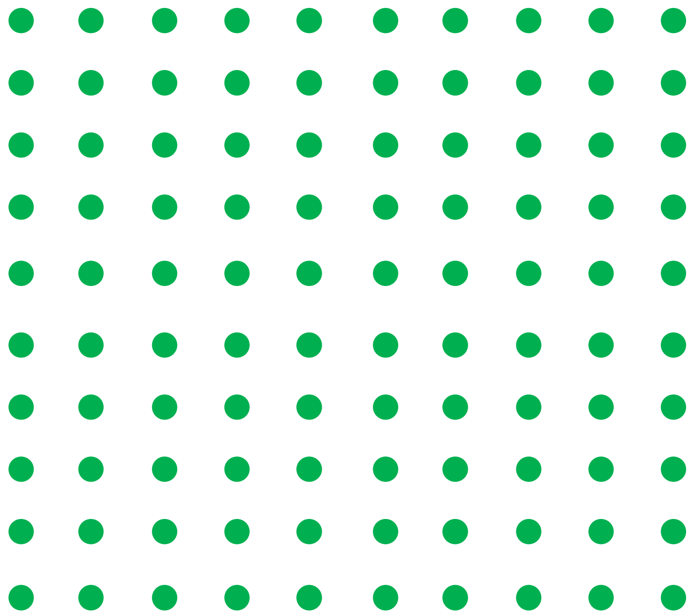
$$p_{\mu} = |b_{\mu}|^2$$

The **entanglement entropy** is defined as

$$S = -\sum_{\mu} p_{\mu} \log p_{\mu}$$

Many-Body Area Law, Handwaving Formulation

"The ground state of a many-body Hamiltonian with local interactions (+ a few more assumptions) **obeys an bipartite entanglement entropy area law."**



See, e.g., Eisert, Cramer, Plenio, RMP 2010

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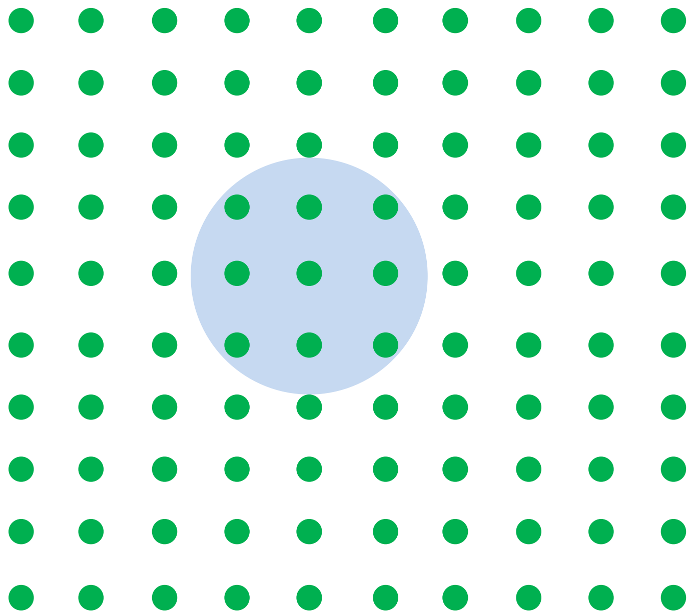

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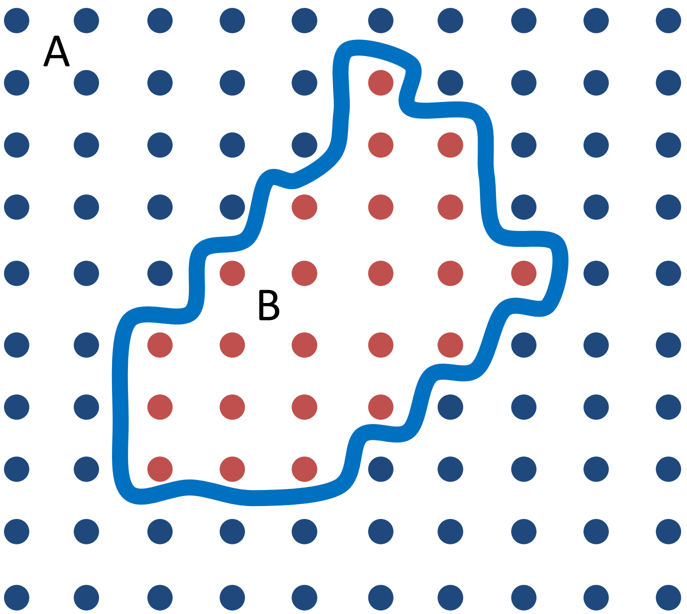

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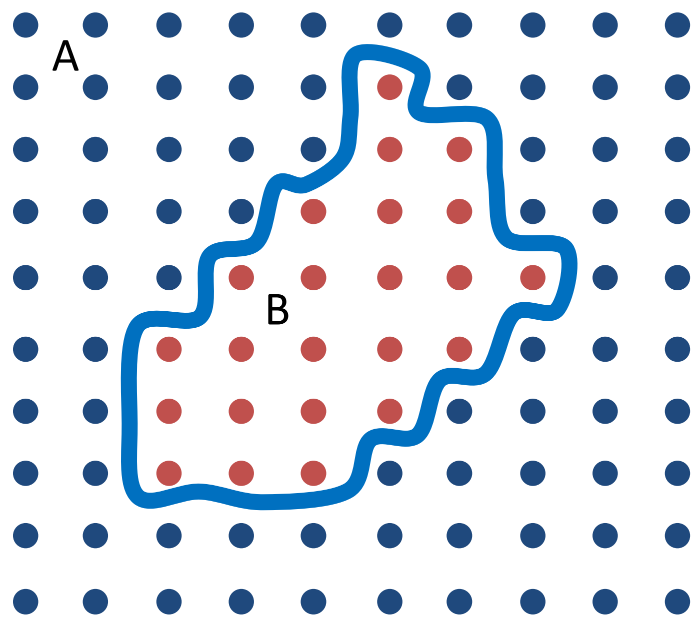

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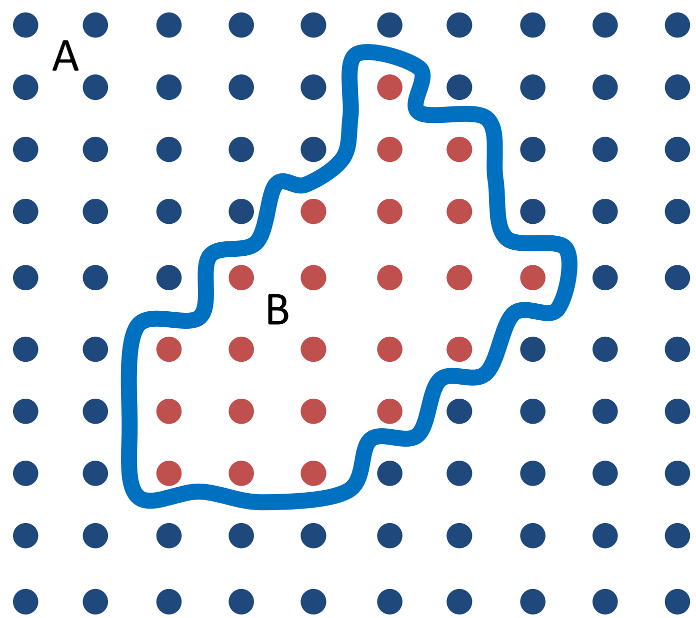
$$p_{\mu} = |b_{\mu}|^2$$

$$S = -\sum_{\mu} p_{\mu} \log p_{\mu}$$

$$S \propto \mathrm{Area\,(Boundary)}$$



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Many-Body Area Law, Handwaving Formulation

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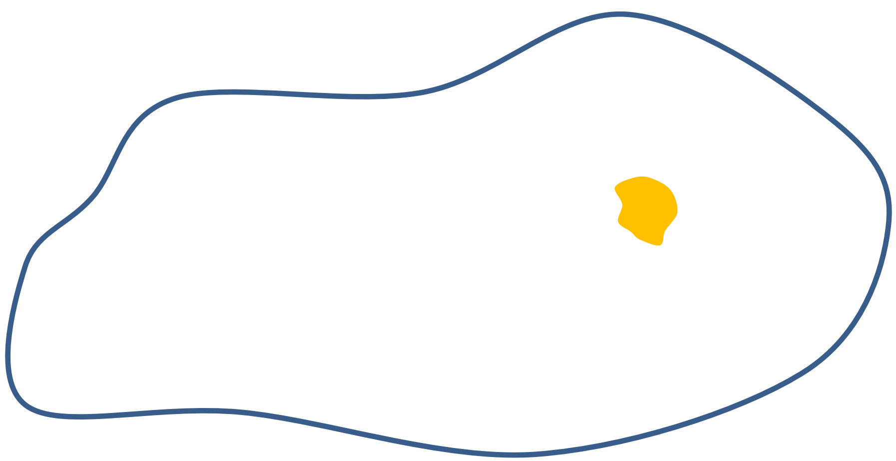
Physically relevant corner of the Hilbert space



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Tensor Network States

- The **number of variables** needed to describe states of a many-body system **scales exponentially** with the system size. This makes it hard to simulate large systems (classically).

- **Tensor network states** are ansatz states for **many body systems**, mostly on a lattice, for either **analytical or numerical** studies, based on contractions of **local tensors that depend on few parameters**.

- In spite of their **simple description**, tensor network states describe and approximate **physically relevant states of many-body systems**.

Relevant reviews:
Schollwöck, Ann. Phys. 2011
Orus, Ann. Phys. 2014
Cirac, Perez-Garcia, Schuch, Verstraete, RMP 2021

Tensor Network Studies of LGTs

- **Real-Time evolution:**
  - Not available in Wick rotated, Euclidean spacetimes, used in conventional Monte-Carlo path integral LGT calculations
  - Calculations in **quantum Hilbert spaces**, where states evolve in real time, instead of in Wick-rotated statistical mechanics analogies.

- **Sign problem:**
  - Appears in several scenarios with fermions (finite density), represented by Grassman variables in a Wick-rotated, Euclidean spacetime
  - Calculations in **quantum Hilbert spaces**: fermions are fermions, no integration over time dimension. If the problem arises, it can be the result of using a particular method, nothing general.

See the review paper by Bañuls and Cichy, Rep. Prog. Phys. 83 024401 (2020)

MPS & LGT – Numerical Approach

- **Mostly in 1+1d**, combining MPS (Matrix Product States) with **DMRG (Density Matrix Renormalization Group)**; have been widely and successfully used for various many body models, mostly from condensed matter.

- **Very successfully applied to 1+1d lattice gauge theories**, including finite chemical potential and real time evolution (string breaking) for Abelian and non-Abelian theories
  - Bañuls (spoke last week), Cichy, Cirac, Kühn, Jansen, Saito…
  - Dalmonte, Montangero, Pichler, Rico (tomorrow), Silvi, Tschirsich, Zoller…
  - Buyens, Haegeman, Hebenstreit, van Acoleyen, Verschelde, Verstraete…
  - Borla, Moroz, Grusdt, Verresen… (rather more CM-like)

- High dimensional generalizations: challenging and demanding scaling; works nicely for ladders, cylinders etc.s

See the review paper by Bañuls and Cichy, Rep. Prog. Phys. 83 024401 (2020) and refs. therein

Tensor Field Theory

A non-Hamiltonian approach

See e.g. the review of Meurice, Sakai and Unmuth-Yockey (Rev. Mod. Phys. 94, 025005) and the talks by Yannick and Judah

* Non-Hamiltonian methods in more than 1+1d: TRG, Tensor Field Theory, …
– see e.g. the review of Meurice, Sakai and Unmuth-Yockey, arXiv:2010.06539 [hep-lat]

Hamiltonian LGT TNs in 2+1d and more

- Tagliacozzo, Vidal, **Entanglement renormalization and gauge symmetry**, PRB 2011
  - Pure gauge, Z2
- Tagliacozzo, Celi, Lewenstein, **Tensor Networks for Lattice Gauge Theories with Continuous Groups**, PRX 2014
  - Pure gauge, continuous groups

Gauging globally invariant ("matter") PEPS to locally invariant LGT PEPS – introducing gauge fields which lift the symmetry to a local one

- Haegeman, Van Acoleyen, Schuch, Cirac, Verstraete, **Gauging Quantum States: From Global to Local Symmetries in Many-Body Systems**, PRX 2015
  - Gauge field Hilbert space = Matter Hilbert Space (Higgs-like theories)

- Zohar, Burrello, **Building projected entangled pair states with a local gauge symmetry**, NJP 2016
  - Different Gauge Field Hilbert spaces, allowing for fermionic constructions (matching the standard model content)

Hamiltonian LGT TNs in 2+1d and more

iPEPS:

- **Zapp and Orus, Tensor network simulation of QED on infinite lattices: learning from (1+1)d, and prospects for (2+1)d**, PRD 2014
  - General discussion
- **Robaina, Bañuls, Cirac, Simulating 2+1d Z3 lattice gauge theory with iPEPS**, PRL 2021
  - Introduced an iPEPS algorithm and applied it to the Z3 case

Tree tensor networks:

- **Felser, Silvi, Collura, Montangero, Two-dimensional quantum-link lattice Quantum Electrodynamics at finite density**, PRX 2020
- **Magnifico, Felser, Silvi, Montangero, Lattice Quantum Electrodynamics in (3+1)-dimensions at finite density with Tensor Networks**, Nat. Comm. 2021
- **Montangero, Rico, Silvi, Loop-free tensor networks for high-energy physics**, Phil. Trans. R. Soc. A, 2022

Gauged Gaussian Fermionic PEPS:

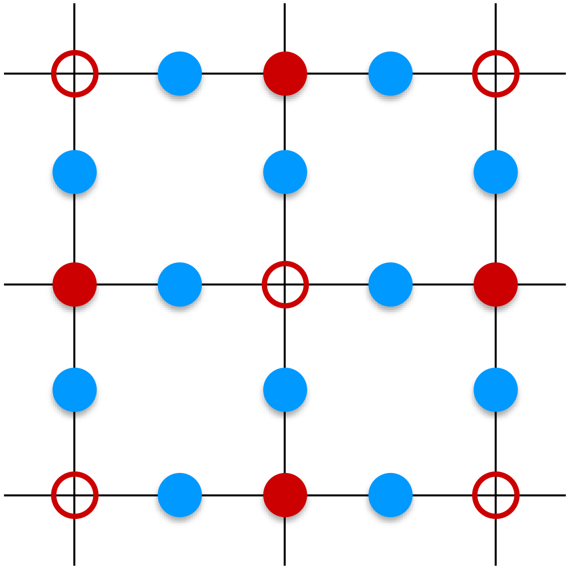
- **Zohar, Burrello, Wahl, Cirac**, **Fermionic projected entangled pair states and local U(1) gauge theories**, Ann. Phys. 2015
- **Zohar, Wahl, Burrello, Cirac**, **Projected Entangled Pair States with non-Abelian gauge symmetries: an SU (2) study**,
  Ann. Phys. 2016
- **Zohar, Cirac, Combining tensor networks with Monte Carlo methods for lattice gauge theories**, PRD 2018
- **Emonts, Bañuls, Cirac, Zohar, Variational Monte Carlo simulation with tensor networks of a pure gauge Z3 theory in 2+1d**, PRD 2020
- **Emonts, Kelman, Borla, Moroz, Gazit, Zohar, Finding the ground state of a lattice gauge theory with fermionic tensor networks: A 2+1-D Z2 demonstration**, PRD 2023

The LGT Hilbert Space

- **The lattice is spatial**: time is a continuous, real coordinate.

- **Matter particles** (fermions) – on the **vertices**.

- **Gauge fields** – on the lattice's **links**



- Hamiltonian picture → Hilbert space
  → Natural way to describe constraints

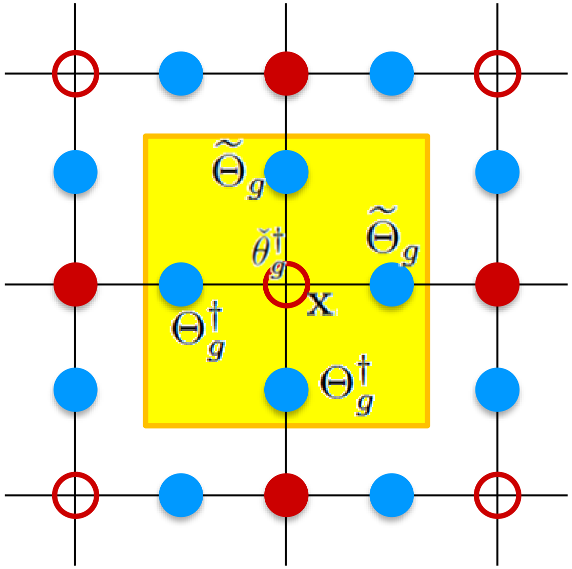
Gauge Transformations

- Act on both the matter and gauge degrees of freedom.

- **Local** : a unique transformation (depending on a unique element of the **gauge group**) may be chosen for each site

- The states are **invariant under each local transformation separately**.

$$\hat{\Theta}_g(\mathbf{x}) = \prod_{k=1\ldots d} \left(\widetilde{\Theta}_g(\mathbf{x},k)\, \Theta_g^\dagger\left(\mathbf{x}-\hat{\mathbf{k}},k\right)\right) \check{\theta}_g^\dagger(\mathbf{x})$$



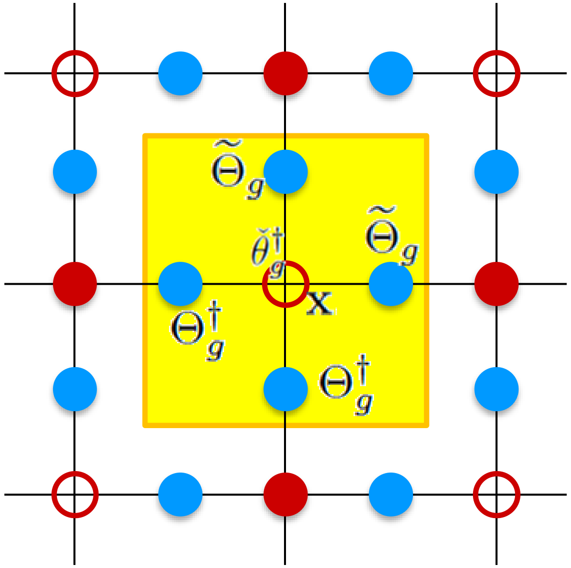
Symmetry → Conserved Charge

– **Transformation rules on the links**

$$\{|g\rangle\}_{g\in G}$$

$$\Theta_g |h\rangle = \left|hg^{-1}\right\rangle \quad \Theta_g = e^{i\phi_a(g)R_a}$$

$$\widetilde{\Theta}_g |h\rangle = \left|g^{-1}h\right\rangle \quad \widetilde{\Theta}_g = e^{i\phi_a(g)L_a}$$



– **Gauge Transformations:**

$$\hat{\Theta}_g(\mathbf{x}) = \prod_{k=1\ldots d}\left(\widetilde{\Theta}_g(\mathbf{x},k)\,\Theta_g^\dagger\left(\mathbf{x}-\hat{\mathbf{k}},k\right)\right)\breve{\theta}_g^\dagger(\mathbf{x})$$

$$\hat{\Theta}_g(\mathbf{x})|\Psi\rangle = |\Psi\rangle \quad \forall \mathbf{x}, g$$

– **Generators → Gauss law , left and right E fields:**

$$G_a(\mathbf{x}) = \sum_{k=1\ldots d}\left(L_a(\mathbf{x},k) - R_a\left(\mathbf{x}-\hat{\mathbf{k}},k\right)\right) - Q_a(\mathbf{x})$$

$$G_a(\mathbf{x})|\Psi\rangle = 0 \quad [G_a(\mathbf{x}),H] = 0 \quad \forall \mathbf{x}, a$$

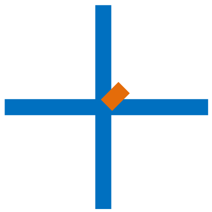
PEPS

- **Projected Entangled Pair States**: a particular tensor network construction, that
  - Allows to **encode and treat symmetries** in a very natural way.
  - Has, by construction, a **bipartite entanglement area law**, and therefore is suitable for describing "physically relevant" states.
  - Offers new approaches for the **study of phase diagrams and other properties of many body systems**.

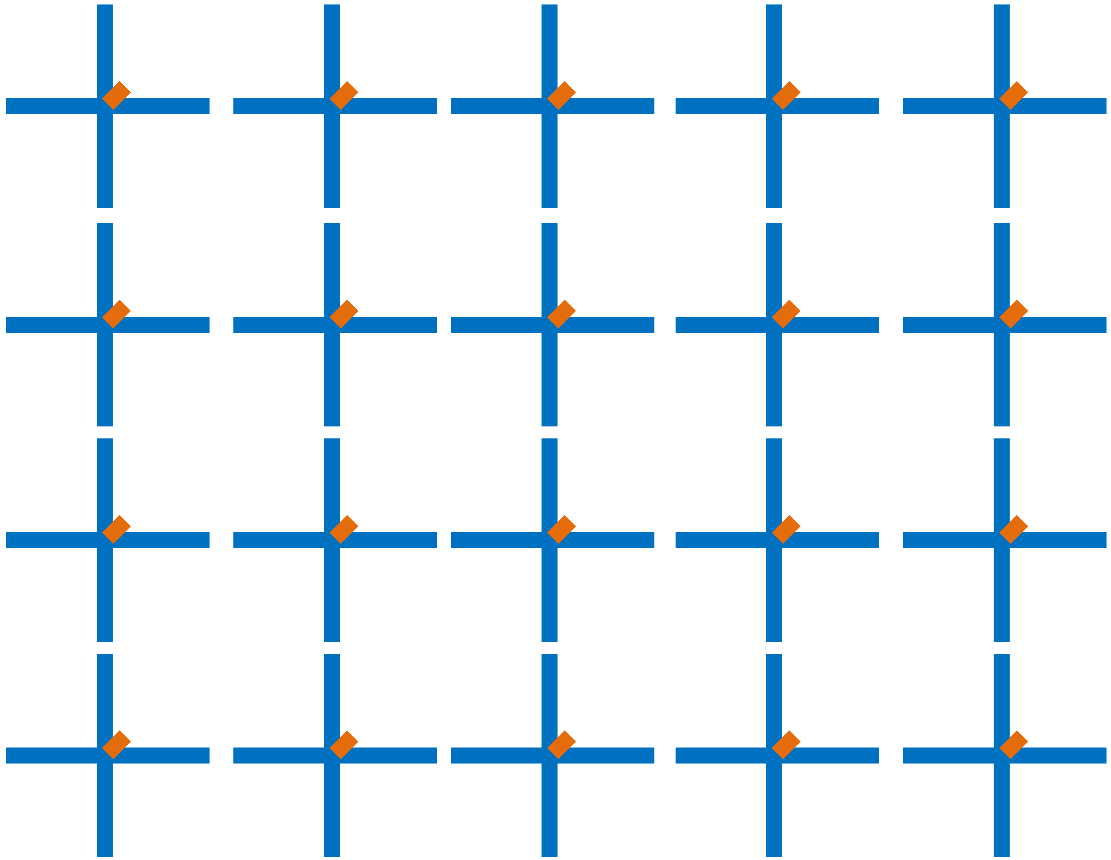
- In 1 space dimension – **MPS (Matrix Product States)**

PEPS

- Constructed out of local ingredients that include **physical** and **auxiliary** degrees of freedom.

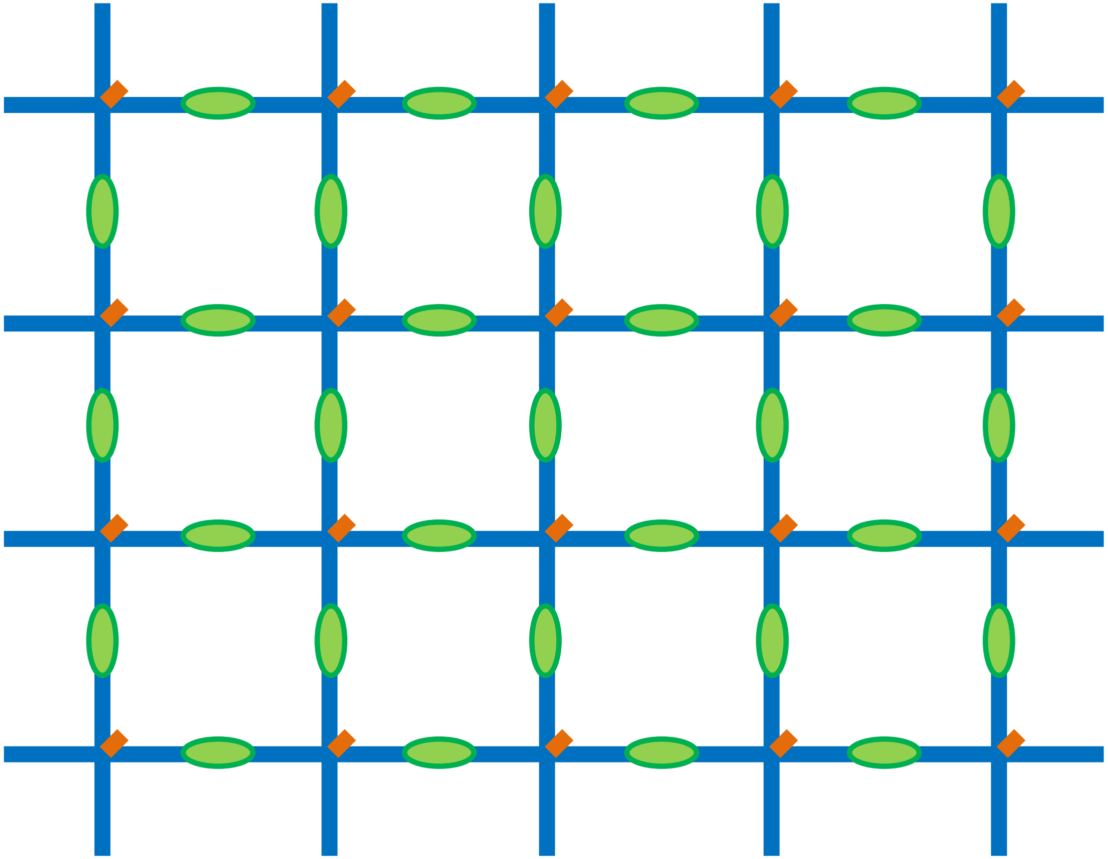


$$A(\mathbf{x})\,|\Omega\rangle$$



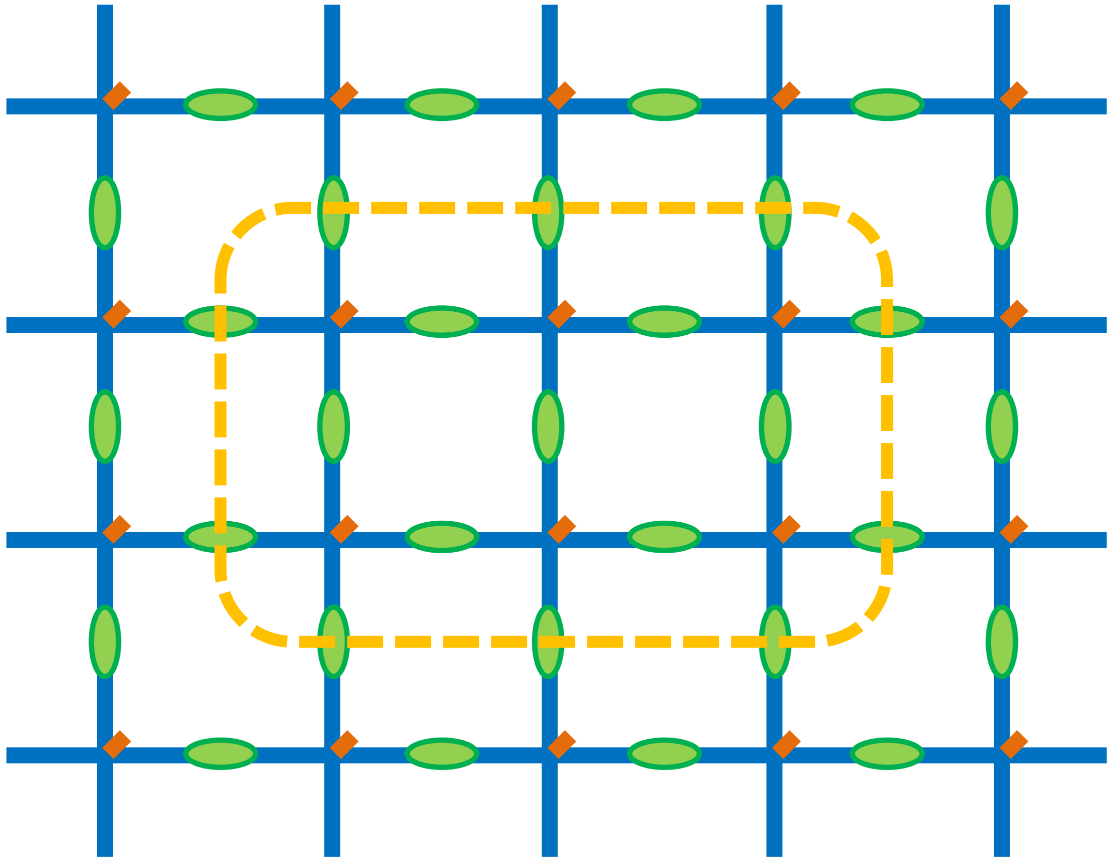
$$\prod_{\mathbf{x}} A\left(\mathbf{x}\right)|\Omega\rangle$$

- A **physical** only state is obtained out of projecting pairs of **auxiliary** degrees of freedom, on the two sides of a link, onto maximally entangled states.

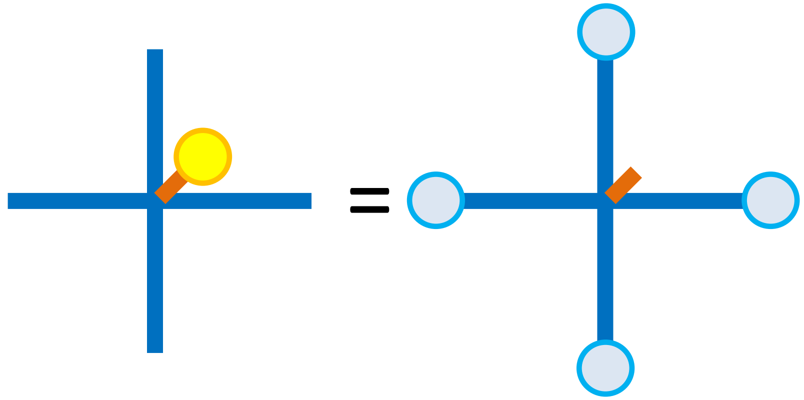


$$\left|\psi_0\right\rangle = \left\langle \Omega_v \right| \prod_{\ell} \omega_{\ell} \prod_{\mathbf{x}} A\left(\mathbf{x}\right) \left|\Omega\right\rangle$$

- An entanglement area law is satisfied by construction.

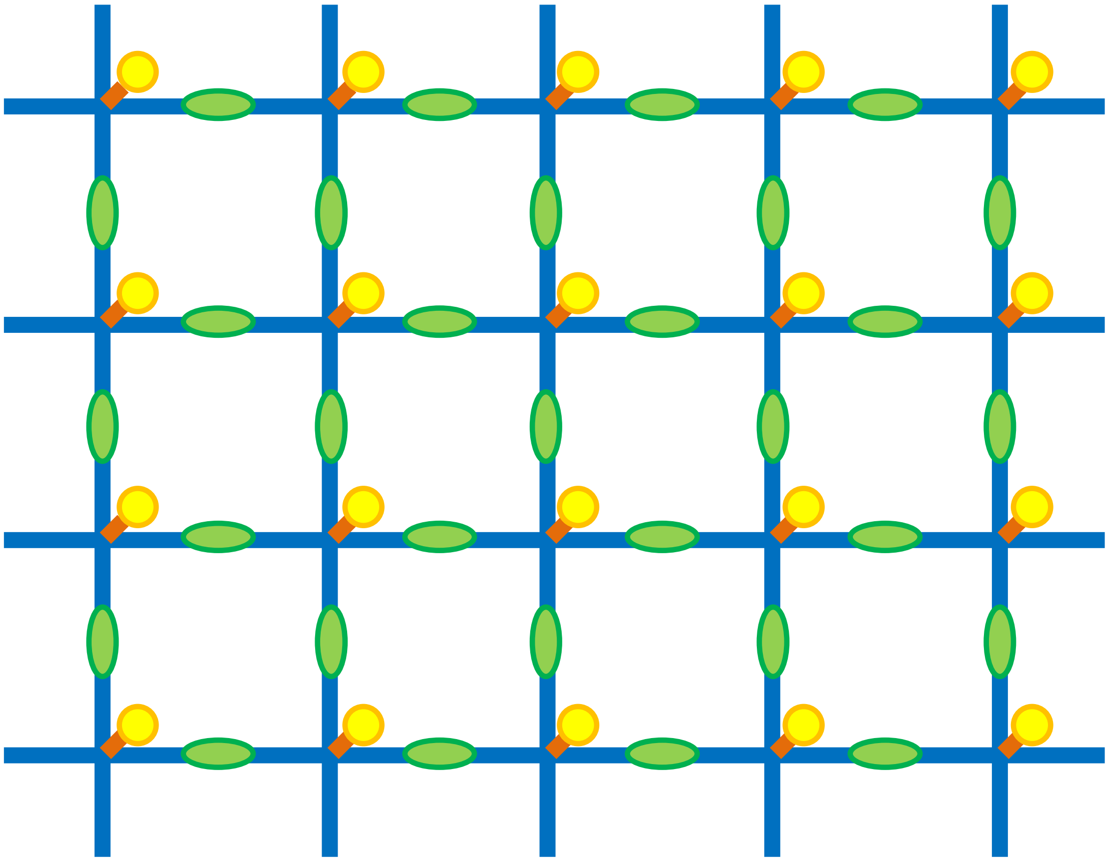


- Demanding global symmetry:
  - Acting with a group transformation on the physical degrees of freedom is equivalent to acting on the auxiliary ones.



  - Projectors are invariant under group actions from both sides.

Global Transformation: $e^{i\Lambda \sum_{\mathbf{x}} Q(\mathbf{x})} |\psi_0\rangle$



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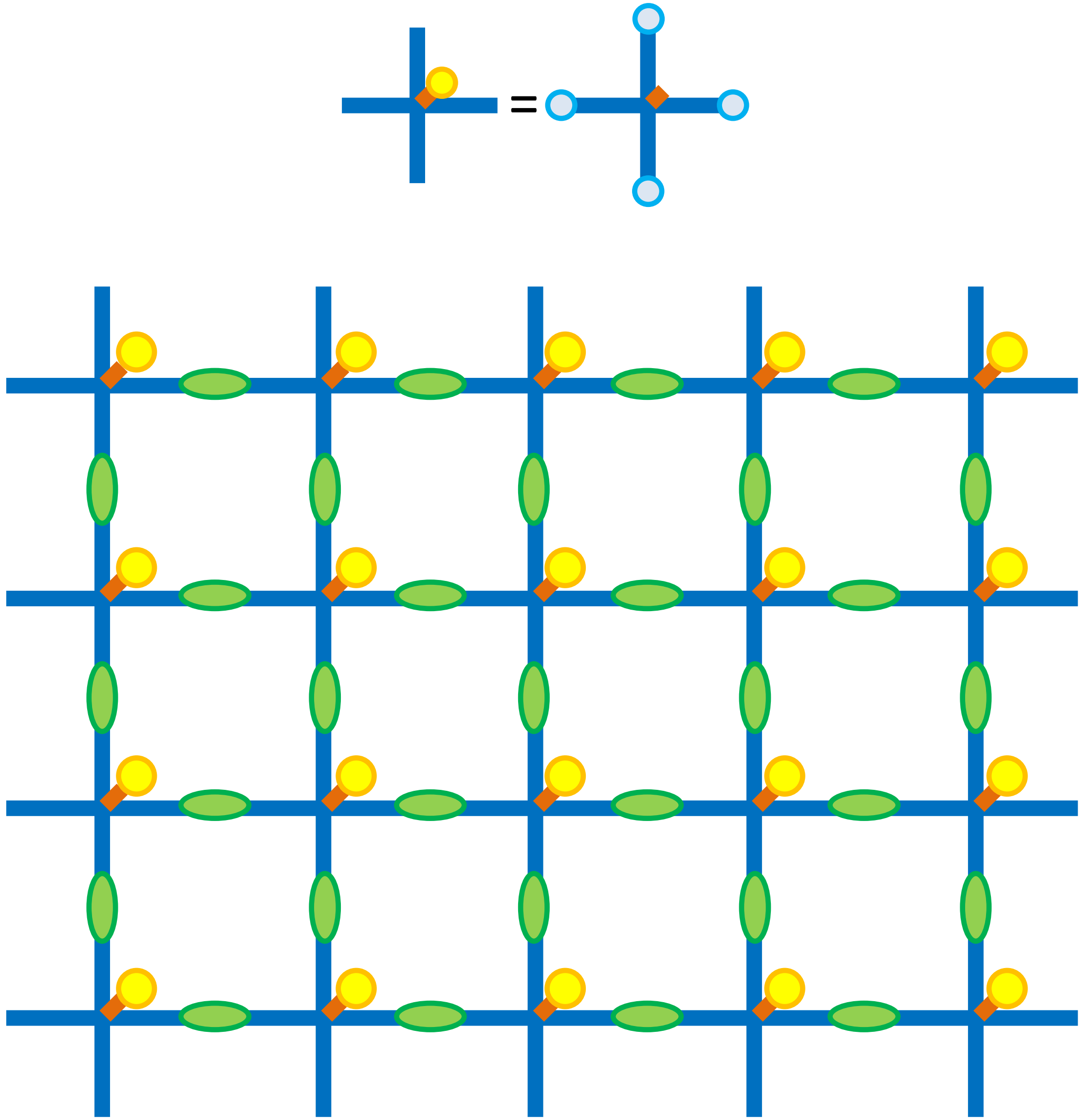

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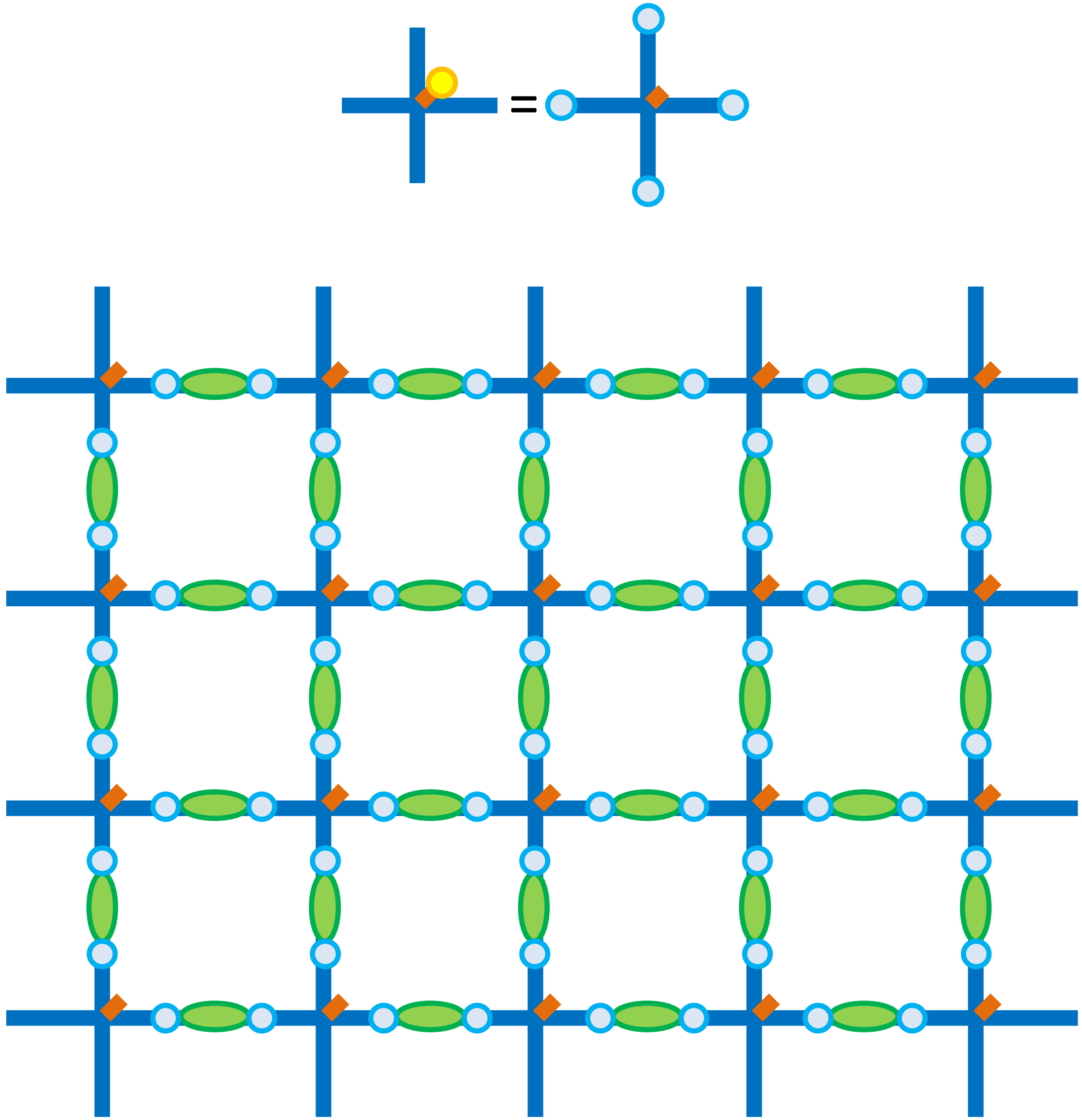

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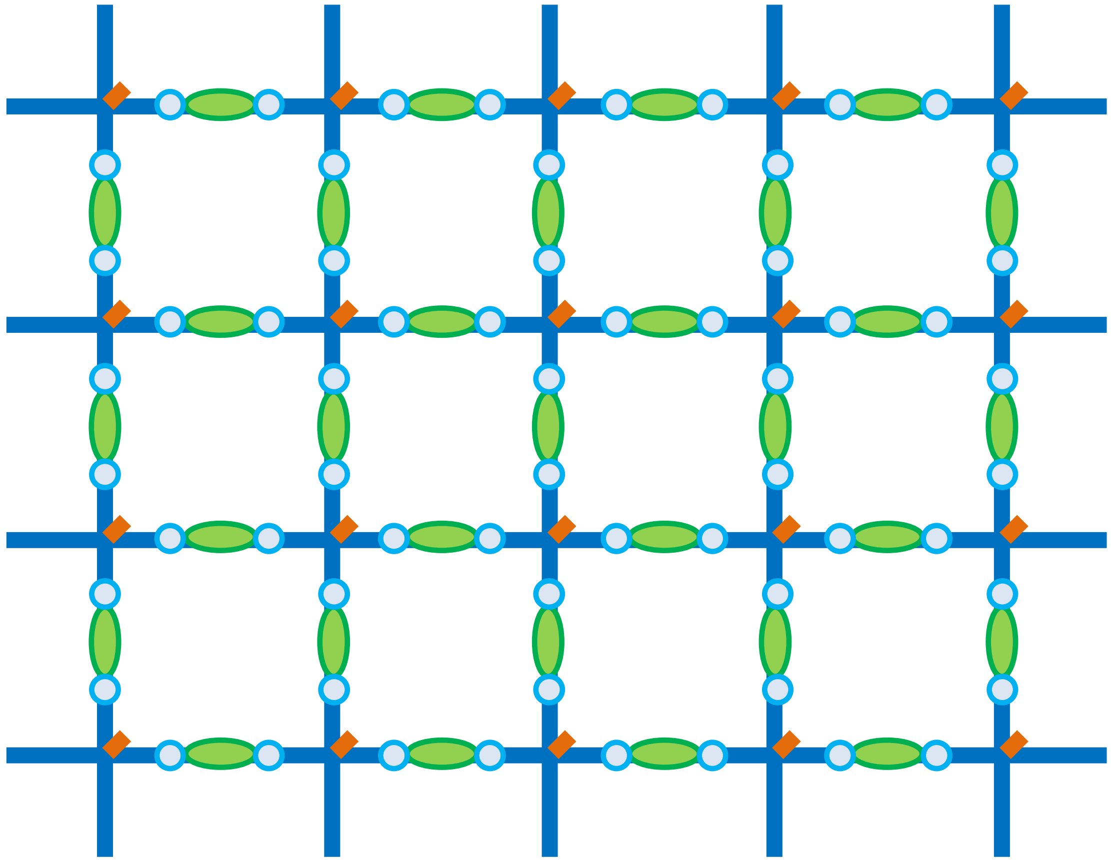

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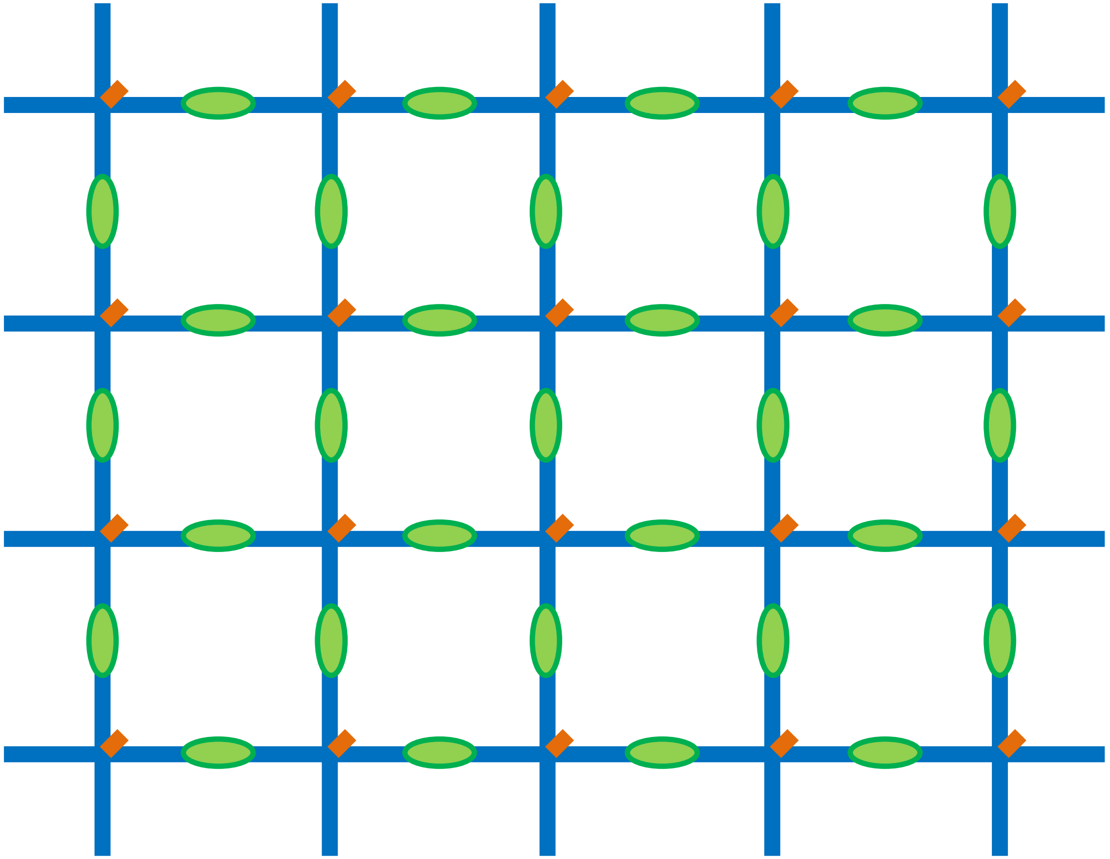

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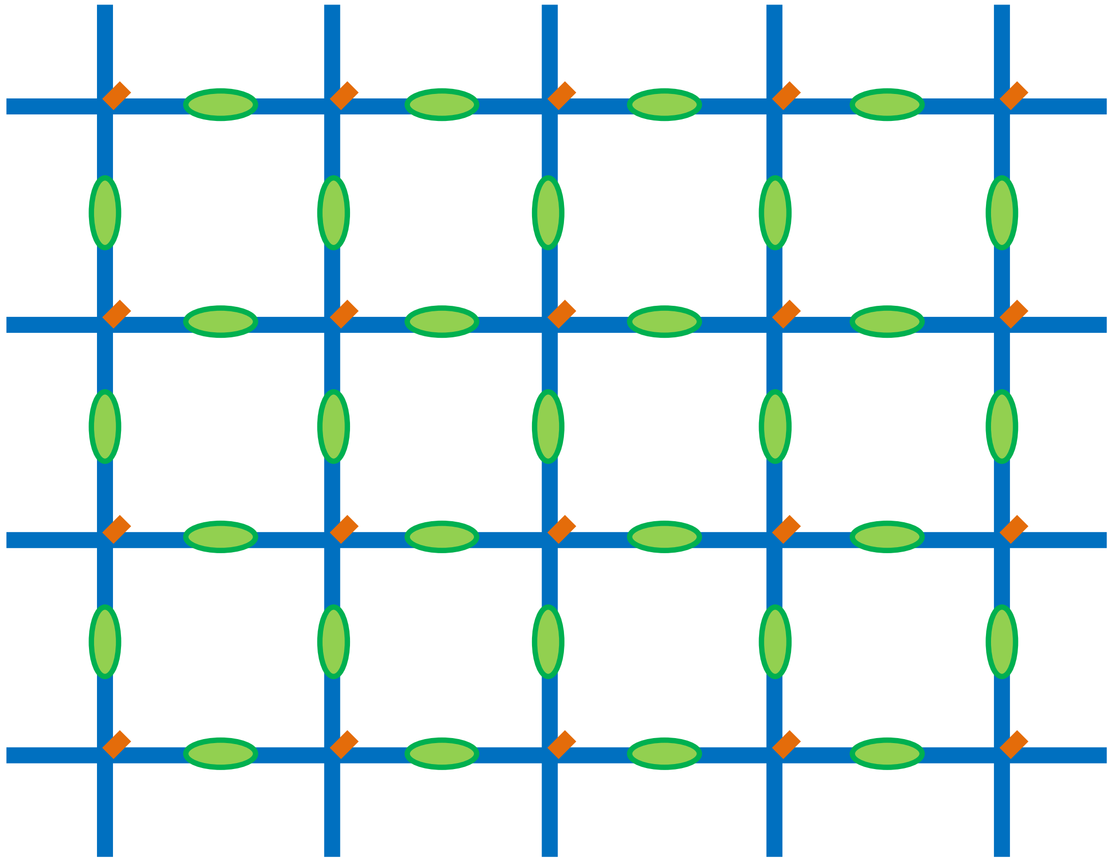
Global Symmetry: $$e^{i\Lambda\sum_{\mathbf{x}} Q(\mathbf{x})}|\psi_0\rangle = |\psi_0\rangle$$



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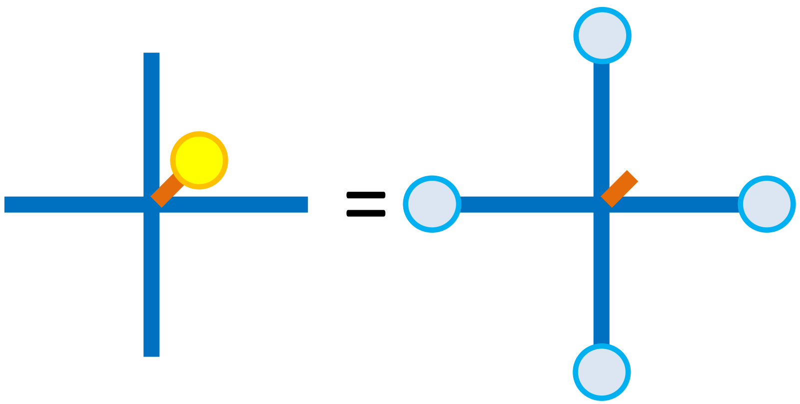
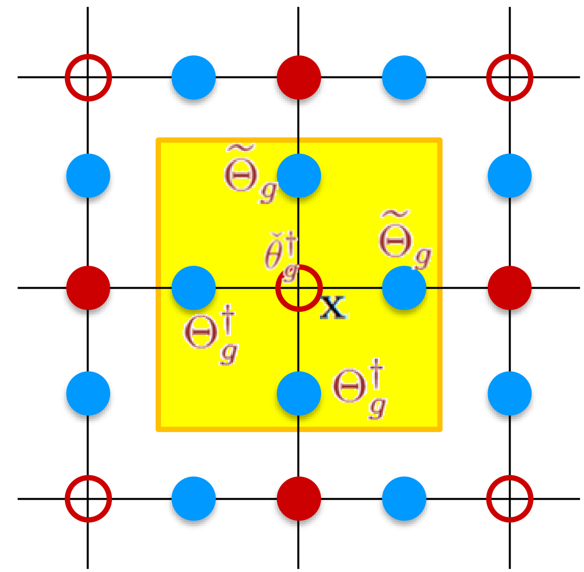
Virtual vs. Physical Gauge Invariance

Virtual - PEPS



Physical charge, but auxiliary electric fields: local symmetry exists, but it is auxiliary/virtual. The physical symmetry is global, after the bonds projection.

Physical – LGT states



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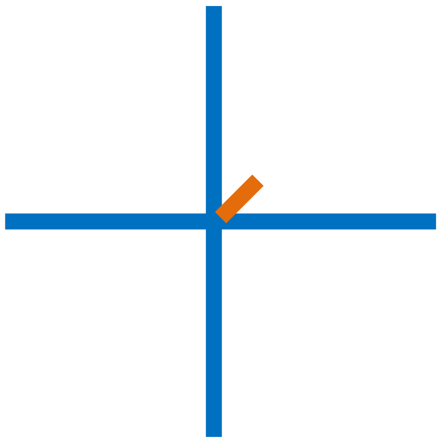
Gauging the PEPS: minimal coupling of a state

- Lift the virtual symmetry to be physical:
  Lift the global symmetry to a local one.

$$\left|\psi_0\right\rangle = \left\langle \Omega_v \right| \prod_{\ell} \omega_\ell \prod_{\mathbf{x}} A\left(\mathbf{x}\right) \left|\Omega\right\rangle$$



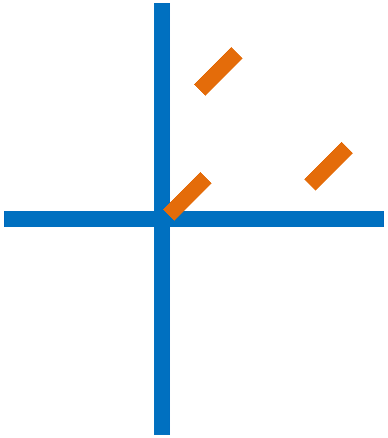
E. Zohar and M. Burrello, New J. Phys. 18 043008 (2016)
E. Zohar, J.I. Cirac, Phys. Rev. D 97, 034510 (2018)

Gauging the PEPS: minimal coupling of a state

- Lift the **virtual** symmetry to be **physical**:
  The **global** to **local**.
- Step 1: Introduce **gauge field Hilbert spaces** on the links. Add (by a tensor product) the gauge field singlet states:

$$\left|\psi_0\right\rangle = \left\langle\Omega_v\right| \prod_{\ell} \omega_\ell \prod_{\mathbf{x}} A\left(\mathbf{x}\right)\left|\Omega\right\rangle$$

$$\left\langle\Omega_v\right| \prod_{\ell} \omega_\ell \prod_{\mathbf{x}} A\left(\mathbf{x}\right) \left|0\right\rangle_{\mathbf{x},1} \left|0\right\rangle_{\mathbf{x},2} \left|\Omega\right\rangle$$



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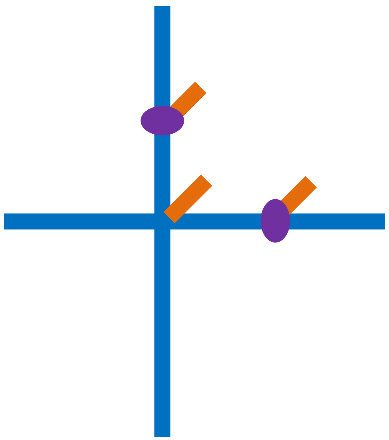
Gauging the PEPS: minimal coupling of a state

- Lift the virtual symmetry to be physical:
  The global to local.
- Step 2: Entangle the auxiliary degrees on the outgoing links with the gauge fields, by a unitary gauging transformation (map the auxiliary electric field information to the physical one).

$$|\psi_0\rangle = \langle\Omega_v| \prod_{\ell} \omega_\ell \prod_{\mathbf{x}} A(\mathbf{x}) |\Omega\rangle$$

$$\downarrow$$

$$\langle\Omega_v| \prod_{\ell} \omega_\ell \prod_{\mathbf{x}} A(\mathbf{x}) |0\rangle_{\mathbf{x},1} |0\rangle_{\mathbf{x},2} |\Omega\rangle$$



$$\downarrow$$

$$|\psi\rangle = \langle\Omega_v| \prod_{\ell} \omega_\ell \prod_{\mathbf{x}} \mathcal{U}_G(\mathbf{x},1)\, \mathcal{U}_G(\mathbf{x},2) A(\mathbf{x}) |0\rangle_{\mathbf{x},1} |0\rangle_{\mathbf{x},2} |\Omega\rangle$$

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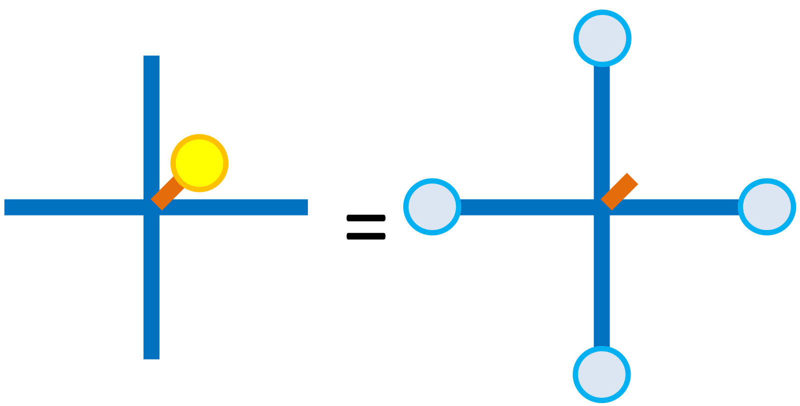
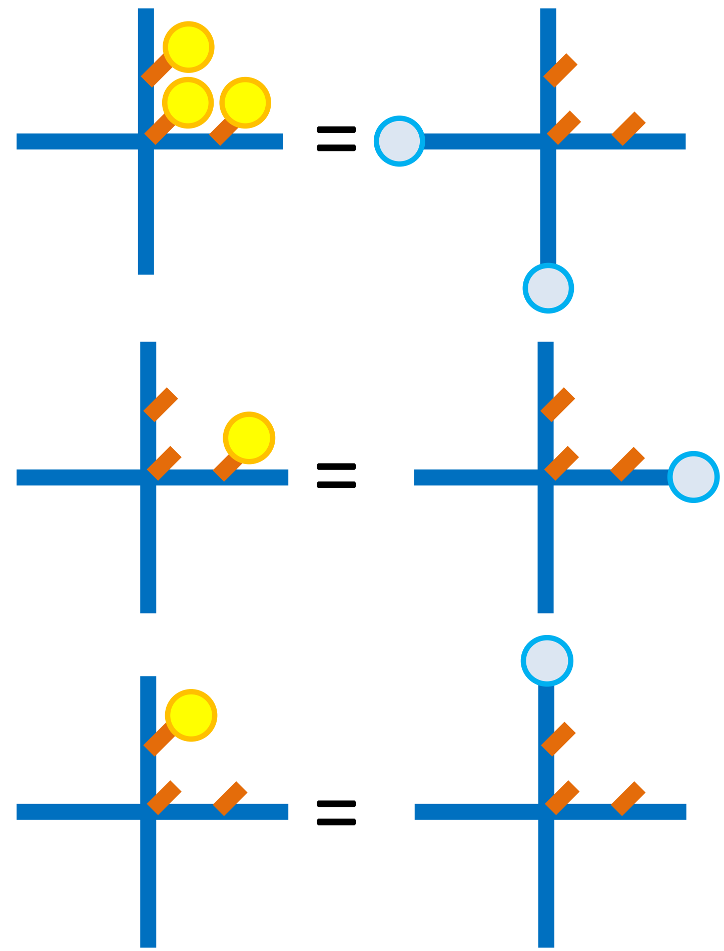
Gauging the PEPS: minimal coupling of a state

Building block of a globally invariant PEPS



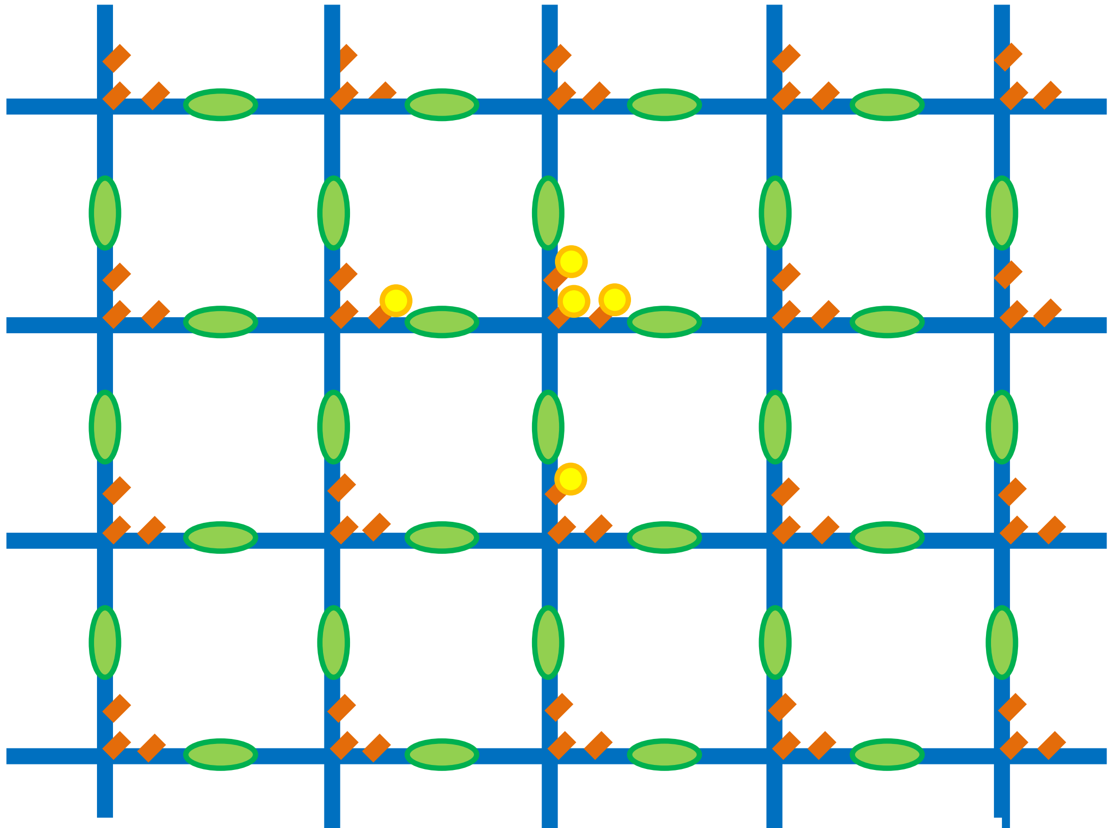
Gauging Transformation

Building block of a globally invariant PEPS (gluing together the matter and gauge field tensors)



E. Zohar and M. Burrello, New J. Phys. 18 043008 (2016)
E. Zohar, J.I. Cirac, Phys. Rev. D 97, 034510 (2018)

Local Transformation: $e^{i\Lambda\mathcal{G}(\mathbf{x}_0)}\left|\psi\right\rangle$



$$\left|\psi\right\rangle = \left\langle\Omega_v\right|\prod_{\ell}\omega_{\ell}\prod_{\mathbf{x}}\mathcal{U}_G\left(\mathbf{x},1\right)\mathcal{U}_G\left(\mathbf{x},2\right)A\left(\mathbf{x}\right)\left|0\right\rangle_{\mathbf{x},1}\left|0\right\rangle_{\mathbf{x},2}\left|\Omega\right\rangle$$

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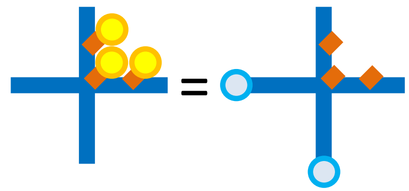

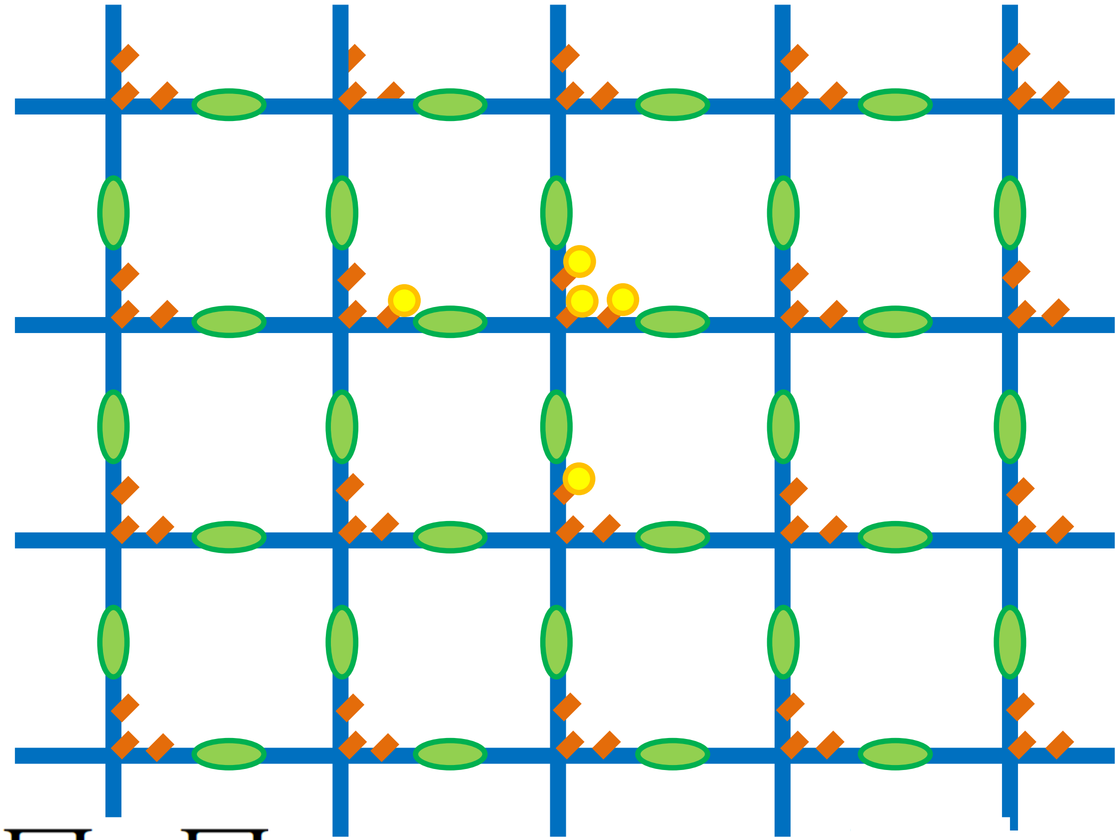

$$|\psi\rangle = \langle \Omega_v | \prod_{\ell} \omega_\ell \prod_{\mathbf{x}} \mathcal{U}_G(\mathbf{x},1)\, \mathcal{U}_G(\mathbf{x},2) A(\mathbf{x}) |0\rangle_{\mathbf{x},1} |0\rangle_{\mathbf{x},2} |\Omega\rangle$$

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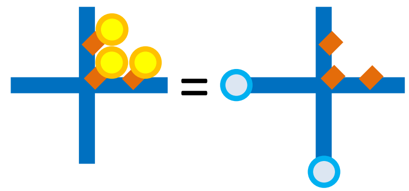

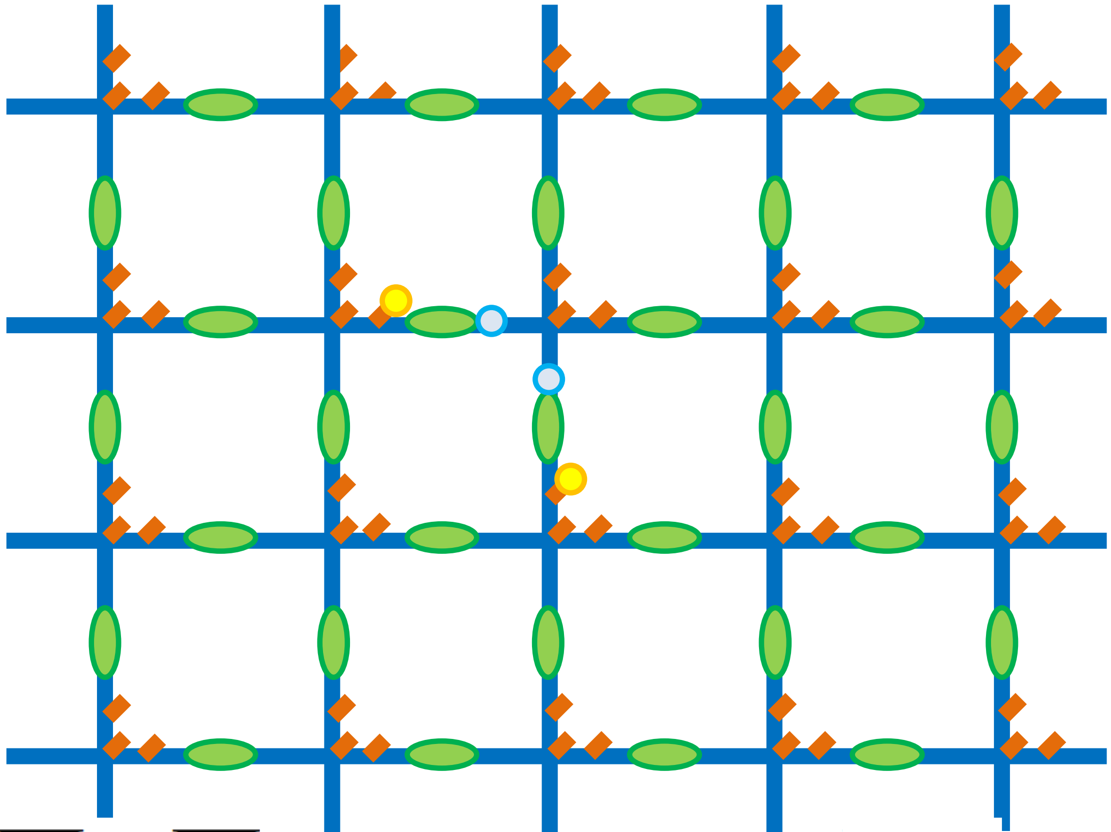

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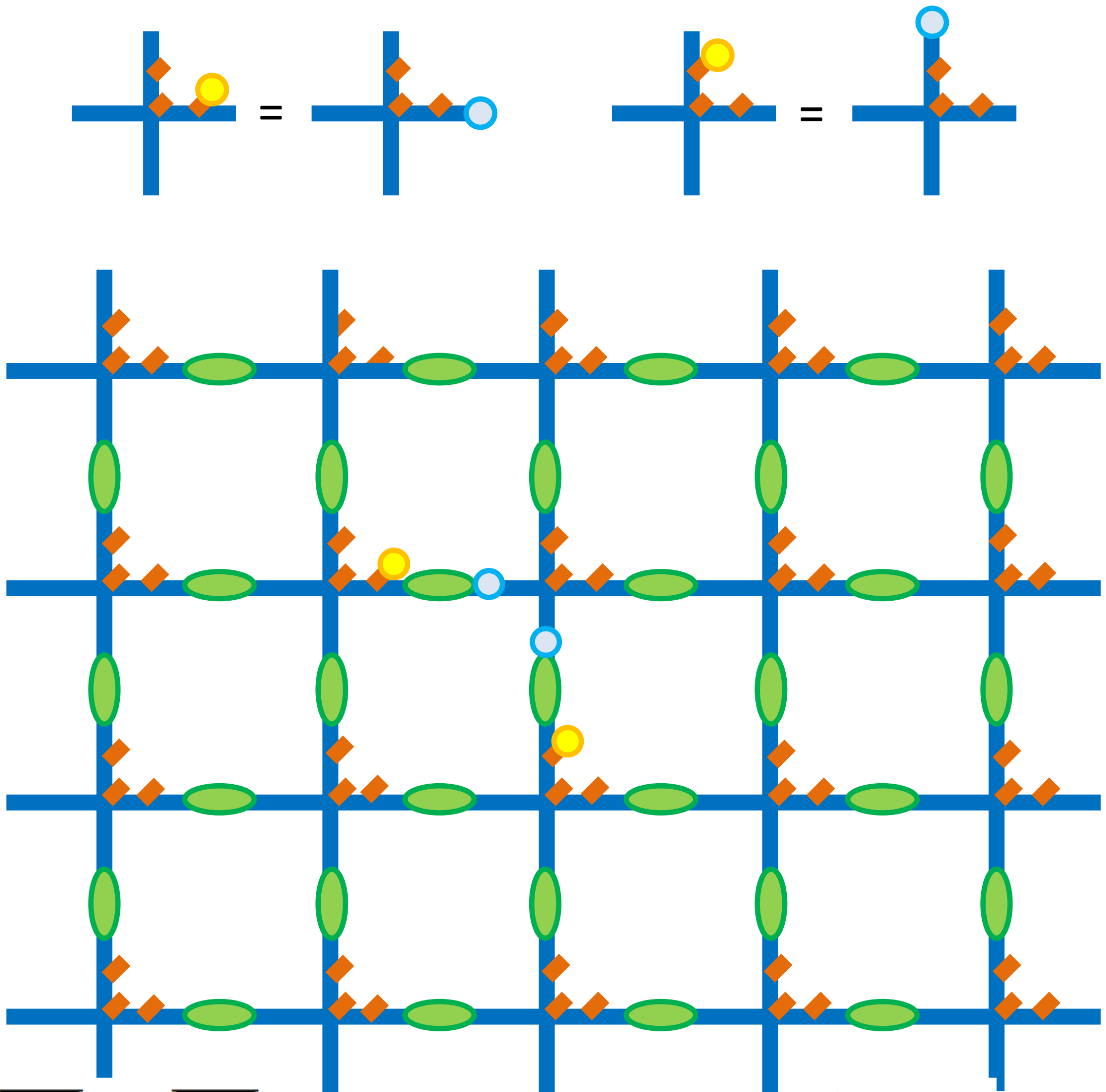

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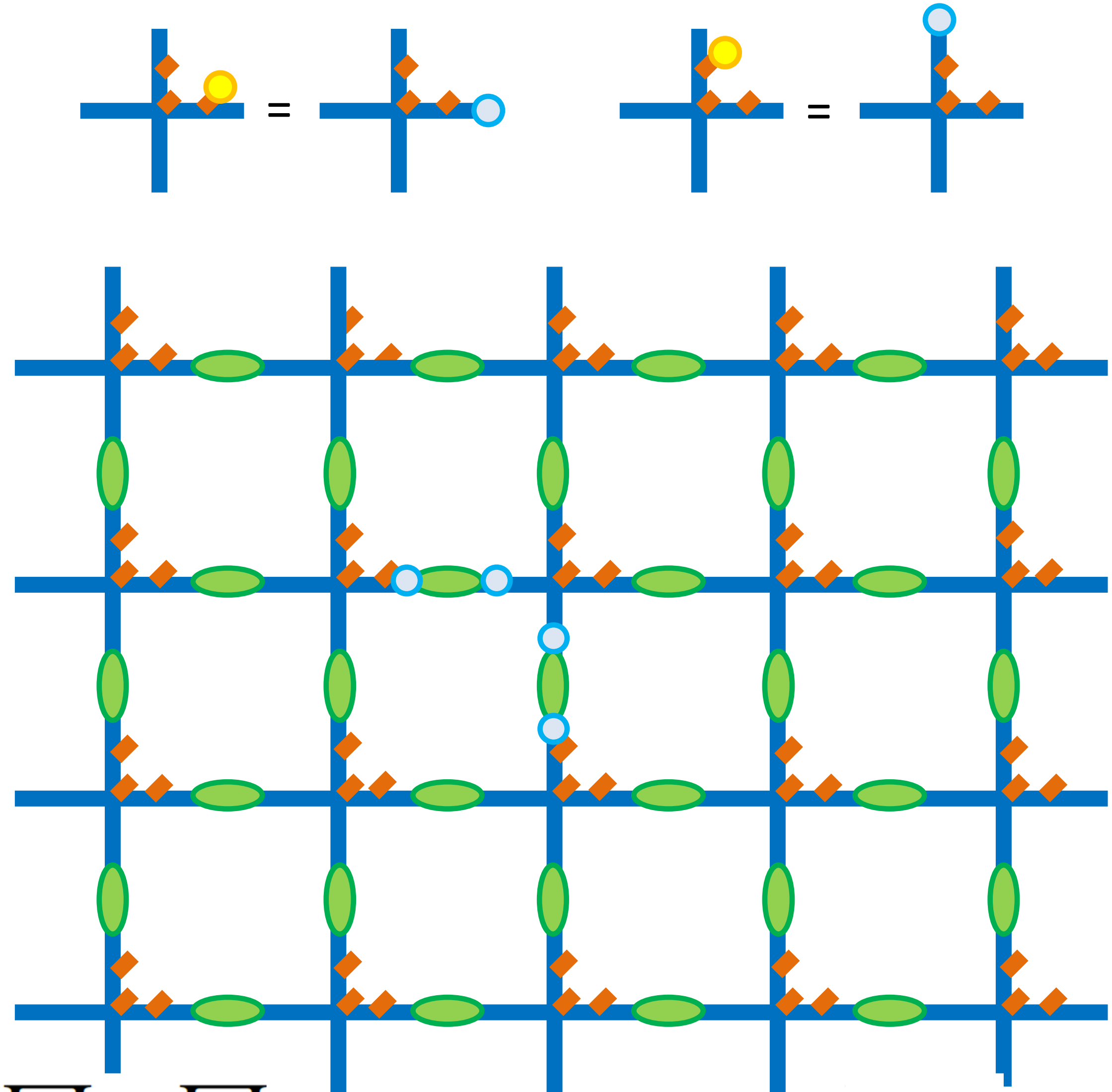

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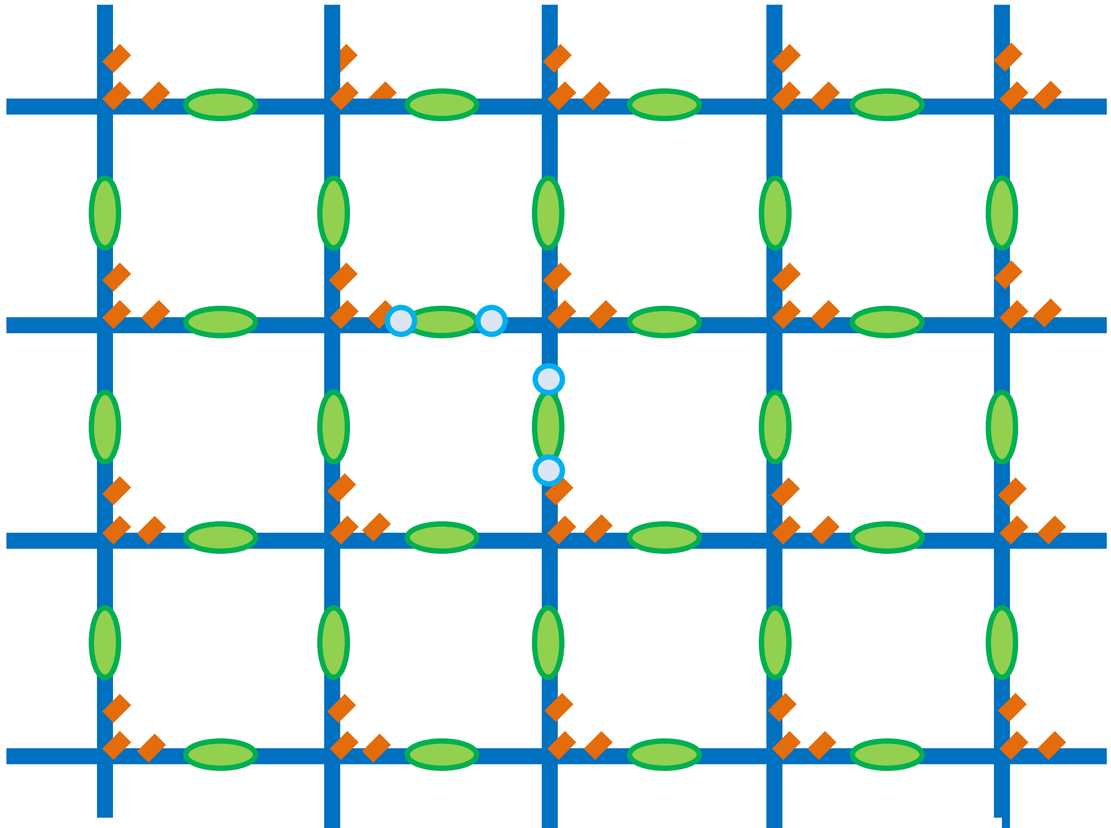

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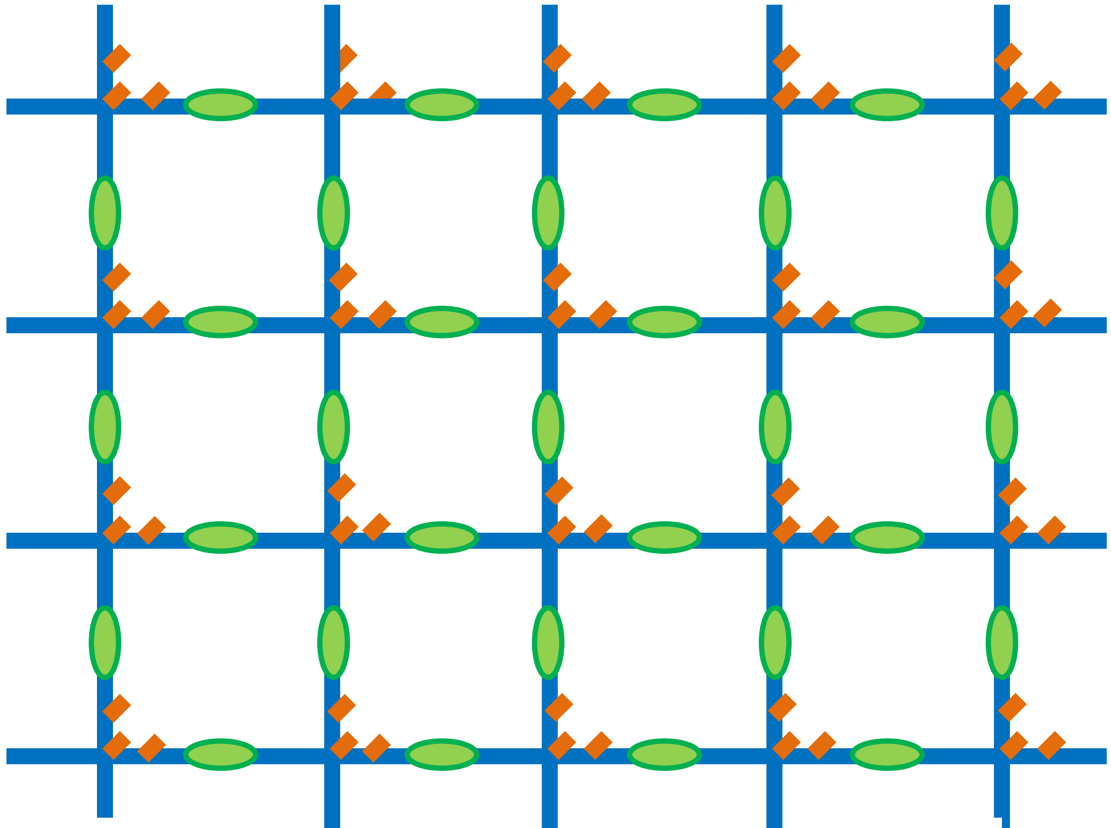
Local Symmetry: $$e^{i\Lambda\mathcal{G}(\mathbf{x}_0)}\left|\psi\right\rangle = \left|\psi\right\rangle$$



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Locally gauge invariant fermionic PEPS

- We We wish to describe PEPS of fermionic matter coupled to dynamical gauge fields.

- Starting point – **Gaussian fermionic PEPS** with a global symmetry.
  - **Gaussian states** – ground states of quadratic Hamiltonians, completely described by their covariance matrix. Very easy to handle analytically with the use of the Gaussian formalism.
  - **Fermionic PEPS** – defined with fermionic creation operators acting on the Fock vacuum. Easy to parameterize if they are Gaussian (Kraus, Schuch, Verstraete, Cirac, PRA 2011)

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- **Start with these, then make the symmetry local and add the gauge field.**

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- **Start with these, then make the symmetry local and add the gauge field.** Similar to **minimal coupling**: **Gauge a free matter state → obtain an interacting matter-gauge field state** without introducing further parameters.

E. Zohar, M. Burrello, T.B. Wahl, and J.I. Cirac, Ann. Phys. 363, 385-439 (2015)
E. Zohar, T.B. Wahl, M. Burrello, and J.I. Cirac, Ann. Phys. 374, 84-137 (2016)
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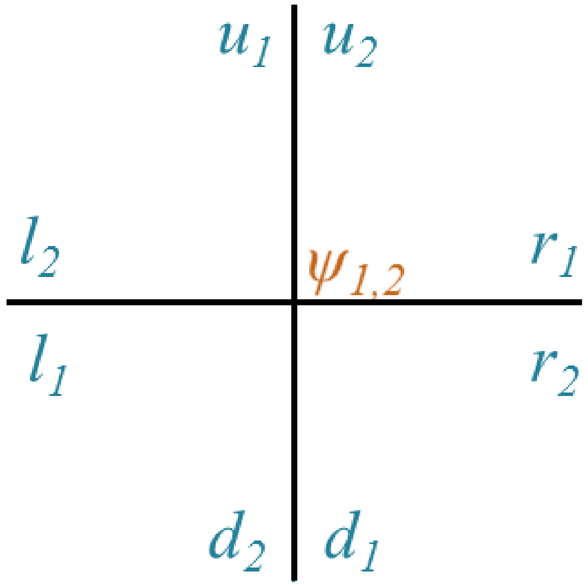
The PEPS ingredients



- At each vertex –
  - Physical fermions
    - For example,
      - 1 For $U(1)$
      - 2 For $SU(2)$, in the fundamental rep.
  - Virtual fermions on the legs (e.g. 2 per leg, 8 in total)
- The local **Gaussian state** involving both physical and virtual modes is created from the Fock vacuum with

$$A = \exp\left(\sum_{ij} T_{ij}\alpha_i^\dagger \alpha_j^\dagger\right)$$

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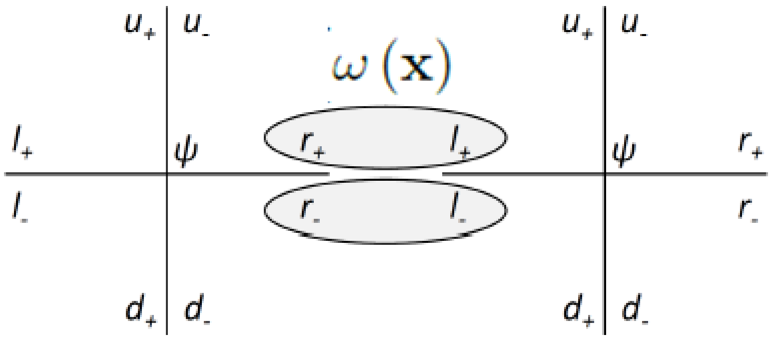
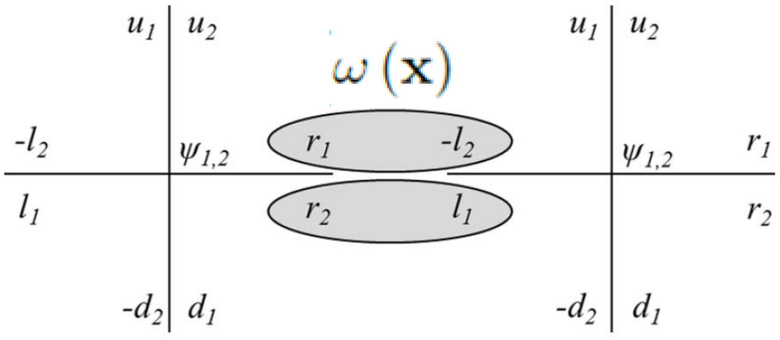
The PEPS

$$A = \exp\left(\sum_{ij} \hat{T}_{ij} \alpha_i^\dagger \alpha_j^\dagger\right)$$



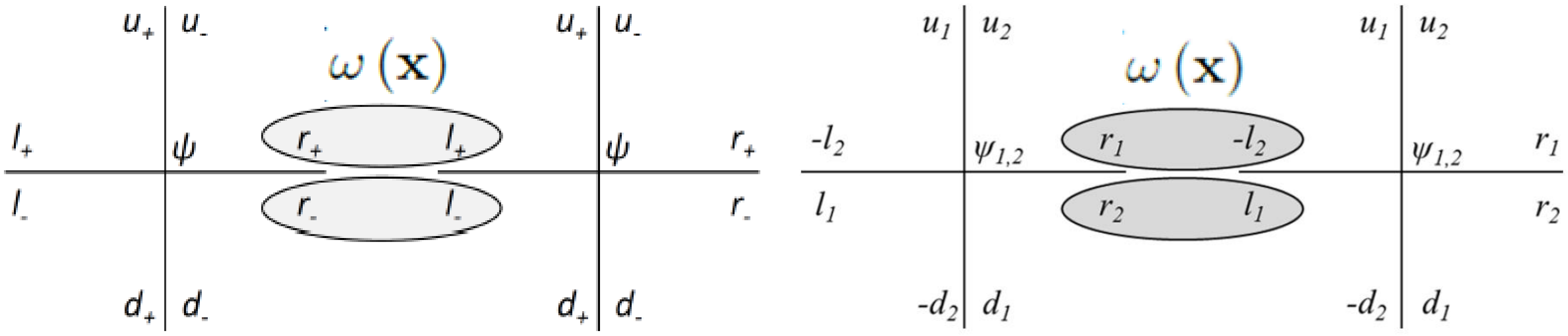
$U(1)$



$SU(2)$

The PEPS

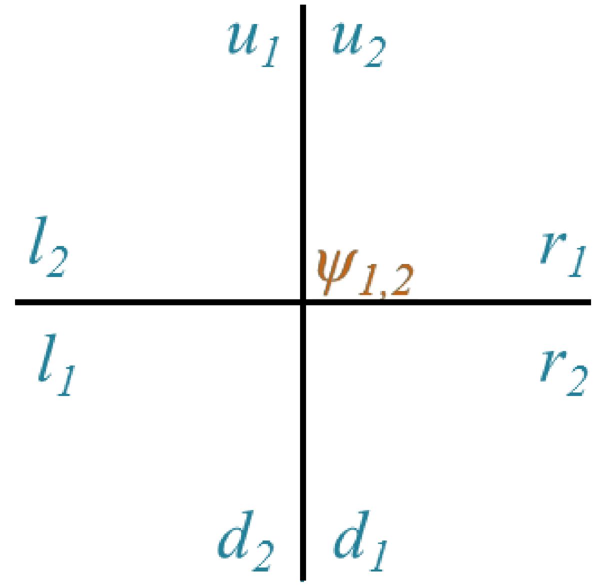
$$A = \exp\left(\sum_{ij} \hat{T}_{ij} \alpha_i^\dagger \alpha_j^\dagger\right)$$



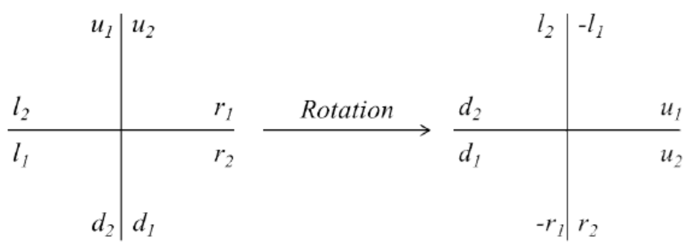
$U(1)$ $SU(2)$

$$|\psi_0\rangle = \langle \Omega_v| \prod_{\ell} \omega_\ell \prod_{\mathbf{x}} A(\mathbf{x}) |\Omega\rangle$$

Parameterizing: Demanding global symmetries



- Translation invariance – same *T* everywhere.
- Rotation invariance: introduces some relations between the *T* matrix elements.



$$A = \exp\left(\sum_{ij} T_{ij} \alpha_i^\dagger \alpha_j^\dagger\right)$$

- Physical fermionic number conservation (global $U(1)$, relevant for the $SU(2)$ case).

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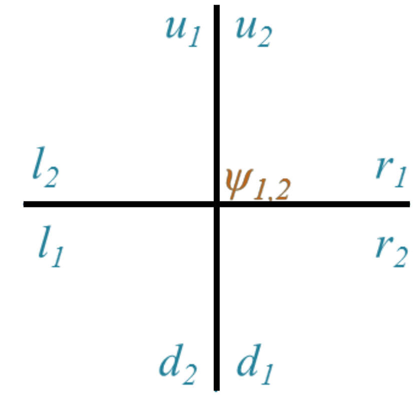


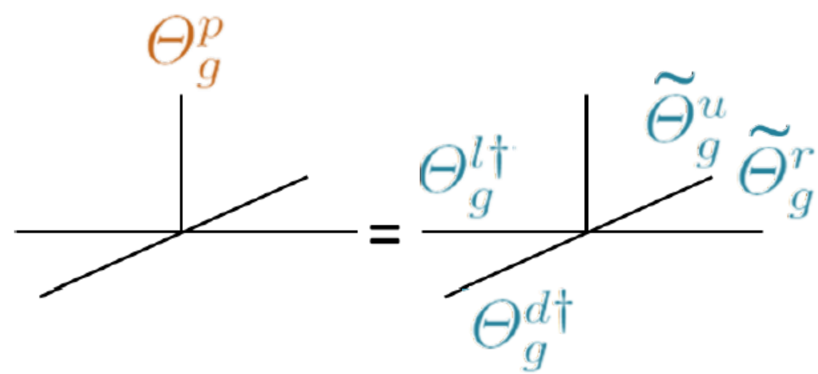
- Global (or virtual gauge) invariance: under transformations of the form

$$\hat{\Theta}_g = \widetilde{\Theta}^r_g \widetilde{\Theta}^u_g \Theta^{l\dagger}_g \Theta^{d\dagger}_g \Theta^{\dagger p}_g = e^{i\phi_a G^a}$$

- generated by bilinears of fermionic operators.

$$A = \exp\left(\sum_{ij} T_{ij} \alpha_i^\dagger \alpha_j^\dagger\right)$$

$$\hat{\Theta}_g A \hat{\Theta}_g^\dagger = A \quad \Longleftrightarrow$$



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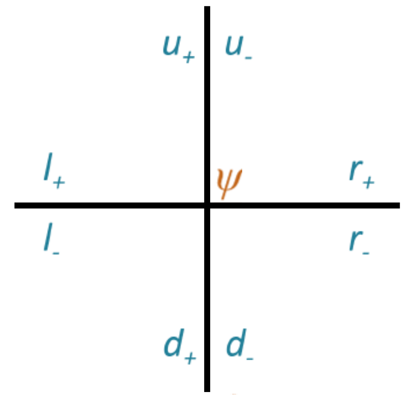
The $U(1)$ case



- Define a "virtual Gauss law"

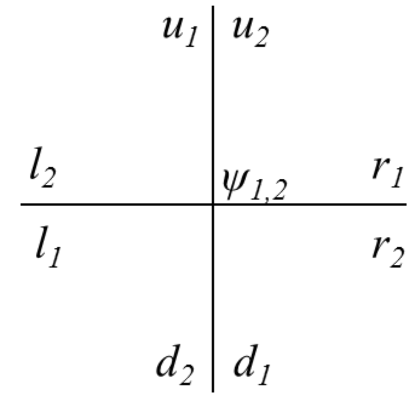
$$G_0 = \mathrm{div}E - Q = E_r + E_u - E_l - E_d - Q =$$
$$r_+^\dagger r_+ - r_-^\dagger r_- + u_+^\dagger u_+ - u_-^\dagger u_- - l_+^\dagger l_+ + l_-^\dagger l_- - d_+^\dagger d_+ + d_-^\dagger d_- - s_{\mathbf{x}} \psi^\dagger \psi$$

- Electric field truncation imposed by the fermions
- Negative virtual modes: $\{a_i\}_{i=1}^4 = \{l_+, r_-, u_-, d_+\}$
- Positive virtual modes: $\{b_i\}_{i=1}^4 = \{l_-, r_+, u_+, d_-\}$
- Physical mode - staggered a_0 or b_0

$$A(\mathbf{x}) = \begin{cases} \exp\left(\sum\limits_{ij} T_{ij} a_i^\dagger b_j^\dagger\right), & \mathbf{x} \textit{ even;} \\ \exp\left(\sum\limits_{ij} T_{ij} b_i^\dagger a_j^\dagger\right), & \mathbf{x} \textit{ odd.} \end{cases}$$

$$e^{i\phi G_0} A e^{-i\phi G_0} = A$$

The *SU*(2) case



- Global group invariance:
  under transformations of the form

$$\hat{\Theta}_g = \widetilde{\Theta}^r_g \widetilde{\Theta}^u_g \Theta^{l\dagger}_g \Theta^{d\dagger}_g \Theta^{\dagger p}_g = e^{i\phi_a G^a}$$

- generated by bilinears of fermionic operators.

$$A = \exp\left(\sum_{ij} T_{ij} \alpha_i^\dagger \alpha_j^\dagger\right)$$

$$G^a = L^a(r) + L^a(U) - R^a(l) - R^a(d) - R^a(\psi)$$

$$R^a(\alpha) = \frac{1}{2}\alpha^\dagger_m \sigma^a_{mn} \alpha_n \qquad \left[R^a, R^b\right] = i\epsilon^{abc} R^c$$

$$L^a(\alpha) = \frac{1}{2}\alpha^\dagger_m \sigma^a_{nm} \alpha_n \qquad \left[L^a, L^b\right] = -i\epsilon^{abc} L^c$$

All the symmetries

- Translation invariance
- Rotation (lattice) invariance
- **Global** *U*(1) **or** SU(2) invariance

***U*(1)**

$$T=\begin{pmatrix} t & \eta_p^2 t & \eta_p t & \eta_p^3 t \\ 0 & y & z/\sqrt{2} & z/\sqrt{2} \\ -y & 0 & -z/\sqrt{2} & z/\sqrt{2} \\ -z/\sqrt{2} & z/\sqrt{2} & 0 & y \\ -z/\sqrt{2} & -z/\sqrt{2} & -y & 0 \end{pmatrix}$$

$$z, y \in \mathbb{C} \text{ and } t > 0,\ \eta_p = e^{i\pi/4}$$

***SU*(2)**

$$T=\begin{pmatrix} 0 & t & \eta_p^{-1} t & -\eta_p^{-2} t & -\eta_p^{-3} t \\ t & x & z/\sqrt{2} & 0 & z/\sqrt{2} \\ \eta_p^{-1} t & z/\sqrt{2} & x & -z/\sqrt{2} & 0 \\ \eta_p^{-2} t & 0 & z/\sqrt{2} & -x & -z/\sqrt{2} \\ \eta_p^{-3} t & -z/\sqrt{2} & 0 & -z/\sqrt{2} & -x \end{pmatrix}$$

$$t > 0, z \geq 0, x \in \mathbb{C} \text{ and } \eta_p = e^{i\frac{\pi}{4}}$$

E. Zohar, M. Burrello, T.B. Wahl, and J.I. Cirac, Ann. Phys. 363, 385-439 (2015)

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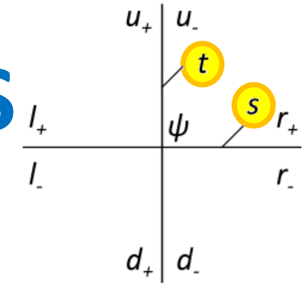
Gauging the Gaussian fermionic PEPS



- The state is not Gaussian anymore, but rather a "generalized Gaussian state"
- Gaussian mapping and formalism are generally not valid, but **the parameterization of the original states "survives"**:
  - Translation invariance → Charge conjugation
  - Rotation invariance → Rotation invariance
  - Global invariance → Local gauge invariance:
    - "Virtual Gauss law" → Physical Gauss laws

$$\widetilde{\Theta}_g^r \widetilde{\Theta}_g^u \Theta_g^{l\dagger} \Theta_g^{d\dagger} \Theta_g^{\dagger p}$$

$$\hat{\Theta}_g(\mathbf{x}) = \prod_{k=1\dots d} \left(\widetilde{\Theta}_g(\mathbf{x},k)\,\Theta_g^{\dagger}\left(\mathbf{x}-\hat{\mathbf{k}},k\right)\right)\breve{\theta}_g^{\dagger}(\mathbf{x})$$

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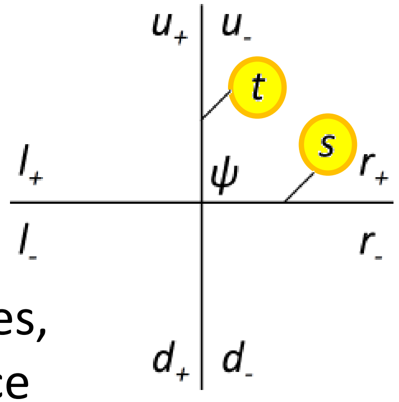
$$\hat{\Theta}_g(\mathbf{x})\left|\Psi\right\rangle = \left|\Psi\right\rangle \quad \forall \mathbf{x}, g$$

Gauging the PEPS



- The physical electric fields are identified with the virtual ones, which are truncated since we have finite virtual Hilbert space on the links (the fermions force a truncation)

- In the $U(1)$ minimal example, the electric field takes the values 0,±1 - instead of all integers (3d Hilbert space on each link).

- In the $SU(2)$ minimal example, the link Hilbert space is 5 dimensional, including the singlet state and four states of j=1/2 (right and left).

- One may use more virtual modes and gauge them with respect to the same physical gauge fields, to increase the truncation as well as the number of (variational) parameters

E. Zohar, E., M. Burello, Phys. Rev. D. 91, 054506 (2015)
E. Zohar, M. Burrello, T.B. Wahl, and J.I. Cirac, Ann. Phys. 363, 385-439 (2015)
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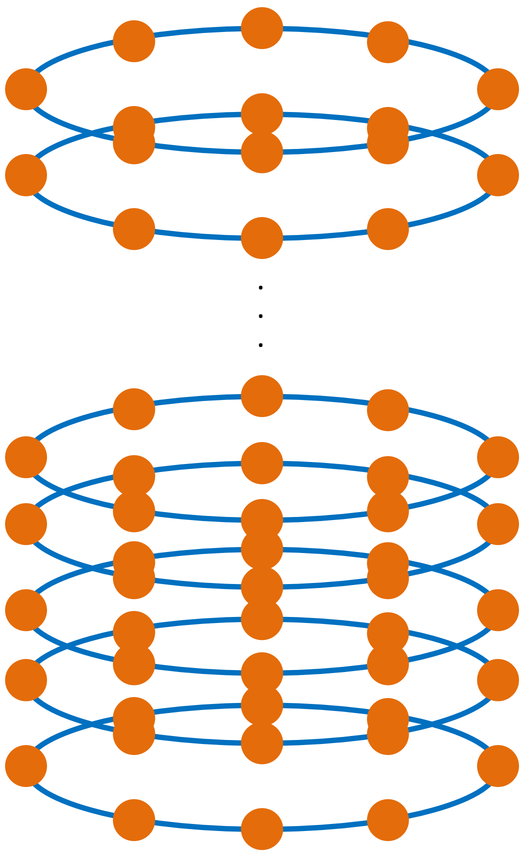
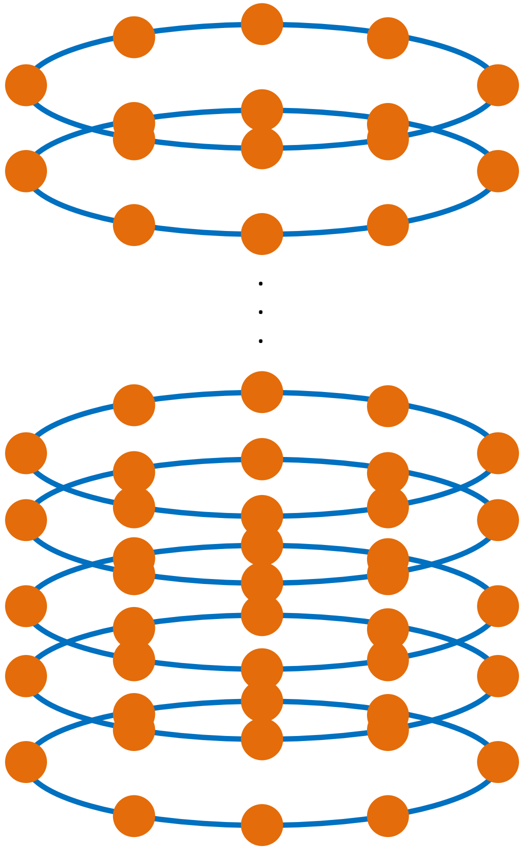
Phase Diagram of the PEPS

- We use a cylindrical geometry.



Phase Diagram of the PEPS

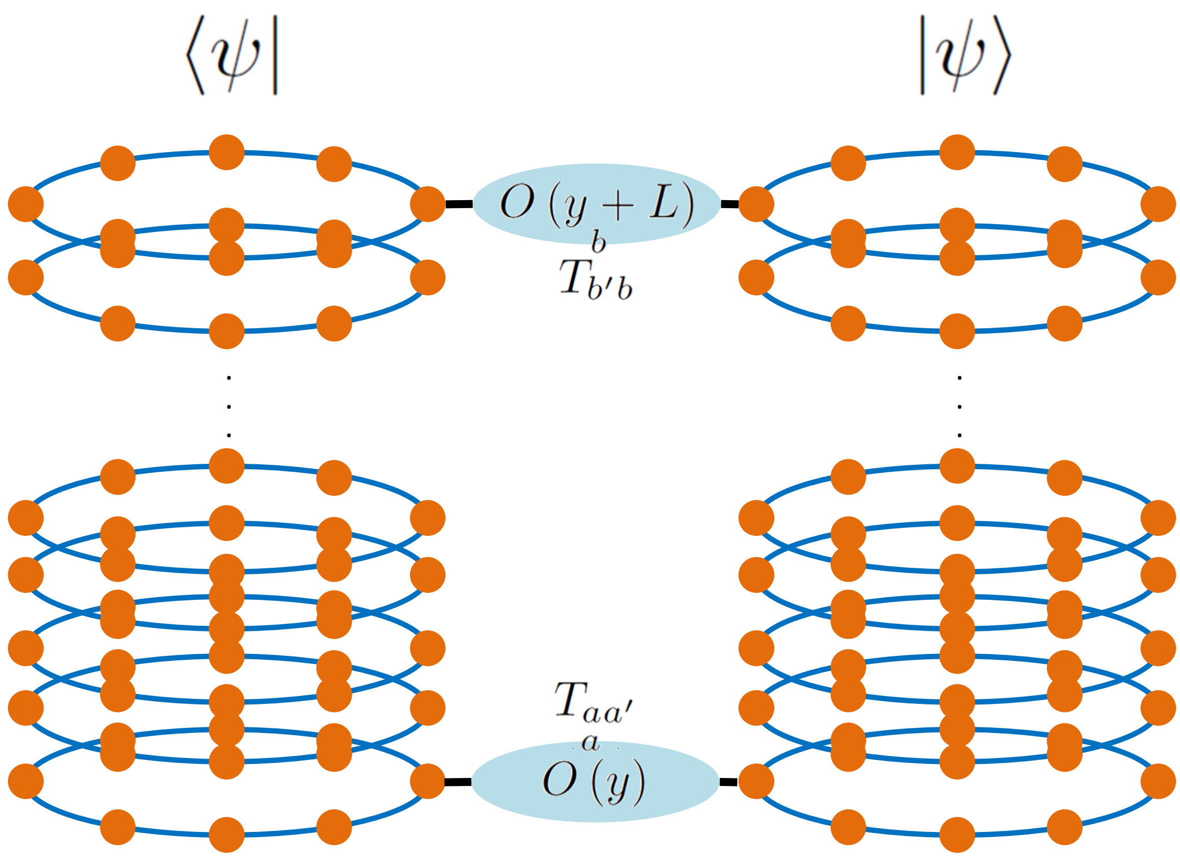
- The system may be converted effectively to 1d by contracting the rows.



Phase Diagram of the PEPS

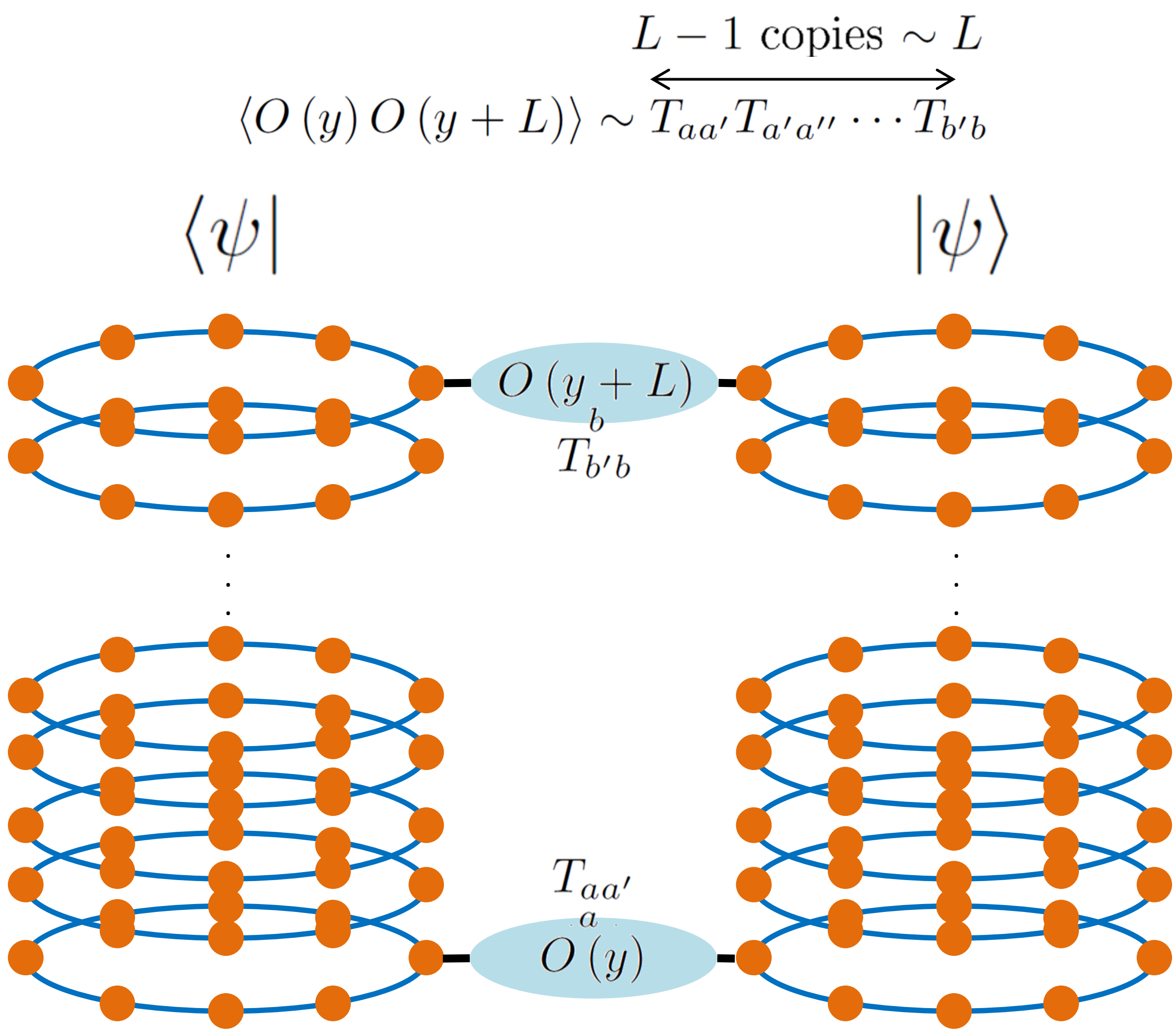
- A row-by-row transfer operator for calculation of correlations may be defined.

$$\langle O(y)\, O(y+L)\rangle \sim \overbrace{T_{aa'}T_{a'a''}\cdots T_{b'b}}^{L-1 \text{ copies} \sim L}$$



Phase Diagram of the PEPS

- A gap between the two highest eigenvalues of the transfer operator corresponds to exponentially decaying correlations in real space.
- The transfer operator may be used as a probe for phase transitions.

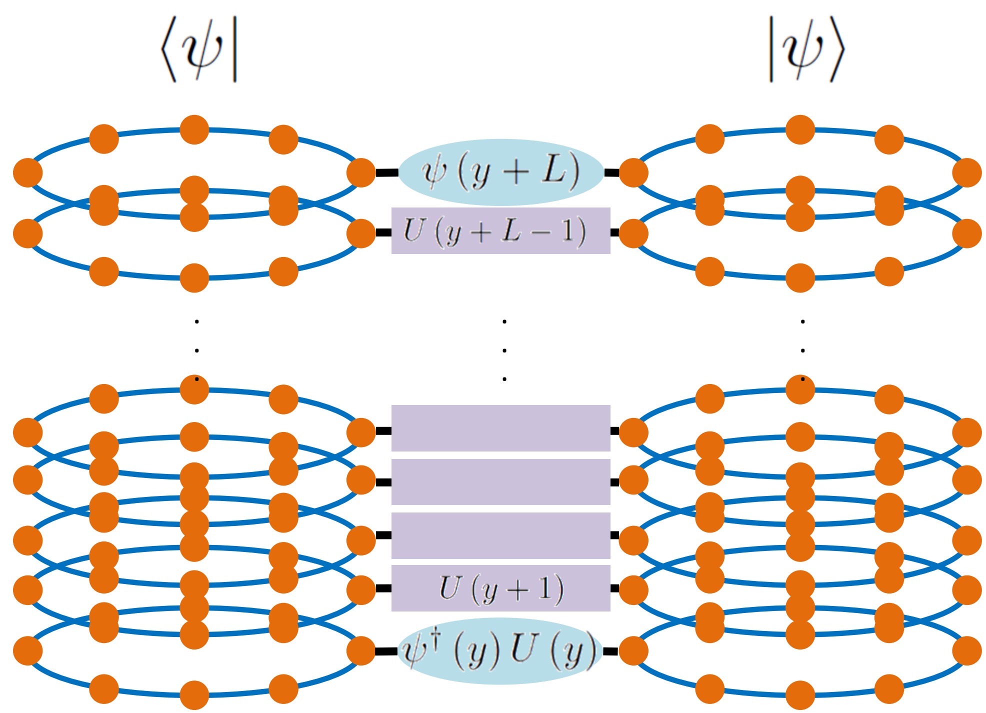


Phase Diagram of the PEPS

- If the state is locally gauge invariant $\langle O(x)\, O(x+L)\rangle \neq 0$ only if each operator is invariant independently. Otherwise, one has to include gauge field strings,

$$\left\langle \psi^{\dagger}(y) \left(\prod_{y \leq z < y+L} U(z) \right) \psi(y+L) \right\rangle$$

which introduce a new type of a transfer matrix, $T_{ab}^{\alpha\beta}$ with extra symmetries which may be studied separately and add further insights on the phase diagram and correlations.



E. Zohar, T.B. Wahl, M. Burrello, and J.I. Cirac, Ann. Phys. 374, 84-137 (2016)

Phase diagrams from the transfer operator – *U*(1)

Width - 6 | Width - 8

Gap of the transfer matrix

Pure gauge
t=0

Gap of the transfer matrix

Interacting
t=1

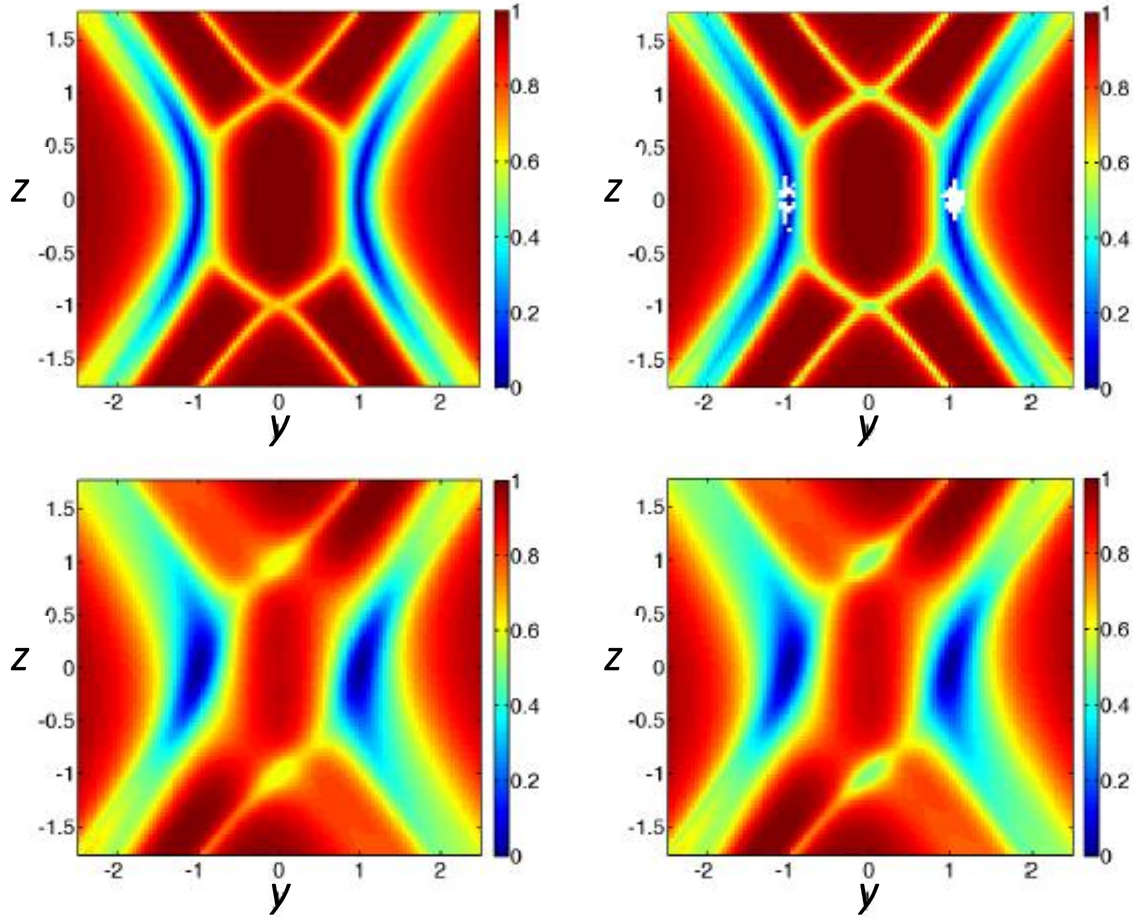


Figure 8: Left: Gap between the highest eigenvalue $\lambda_1 = 1$ and the second highest eigenvalue of the transfer matrix for $t = 0$ (top) and $t = 1$ (bottom) for a cylinder of circumference $L_1 = 6$ as a function of $y, z \in \mathbb{R}$. Right: Same plot for $L_1 = 8$. The lines with low values of the gap indeed seem to be gapless lines in the thermodynamic limit, as the corresponding values of the gap are significantly lower for $L_1 = 8$ than for $L_1 = 6$. Points were the Lanczos algorithm for calculating the eigenvalues of the transfer operator did not converge are marked in white.

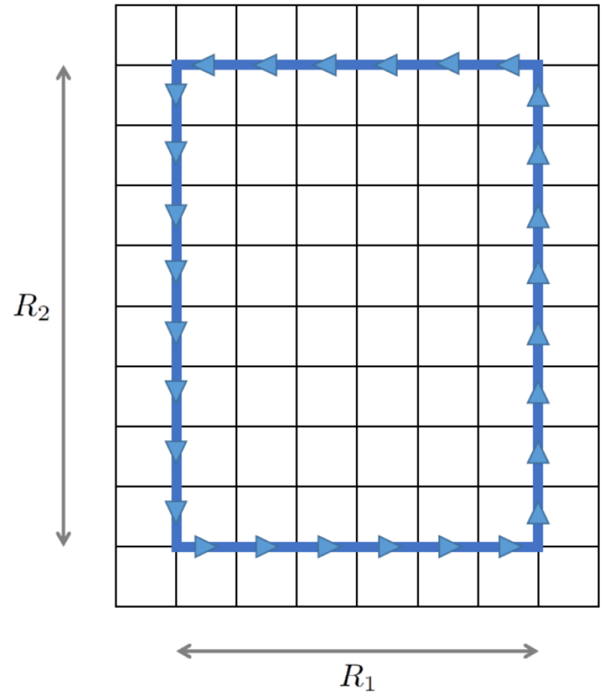
E. Zohar, M. Burrello, T.B. Wahl, and J.I. Cirac, Ann. Phys. 363, 385-439 (2015)

Wilson Loops

- Nonlocal order parameter for confinement (Wilson, 1974):

$$\mathcal{W}^j(\mathcal{C}) = \mathrm{Tr}\left(\prod_{\ell\in\mathcal{C}} U^j(\ell)\right)$$

e.g. in U(1), $\mathcal{W}(\mathcal{C}) = \exp\left(i\sum_{\ell\in\mathcal{C}}\phi(\ell)\right)$

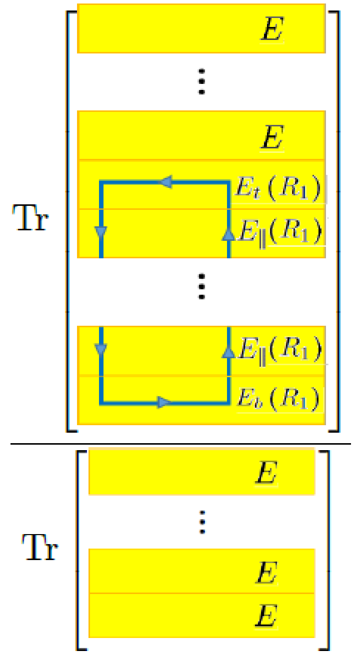


- Confinement – area law: $-\log\langle W(R_1,R_2)\rangle \propto R_1 R_2$

- Deconfinement – perimeter law: $-\log\langle W(R_1,R_2)\rangle \propto R_1 + R_2$

- With LGT PEPS:

$\langle W(R_1, R_2)\rangle =$



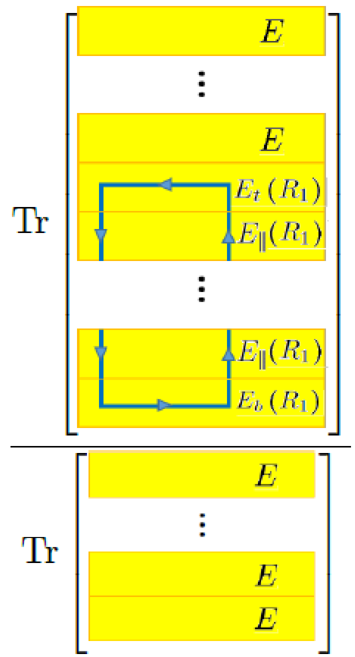
$$= \frac{\mathrm{Tr}\left[\hat{E}_b(R_1)\,\hat{E}_\parallel^{R_2-1}(R_1)\,\hat{E}_t(R_1)\,\hat{E}^{\mathcal{N}-R_2-1}\right]}{\mathrm{Tr}\left[E^{\mathcal{N}}\right]}$$

- When contracting Wilson loops, the local symmetry of the tensors allows one to neglect many elements of the transfer matrix, simplifying the contraction

E. Zohar, Phys. Rev. Research 3, 033179 (2021)

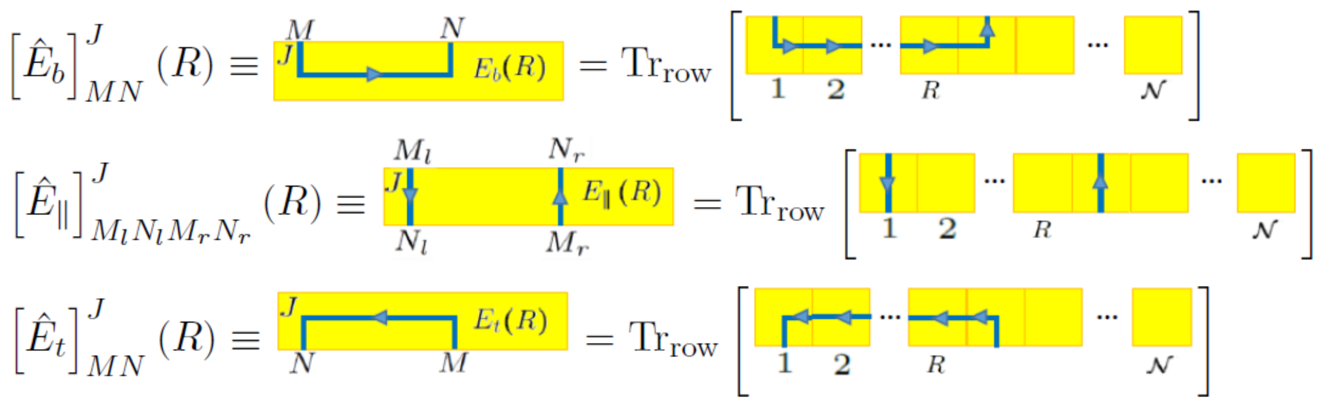
- With LGT PEPS:

$$\langle W(R_1,R_2)\rangle =$$



$$= \frac{\mathrm{Tr}\left[\hat{E}_b(R_1)\,\hat{E}_\parallel^{R_2-1}(R_1)\,\hat{E}_t(R_1)\,\hat{E}^{\mathcal{N}-R_2-1}\right]}{\mathrm{Tr}\left[E^{\mathcal{N}}\right]}$$

- One can formulate conditions for perimeter / area law, based on the spectrum of the building blocks of the transfer matrix:



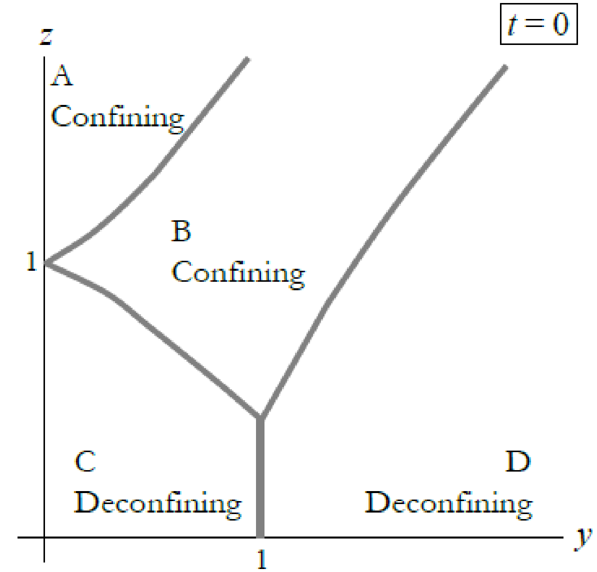
→ Highest eigenvalue proportional to R – necessary condition for an area law (other conditions required!)

E. Zohar, Phys. Rev. Research 3, 033179 (2021)

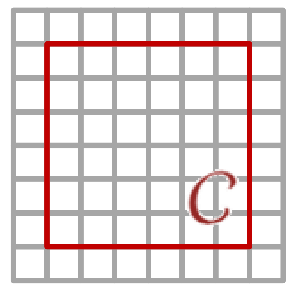
Example: The phases of the pure gauge theory – $U(1)$

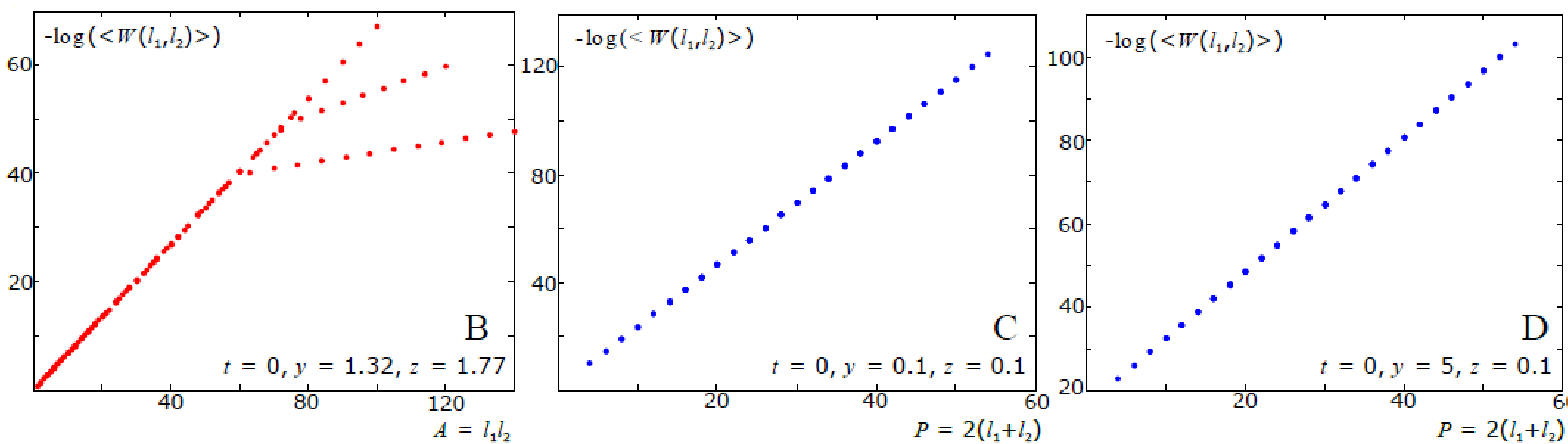
B,C,D – clear results from the Wilson loops (also from other computations, such as the Creutz parameter)

A,D – also some analytical results from $1/z$ or $1/y$ expansions.



$$W(C) = \prod_{\ell \in C} U(\ell)$$





E. Zohar, M. Burrello, T.B. Wahl, and J.I. Cirac, Ann. Phys. 363, 385-439 (2015)

Example: The phases of the pure gauge theory – $U(1)$

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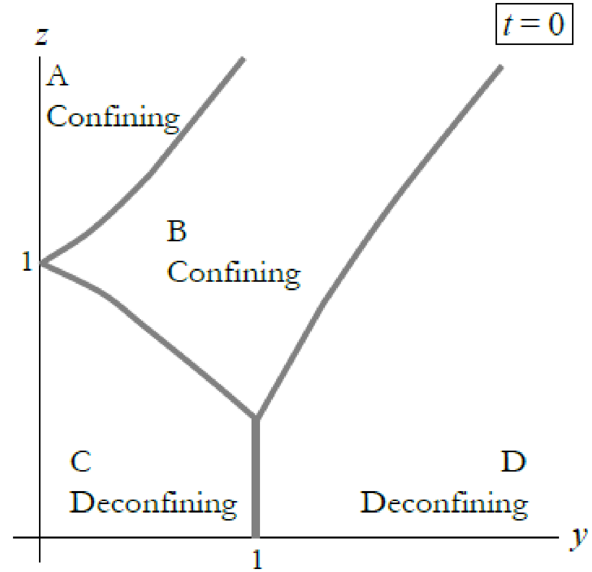
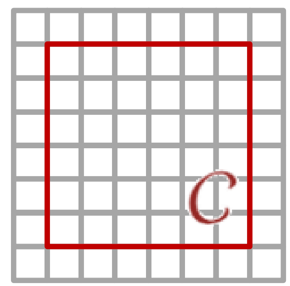
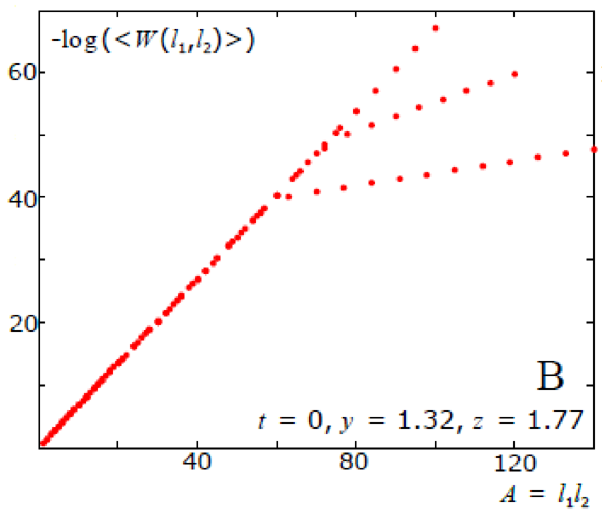
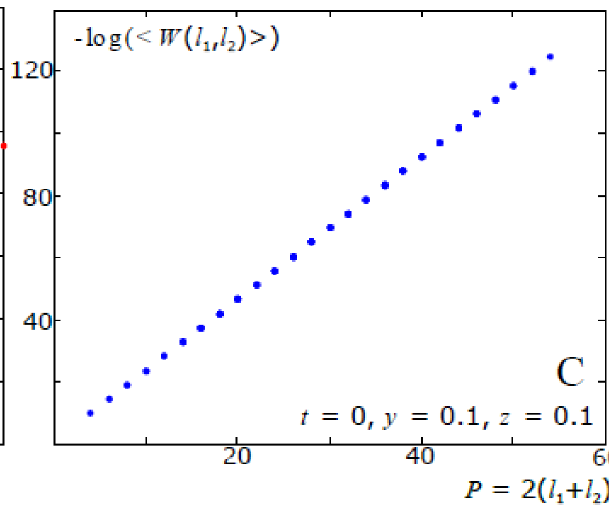
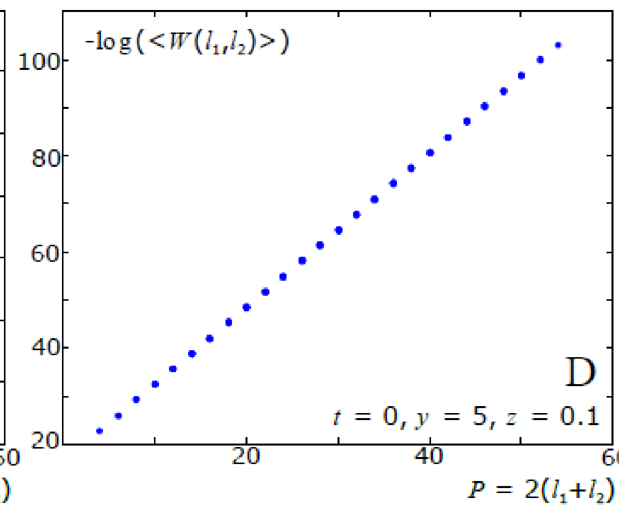


$$W(C) = \prod_{\ell \in C} U(\ell)$$







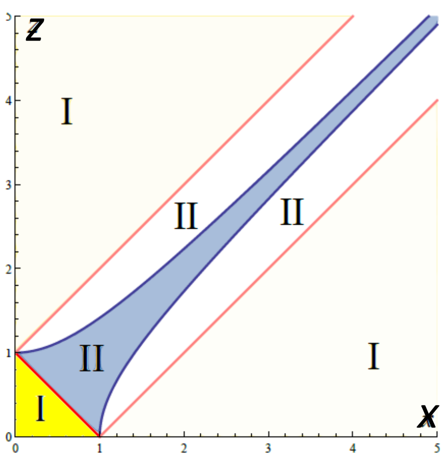


E. Zohar, M. Burrello, T.B. Wahl, and J.I. Cirac, Ann. Phys. 363, 385-439 (2015)

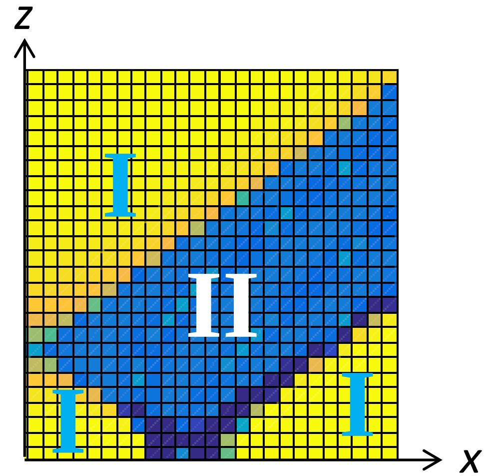
Exploiting the PEPS structure

- The special PEPS construction can also lead to "virtual" symmetries, which, although not involving the physical degrees of freedom, may still have relevance for the physical system.

- For example, in the pure gauge $SU(2)$ case, we had a virtual particle-hole symmetry, introducing transformations which relate physically equivalent points on the phase diagram, whose fixed points are "suspected" as phase boundaries.



$$x \longrightarrow x' = \frac{x}{x^2 - z^2}, \quad z \longrightarrow z' = \frac{z}{x^2 - z^2}$$



Gap of the transfer matrix

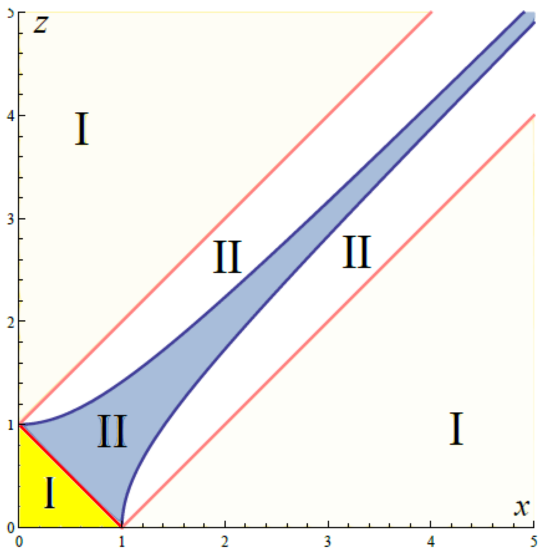
E. Zohar, T.B. Wahl, M. Burrello, and J.I. Cirac, Ann. Phys. 374, 84-137 (2016)

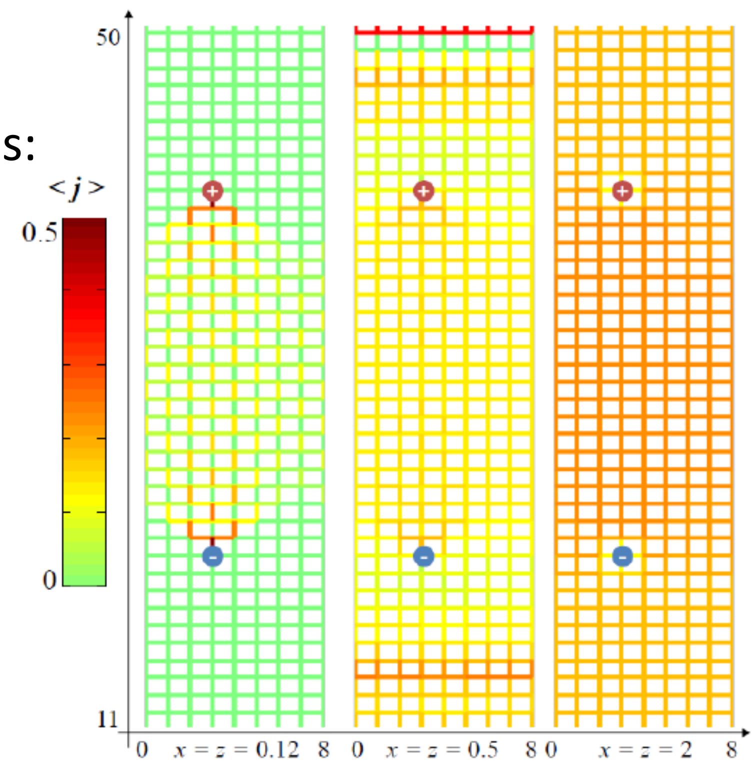
Example: The phases of the pure gauge theory – *SU*(2)

Perimeter law everywhere (Numerical calculation + perturbative expansions where applicable)

I – gapped – "Higgs"-like
II – gapless – "Coulomb"-like

Supported by flux line configuration observations:





E. Zohar, T.B. Wahl, M. Burrello, and J.I. Cirac, Ann. Phys. 374, 84-137 (2016)

Magnetic Basis

- The physical Hilbert space: $\mathcal{H}_{\text{phys}} = \mathcal{H}_{\text{phys}}\left(\{q\left(\mathbf{x}\right)\}\right) \subset \mathcal{H}_{\text{gauge}} \times \mathcal{H}_{\text{matter}}$

- Gauge field configuration states:

$$|\mathcal{G}\rangle = \bigotimes_{\mathbf{x},k} |g\left(\mathbf{x},k\right)\rangle$$
$$\mathcal{D}\mathcal{G} = \bigotimes_{\mathbf{x},k} dg\left(\mathbf{x},k\right)$$

- **General gauge invariant state**:

$$|\psi\rangle = \int \mathcal{D}\mathcal{G}|\mathcal{G}\rangle_{\text{Gauge}} |\psi\left(\mathcal{G}\right)\rangle_{\text{Matter}}$$

Where $|\psi\left(\mathcal{G}\right)\rangle$ represents matter coupled to an external (classical) gauge field $\mathcal{G}$.

- E.g. for *U*(1): $|\Phi\rangle = \bigotimes_{\mathbf{x},k} |\phi\left(\mathbf{x},k\right)\rangle$

$$\mathcal{D}\Phi = \bigotimes_{\mathbf{x},k} d\phi\left(\mathbf{x},k\right)$$
$$|\psi\rangle = \int \mathcal{D}\Phi|\Phi\rangle_{\text{Gauge}} |\psi\left(\Phi\right)\rangle_{\text{Matter}}$$

Monte Carlo with gauged Gaussian fPEPS

- Expressing our states in the **magnetic basis** that allows us to perform **efficient Monte-Carlo calculations**

$$\left|\Psi\right\rangle = \int \mathcal{DG} \left|\mathcal{G}\right\rangle \left|\psi\left(\mathcal{G}\right)\right\rangle$$

  - $\left|\mathcal{G}\right\rangle$ is a **fixed configuration state of the gauge field on the links**.

$$\left|\mathcal{G}\right\rangle \equiv \bigotimes_{\mathbf{x},k} \left|g\left(\mathbf{x},k\right)\right\rangle \qquad \mathcal{DG} = \prod_{\mathbf{x},k} dg\left(\mathbf{x},k\right)$$

$$\left\langle \mathcal{G}' | \mathcal{G} \right\rangle = \delta\left(\mathcal{G}',\mathcal{G}\right)$$

  - $\left|\psi\left(\mathcal{G}\right)\right\rangle$ is a **fermionic Gaussian state**, representing **fermions coupled to a static, background gauge field** $\mathcal{G}$.

E. Zohar, J.I. Cirac, Phys. Rev. D 97, 034510 (2018)

Monte Carlo with gauged Gaussian fPEPS

- Expressing our states in the **magnetic basis** that allows us to perform **efficient Monte-Carlo calculations**

$$\left|\psi\right\rangle = \int \mathcal{D}\Phi \left|\Phi\right\rangle_{\text{Gauge}} \left|\psi\left(\Phi\right)\right\rangle_{\text{Matter}}$$

  - $\left|\Phi\right\rangle$ is a **fixed configuration state of the gauge field on the links**.

$$\left|\Phi\right\rangle = \bigotimes_{\mathbf{x},k} \left|\phi\left(\mathbf{x},k\right)\right\rangle \qquad \mathcal{D}\Phi = \bigotimes_{\mathbf{x},k} d\phi\left(\mathbf{x},k\right)$$

$$\left\langle\Phi'|\Phi\right\rangle = \prod_{\mathbf{x},k} \delta\left(\phi\left(\mathbf{x},k\right) - \phi'\left(\mathbf{x},k\right)\right)$$

  - $\left|\psi\left(\Phi\right)\right\rangle$ is a **fermionic Gaussian state**, representing **fermions coupled to a static, background gauge field** Φ.

E. Zohar, J.I. Cirac, Phys. Rev. D 97, 034510 (2018)

Monte Carlo with gauged Gaussian fPEPS

- Expressing our states in the **magnetic basis** that allows us to perform **efficient Monte-Carlo calculations**

$$\left|\psi\right\rangle = \int \mathcal{D}\Phi \left|\Phi\right\rangle_{\mathrm{Gauge}} \left|\psi\left(\Phi\right)\right\rangle_{\mathrm{Matter}}$$

- Gauge field configuration states are eigenstates of functions of group element operators:

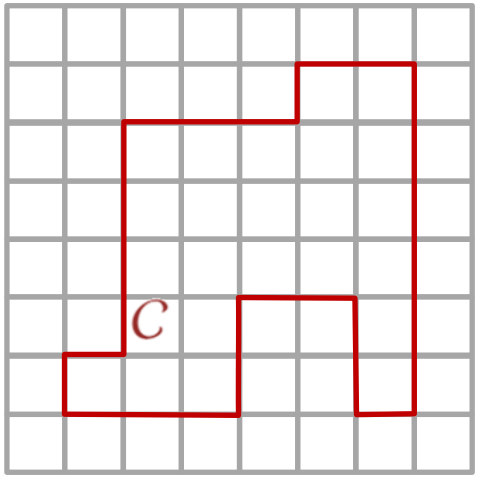
$$U\left|\phi\right\rangle = e^{i\phi}\left|\phi\right\rangle \qquad \left|\Phi\right\rangle = \bigotimes_{\mathbf{x},k}\left|\phi\left(\mathbf{x},k\right)\right\rangle$$

$$F\left(\left\{U\left(\mathbf{x},k\right)\right\}\right)\left|\Phi\right\rangle = F\left(\left\{e^{i\phi\left(\mathbf{x},k\right)}\right\}\right)\left|\Phi\right\rangle$$

E. Zohar, J.I. Cirac, Phys. Rev. D 97, 034510 (2018)

Monte Carlo with gauged Gaussian fPEPS

- Wilson Loops: $W(C) = \prod_{\ell \in C} U(\ell)$



$$W(C)|\Phi\rangle = e^{i \sum_{\ell \in C} \phi(\ell)} |\Phi\rangle$$

- Expectation value for $|\psi\rangle = \int \mathcal{D}\Phi |\Phi\rangle_{\mathrm{Gauge}} |\psi(\Phi)\rangle_{\mathrm{Matter}}$:

$$\langle W(C)\rangle = \langle\psi| W(C) |\psi\rangle / Z = \int \mathcal{D}\Phi \, p(\Phi) \, e^{i \sum_{\ell \in C} \phi(\ell)}$$

$$\langle\psi|\psi\rangle = \int \mathcal{D}\Phi \langle\psi(\Phi)|\psi(\Phi)\rangle \equiv Z$$

$$p(\Phi) = \langle\psi(\Phi)|\psi(\Phi)\rangle / Z$$

E. Zohar, J.I. Cirac, Phys. Rev. D 97, 034510 (2018)

Monte Carlo with gauged Gaussian fPEPS

- Wilson Loops: $$W(C) = \mathrm{Tr}\left(\prod_{\{\mathbf{x},k\}\in C} U(\mathbf{x},k)\right)$$

  - exp. value for $|\Psi\rangle = \int \mathcal{D}\mathcal{G} \, |\mathcal{G}\rangle \, |\psi(\mathcal{G})\rangle$:

$$\langle W \rangle = \frac{\int \mathcal{D}\mathcal{G} \mathrm{Tr}\left(\prod_{\{\boldsymbol{x},k\}\in C} D\left(g\left(\boldsymbol{x},k\right)\right)\right) \langle \psi(\mathcal{G}) | \psi(\mathcal{G}) \rangle}{\int \mathcal{D}\mathcal{G} \, \langle \psi(\mathcal{G}) | \psi(\mathcal{G}) \rangle}$$

- The function

$$p(\mathcal{G}) = \frac{\langle \psi(\mathcal{G}) | \psi(\mathcal{G}) \rangle}{\int \mathcal{D}\mathcal{G}' \, \langle \psi(\mathcal{G}') | \psi(\mathcal{G}') \rangle}$$

is a probability density.

E. Zohar, J.I. Cirac, Phys. Rev. D 97, 034510 (2018)

Monte Carlo with gauged Gaussian fPEPS

- Wilson Loops: $$W(C) = \mathrm{Tr}\left(\prod_{\{\mathbf{x},k\}\in C} U(\mathbf{x},k)\right)$$

  - exp. value for $|\Psi\rangle = \int \mathcal{D}\mathcal{G} \, |\mathcal{G}\rangle \, |\psi(\mathcal{G})\rangle$:

$$\langle W \rangle = \frac{\int \mathcal{D}\mathcal{G} \mathrm{Tr}\left(\prod_{\{\boldsymbol{x},k\}\in C} D(g(\boldsymbol{x},k))\right) \langle \psi(\mathcal{G}) | \psi(\mathcal{G}) \rangle}{\int \mathcal{D}\mathcal{G} \langle \psi(\mathcal{G}) | \psi(\mathcal{G}) \rangle}$$

- **The fermionic calculation is easy, through the gaussian formalism: very efficient**
- **No sign problem: the probability density is obtained from a norm of a state, and thus is real and positive.**

$\Rightarrow$ **Monte Carlo integration!**

E. Zohar, J.I. Cirac, Phys. Rev. D 97, 034510 (2018)

Monte Carlo with gauged Gaussian fPEPS

- The method is extendable to further physical observables, always involving the **probability density function**

$$p\left(\mathcal{G}\right)=\frac{\left\langle\psi\left(\mathcal{G}\right)\middle|\psi\left(\mathcal{G}\right)\right\rangle}{\int\mathcal{D}\mathcal{G}'\left\langle\psi\left(\mathcal{G}'\right)\middle|\psi\left(\mathcal{G}'\right)\right\rangle}$$

and possibly elements of the **covariance matrix of the Gaussian state** $\left|\psi\left(\mathcal{G}\right)\right\rangle$, which could be calculated very efficiently.

 – For example, mesonic operators $M\left(\mathbf{x},\mathbf{y},C\right)=\psi^{\dagger}\left(\mathbf{x}\right)\prod_{\ell\in C}U\left(\ell\right)\psi\left(\mathbf{y}\right)$

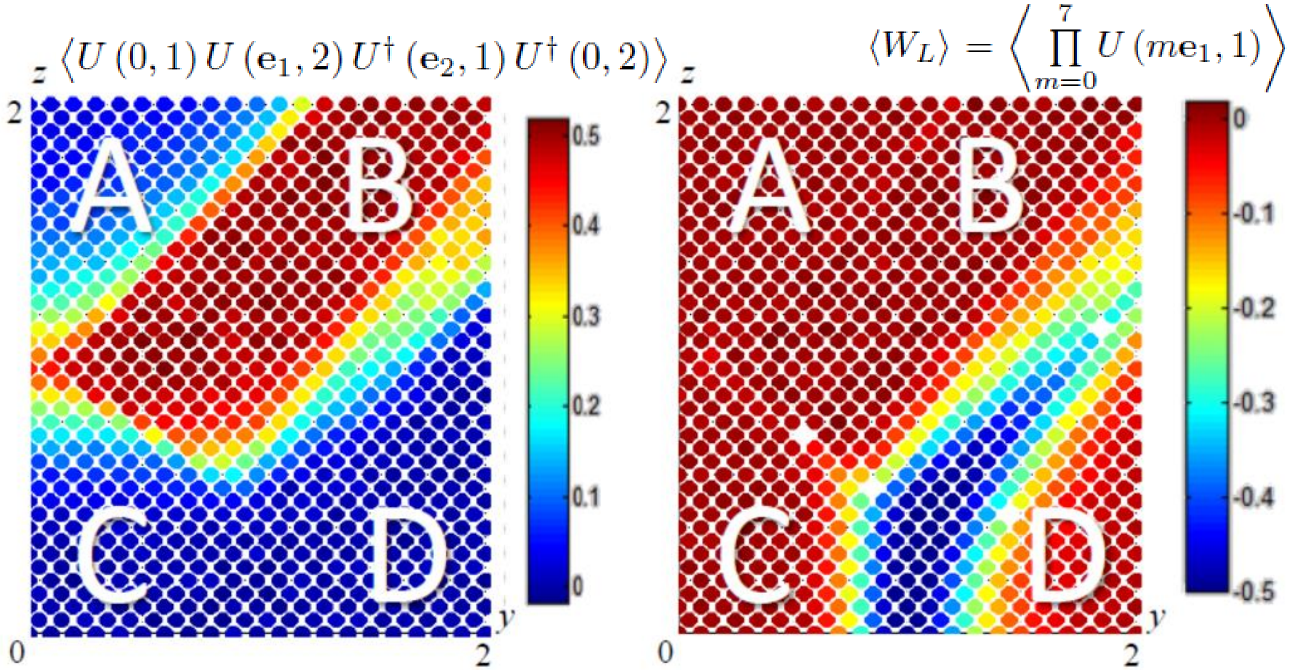
$$\left\langle M\left(\mathbf{x},\mathbf{y},C\right)\right\rangle=\int\mathcal{D}\Phi\, p\left(\Phi\right)e^{i\sum_{\ell\in C}\phi(\ell)}\frac{\left\langle\psi\left(\Phi\right)\right|\psi^{\dagger}\left(\mathbf{x}\right)\psi\left(\mathbf{y}\right)\left|\psi\left(\Phi\right)\right\rangle}{\left\langle\psi\left(\Phi\right)\middle|\psi\left(\Phi\right)\right\rangle}$$

 (given for U(1) for simplicity).

- **It is possible to contract gauged Gaussian fPEPS beyond 1+1d, and without the sign problem of conventional LGT methods (it is not a Euclidean path integral).**

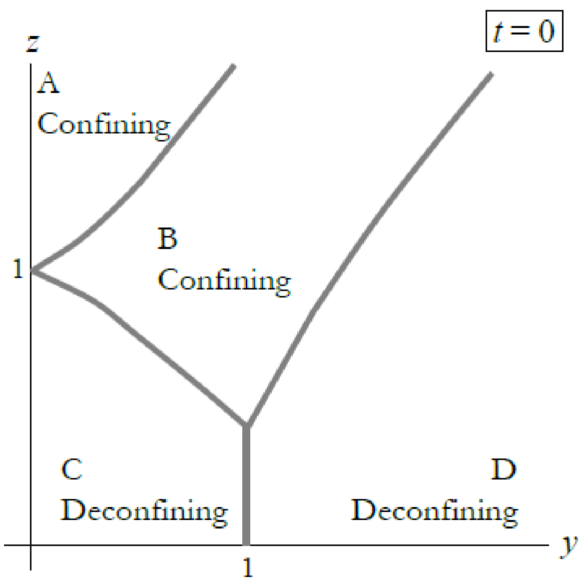
E. Zohar, J.I. Cirac, Phys. Rev. D 97, 034510 (2018)

Illustration: phase diagram of pure gauge Z_3 PEPS in 2+1d



MC, 8x8

E. Zohar, J.I. Cirac, Phys. Rev. D 97, 034510 (2018)



Exact contraction

E. Zohar, M. Burrello, T.B. Wahl, and J.I. Cirac, Ann. Phys. 363, 385-439 (2015)

"Real" benchmark: pure gauge Z_3 PEPS in 2+1d

$$P_\ell^N = Q_\ell^N = 1 \qquad P_\ell^\dagger P_\ell = Q_\ell^\dagger Q_\ell = 1$$

$$P_\ell^\dagger Q_\ell P_\ell = e^{i\delta Q_\ell} \qquad \delta = \frac{2\pi}{N}$$

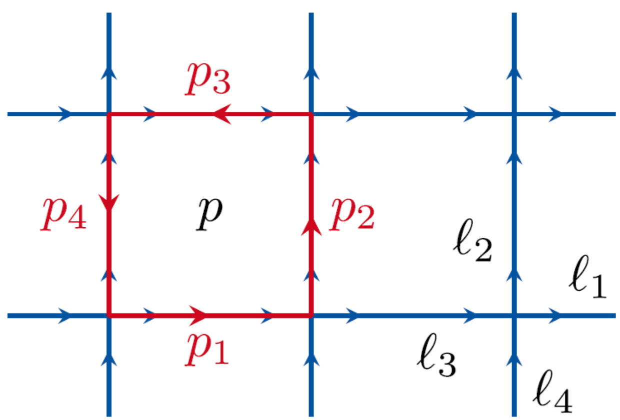
$$H = H_E + H_B$$

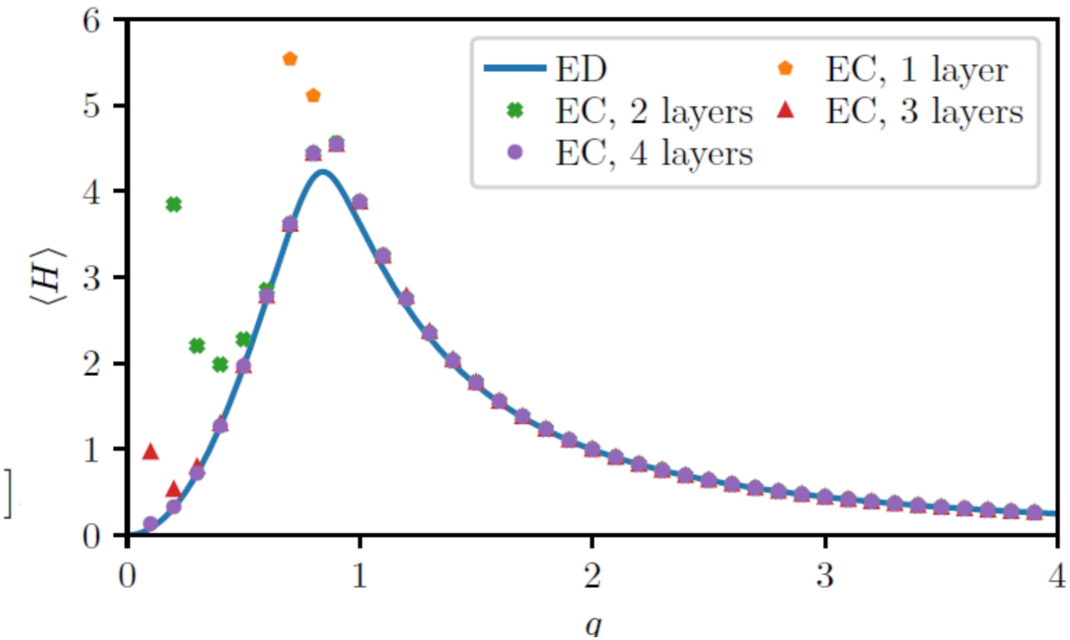
$$= \frac{g^2}{2}\sum_\ell [2 - (P_\ell + P_\ell^\dagger)]$$

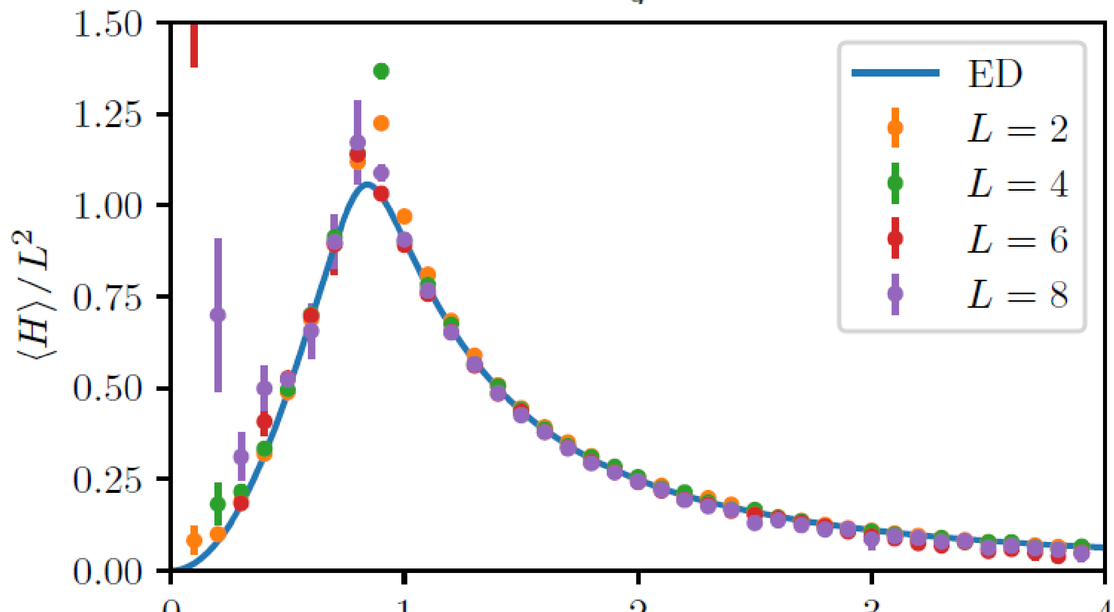
$$+ \frac{1}{2g^2}\sum_p [2 - (Q_{p_1}^\dagger Q_{p_2}^\dagger Q_{p_3} Q_{p_4} + \mathrm{H.c.})]$$

$$\Theta(\mathbf{x}) = P_{\mathbf{x},r} P_{\mathbf{x},u} P^\dagger_{\mathbf{x}-\hat{\mathbf{e}}_1,r} P^\dagger_{\mathbf{x}-\hat{\mathbf{e}}_2,u}$$

$$[\Theta(\mathbf{x}), H] = 0 \quad \forall \;\; \mathbf{x}$$



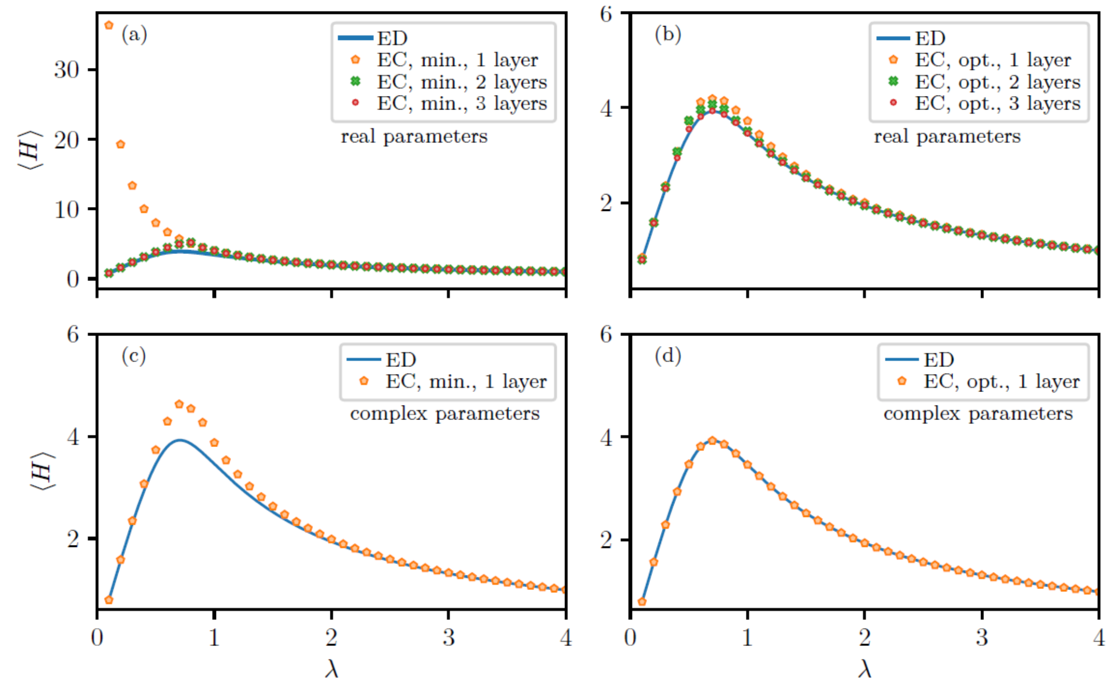




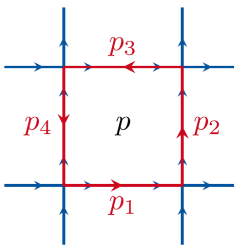
P. Emonts, M.C. Bañuls, J. I. Cirac, **E. Zohar**, Phys. Rev. D 102, 074501 (2020)

Improved Algorithm + Benchmark: pure gauge Z_2 PEPS in 2+1d

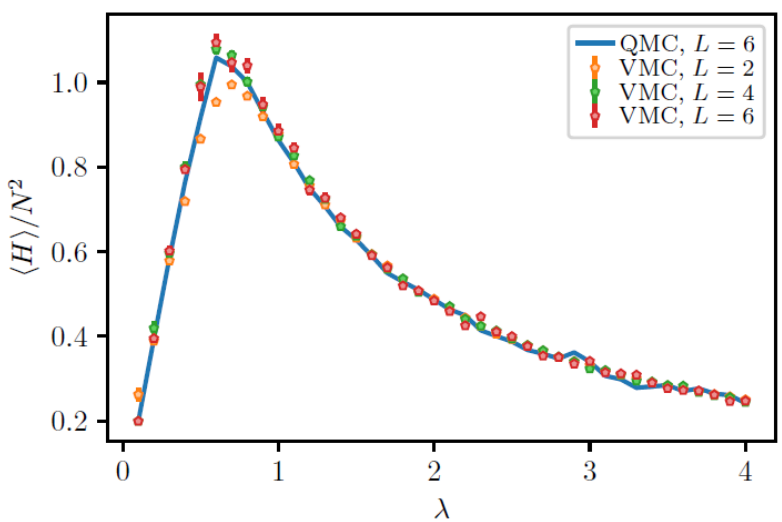
- Solved a numerical bottleneck (Pfaffians)
- Improved the ansatz states analytically

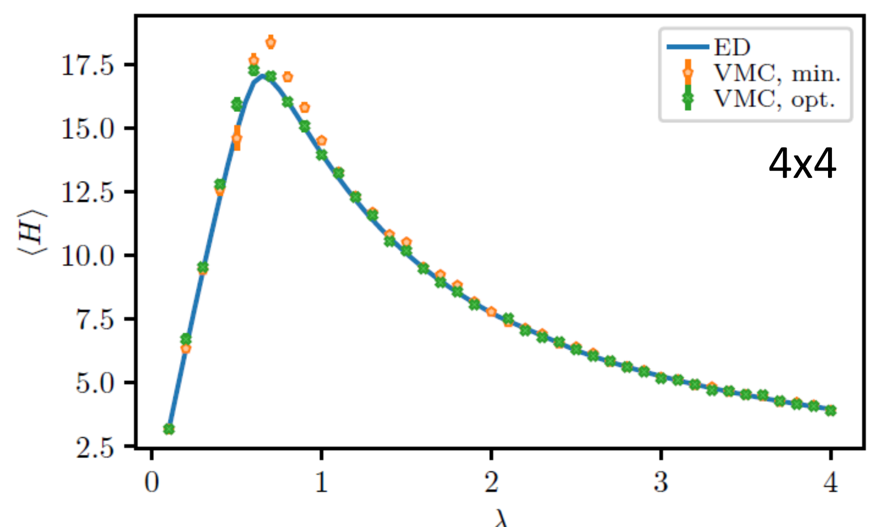


$$H = \lambda \sum_{\ell} [1 - P_\ell] + \frac{1}{\lambda} \sum_{p} [1 - Q_{p_1} Q_{p_2} Q_{p_3} Q_{p_4}]$$



$$P = P^\dagger = \sigma_z$$
$$Q = Q^\dagger = \sigma_x$$





P. Emonts, A. Kelman, U. Borla, S. Moroz, S. Gazit, E. Zohar, Phys. Rev. D 107, 014505 (2023)

3+1-d: fermions ready for gauging

- The Gaussian fermionic PEPS formalism can be extended in a straight forward way to three spatial dimensions.
- As expected, “spin” emerges from demanding lattice rotation invariance; assuming very naively that the fermions are “spinless” leads to Kogut’s staggering.
- It is possible to formally express the ground states of lattice Dirac Hamiltonians exactly as such states.
- Gauging can be done exactly the same, as well as Monte-Carlo sampling; the dependence on the dimension is indirect, only through the number of links to be integrated.
- To be continued

P. Emonts, E. Zohar, arXiv:2304.06744

In conclusion,

- **PEPS** could be generalized to describe lattice **states with a local gauge symmetry**, living in the Hilbert spaces of lattice gauge theories.

- Using the PEPS construction, it is possible to **gauge/minimally couple** matter states with a global symmetry, and properties of these states survive or get modified for the gauge invariant interacting state.

- Gauged Gaussian Fermionic PEPS (**GGFPEPS**) may be used for the numerical study (variational computations) of LGTs in 2+1 and more.

For detailed lecture notes on the topics discussed in this talk, see
Gauss law, Minimal Coupling and Fermionic PEPS for Lattice Gauge Theories
Patrick Emonts and Erez Zohar, SciPost Phys. Lect. Notes 12 (2020)