# Hamiltonian formulations of gauge theories for quantum computation from tensor networks

Judah Unmuth-Yockey

Dept of Theoretical Physics, Fermi National Accelerator Laboratory, Batavia, IL

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# Plan

Team: Haiyuan Zou, Jin Zhang, Shan-Wen Tsai, Yannick Meurice [Meurice et al., 2022]

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  3. Transfer matrix
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  4. Quantum Hamiltonian

# Euclidean lattice

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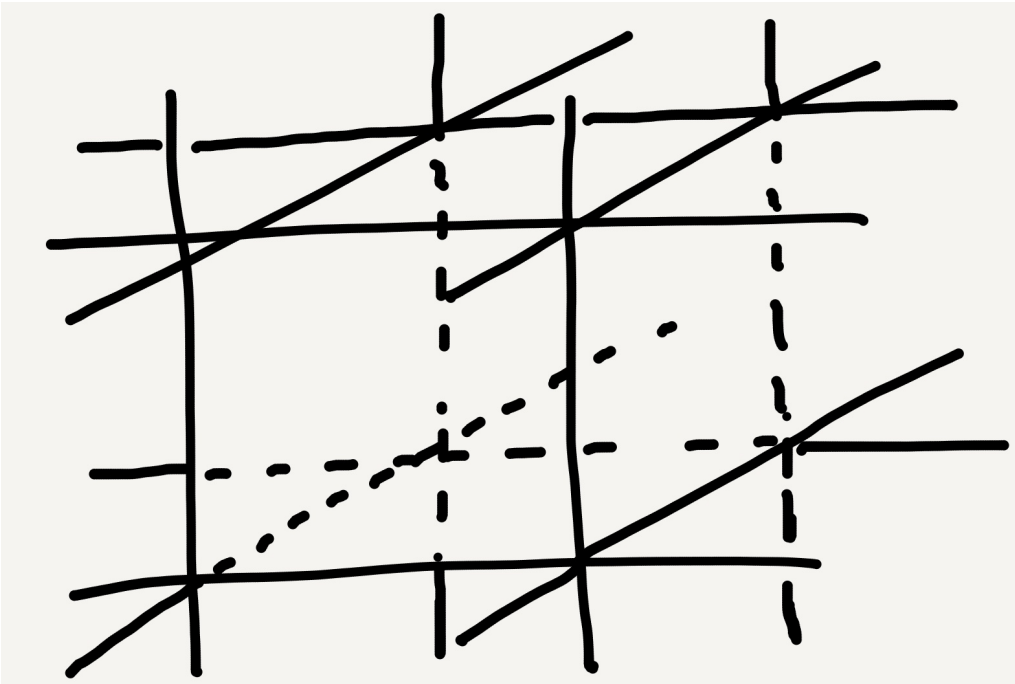

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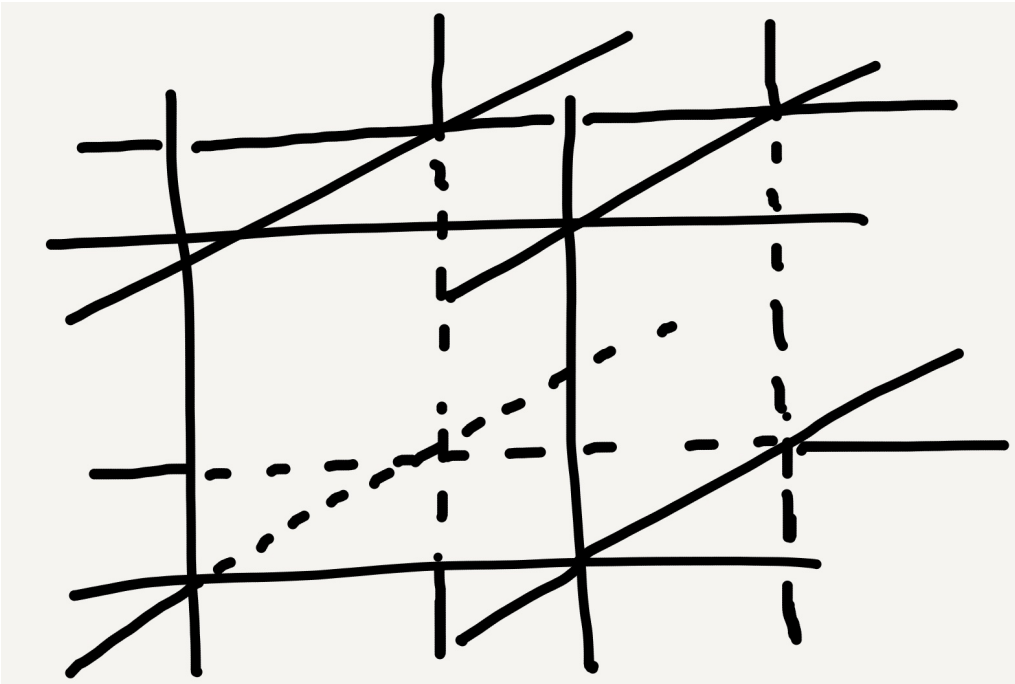
- Cubic lattice
- Infinite volume



# Compact Abelian Higgs model

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# Link tensors

$$\int_0^{2\pi} \frac{d\theta_\mu(x)}{2\pi} e^{i\theta_\mu(x)(m_1+m_2-m_3-m_4+n)}$$
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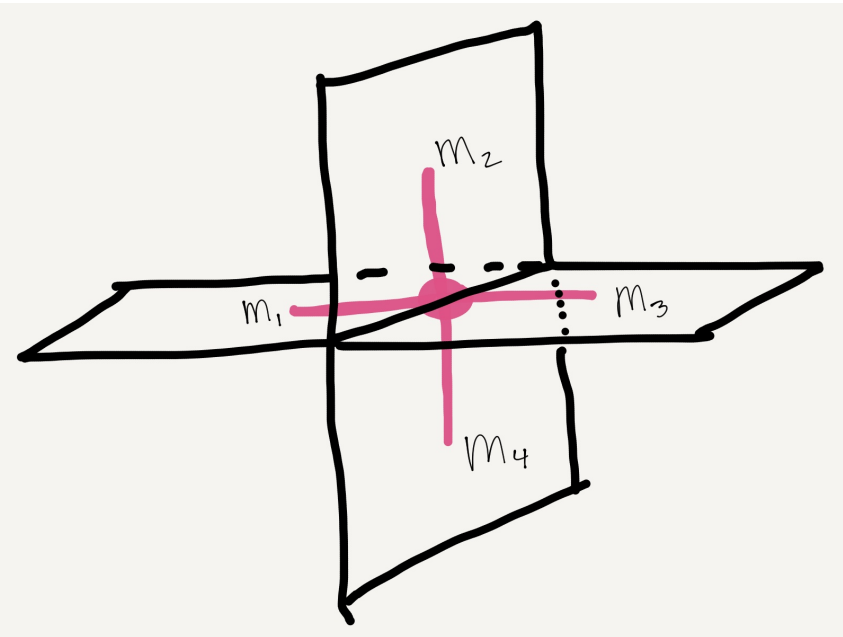

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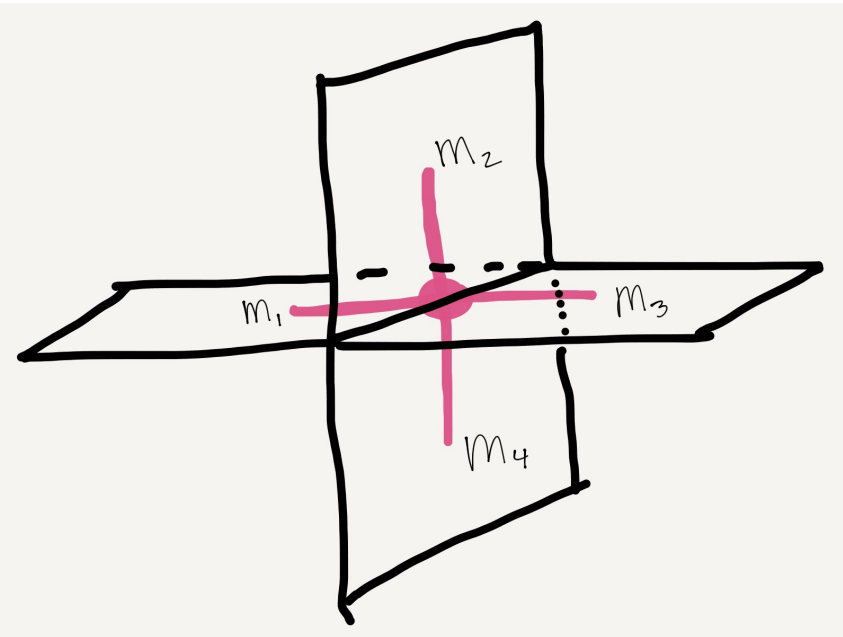


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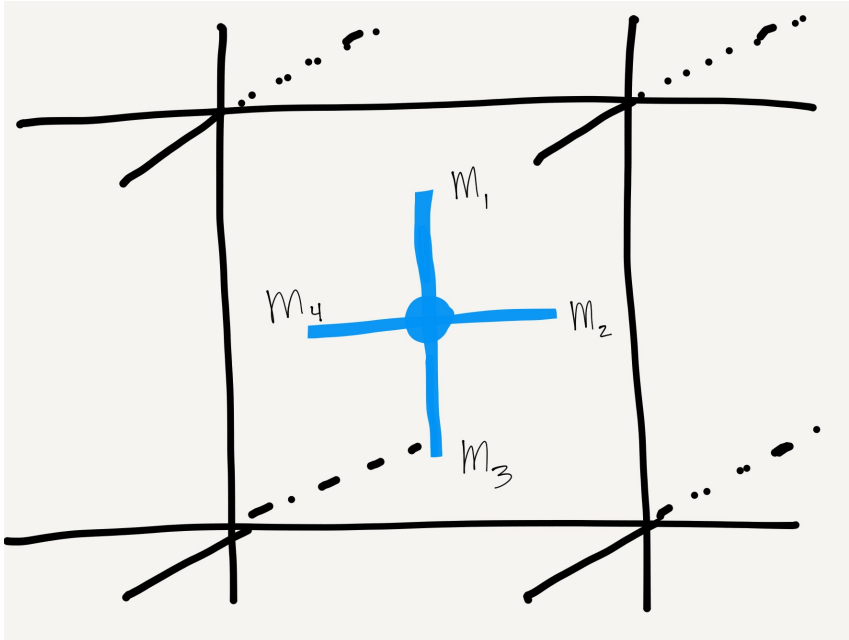

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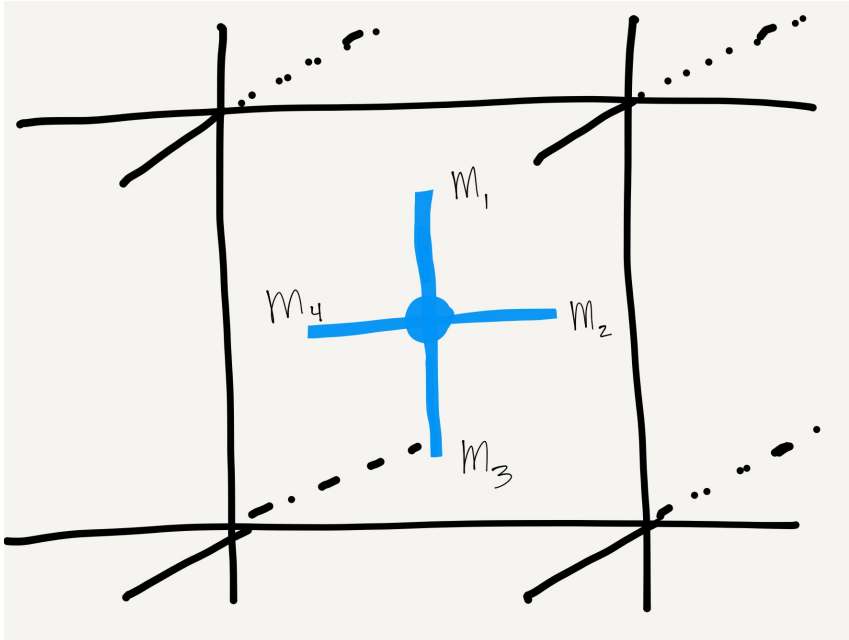
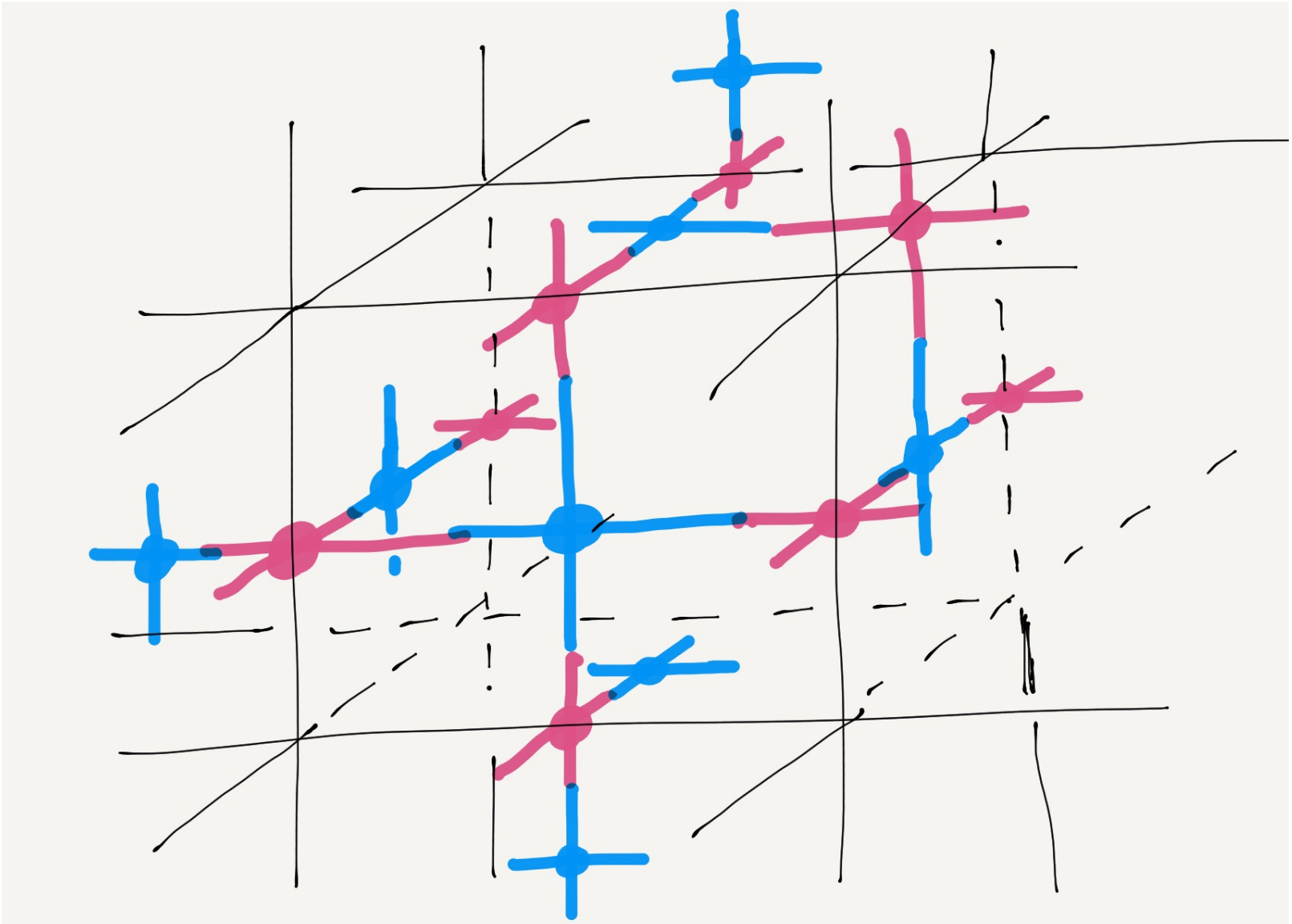


# Tensor network



# Transfer matrix

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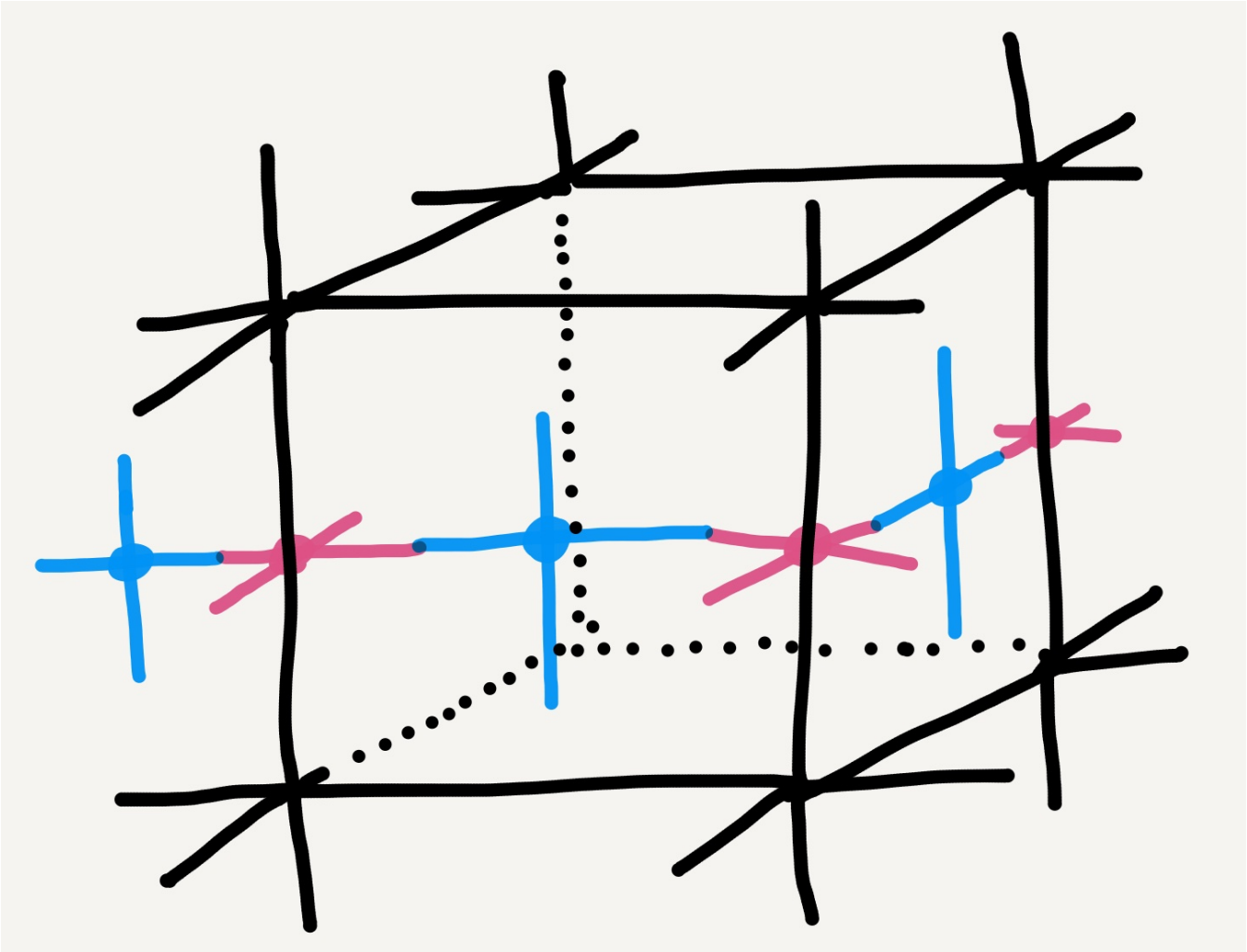
With $H = K + P$

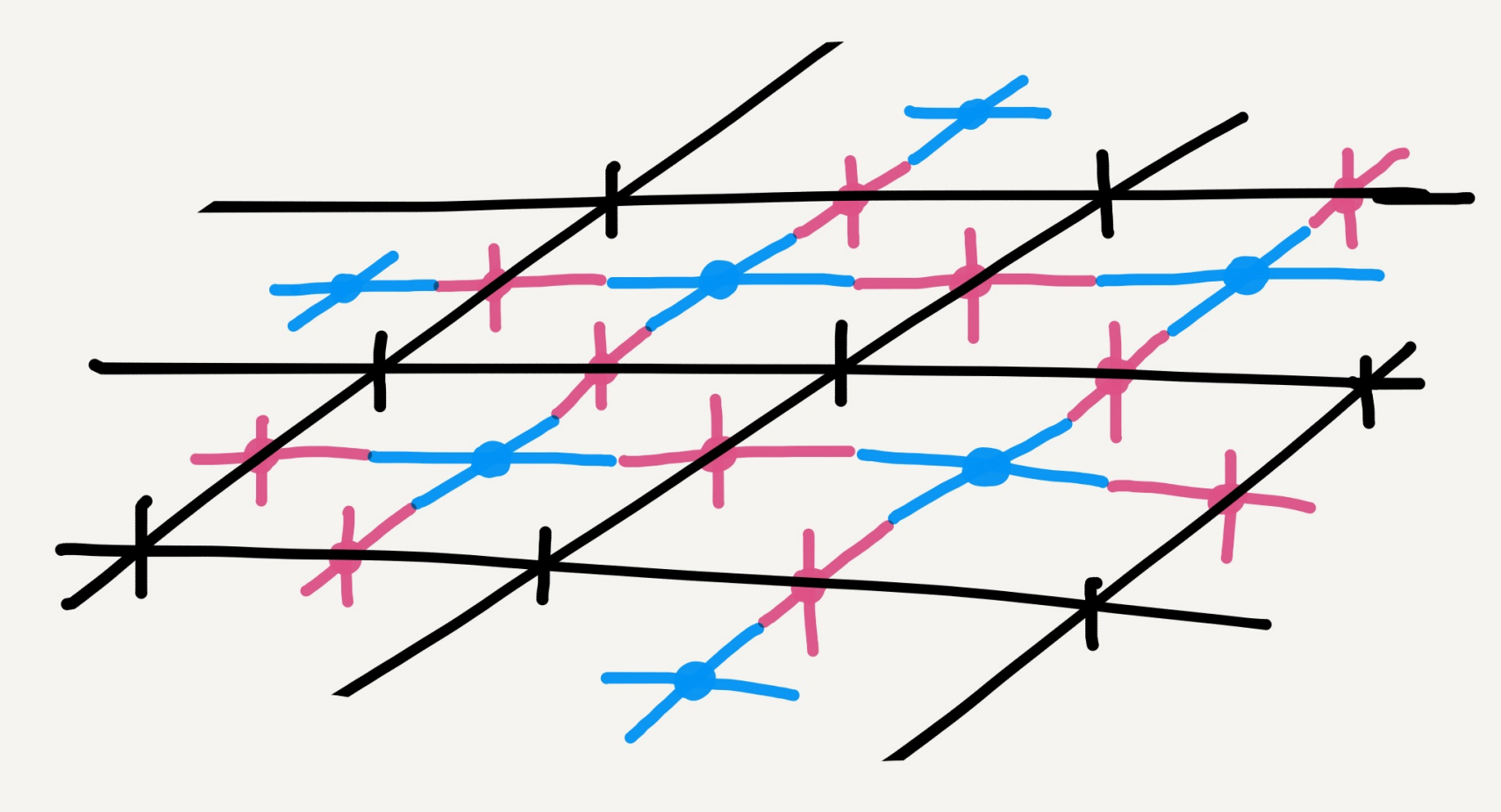
$$\mathbb{T} = \underbrace{e^{-P\tau/2}}_{\text{diagonal}} \underbrace{e^{-K\tau}}_{\text{off-diagonal}} e^{-P\tau/2}$$

# Potential layer

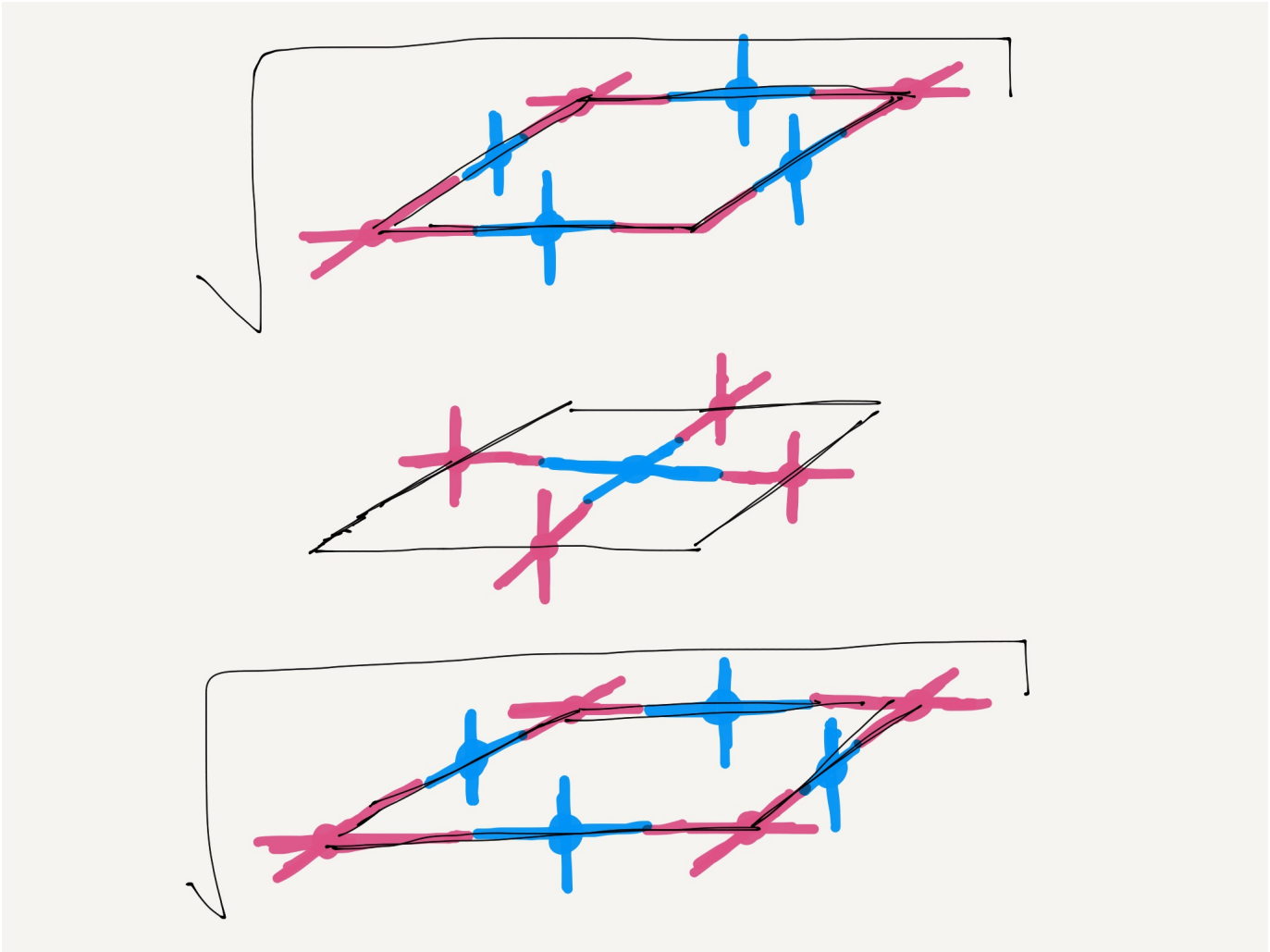


Note: Diagonal

# Kinetic layer



# Sandwich



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[Fradkin and Susskind, 1978]

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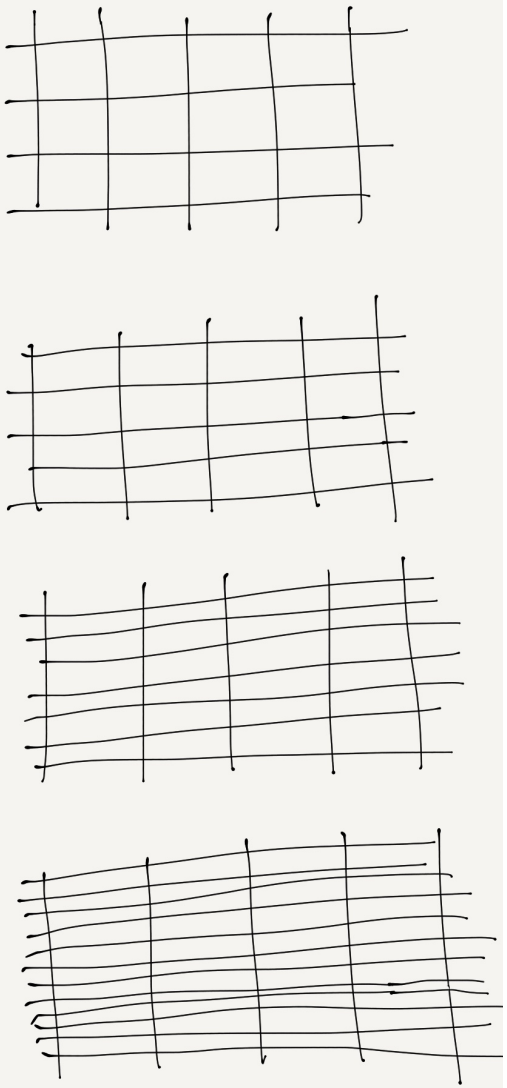

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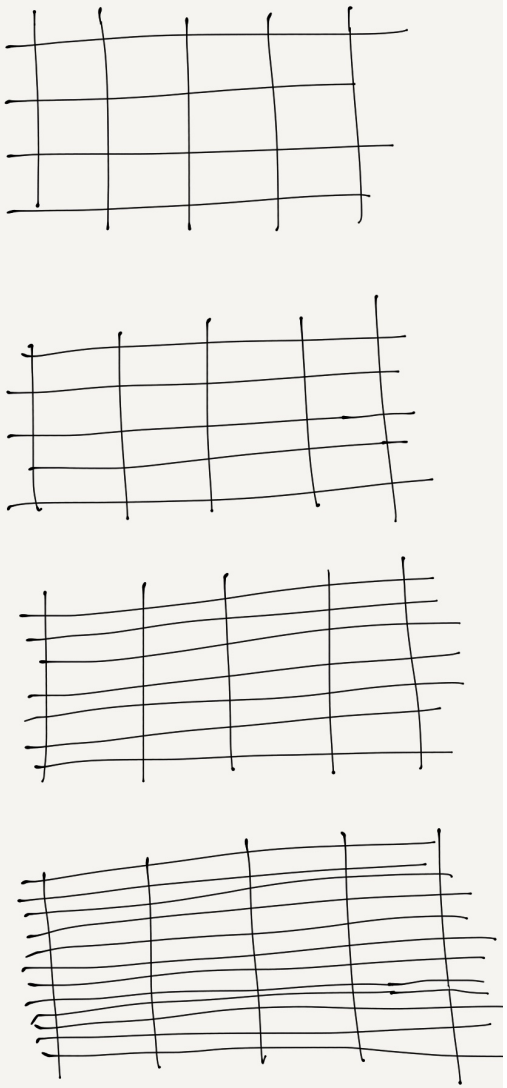

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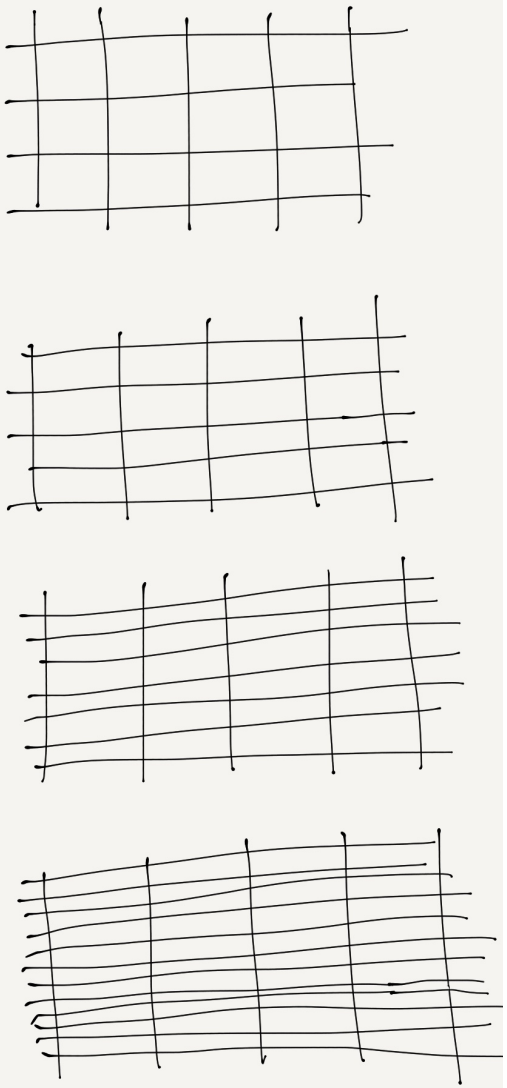

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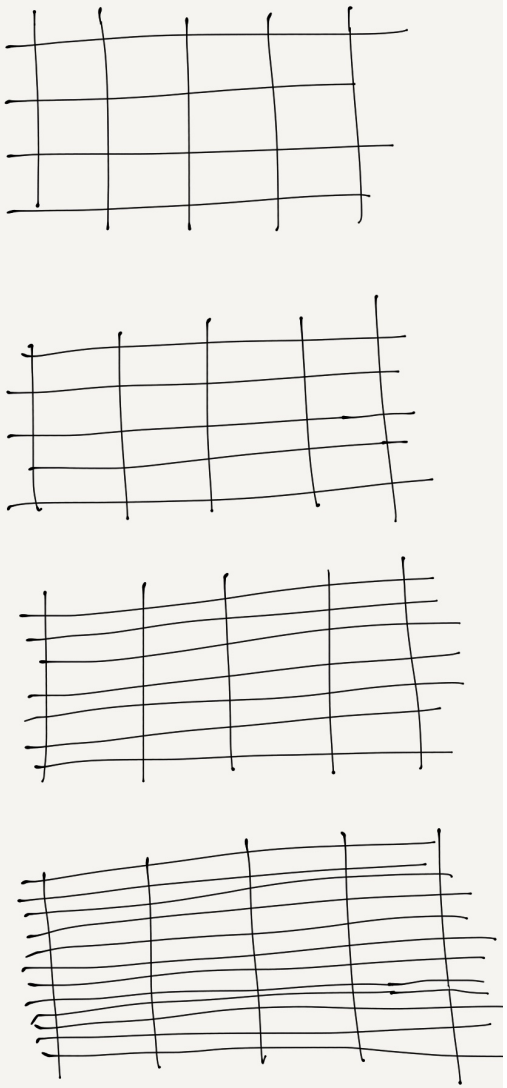

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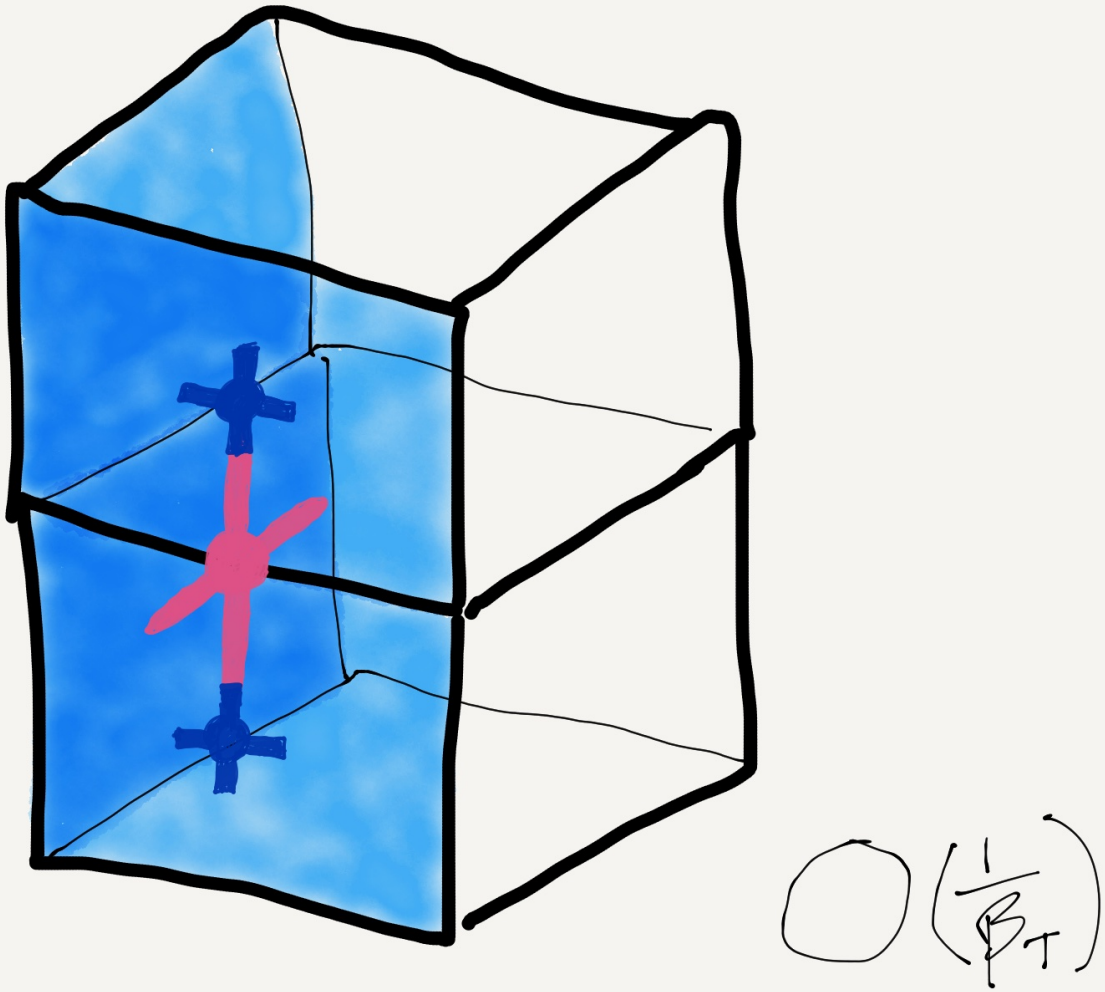
# $\mathcal{O}(1/\beta_{\text{time}})$



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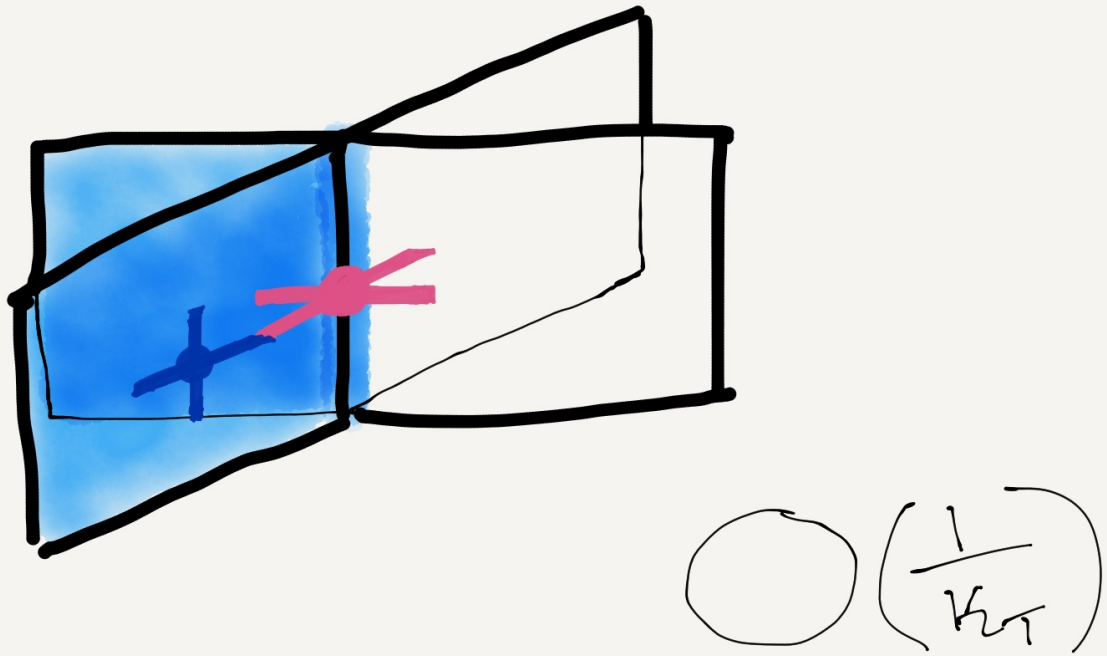

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$$H_{\text{diag}} = \underbrace{\frac{1}{2\beta_{\text{time}}\tau}}_{U/2} \sum_x \sum_i m_{x,i}^2 + \underbrace{\frac{1}{2\kappa_{\text{time}}\tau}}_{Y/2} \sum_x (\vec{\nabla} \cdot \vec{m})_x^2$$

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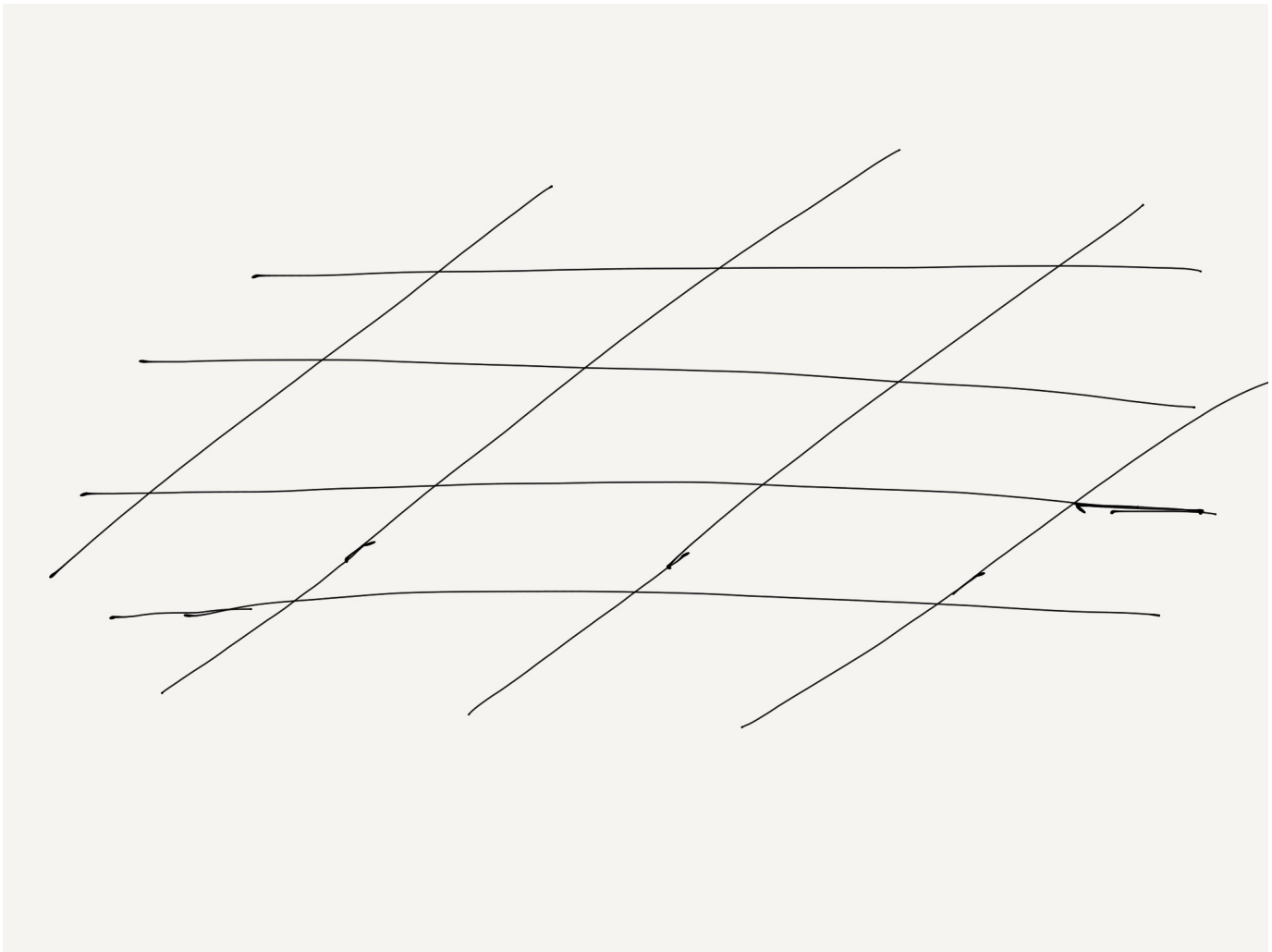

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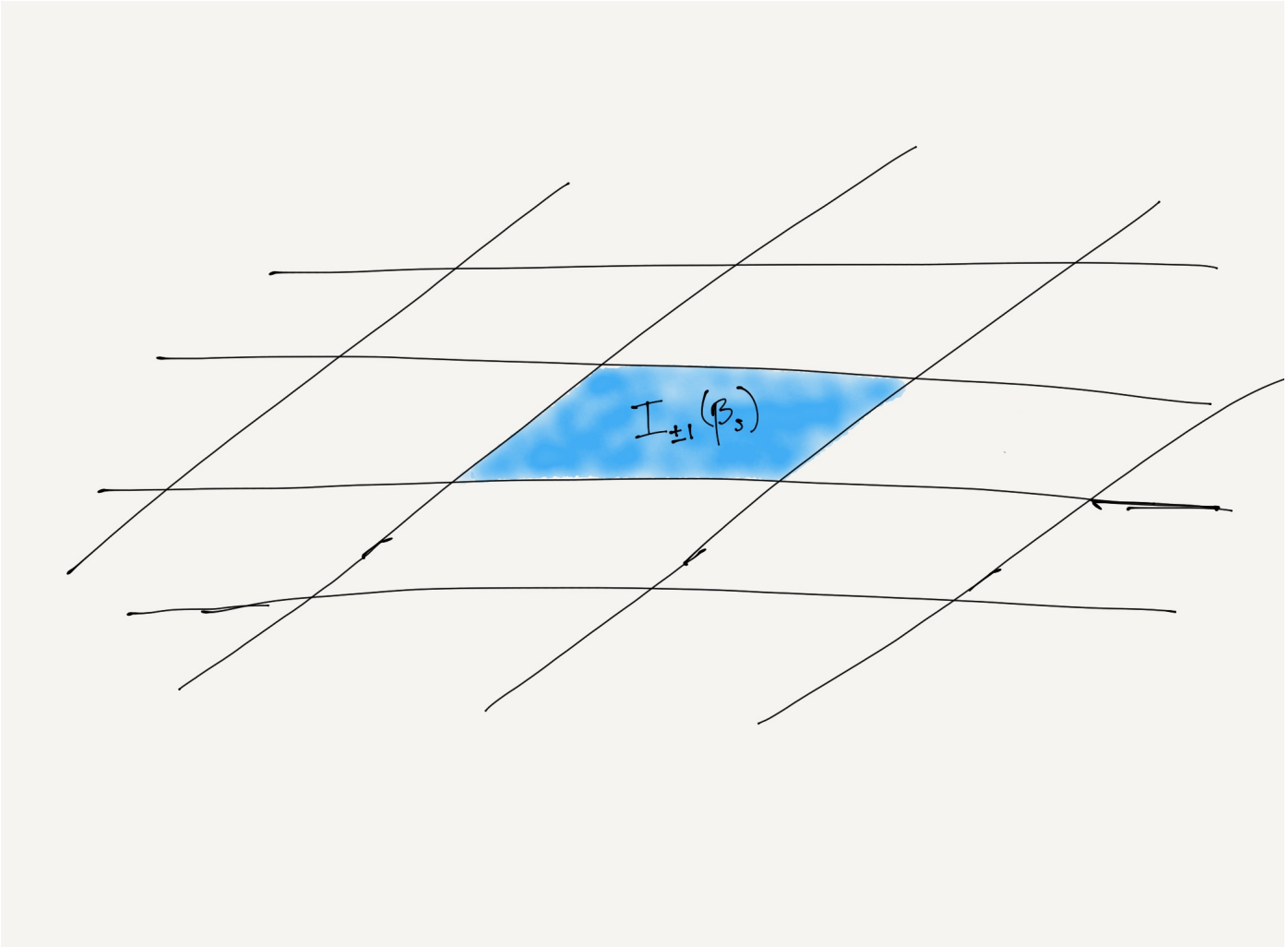

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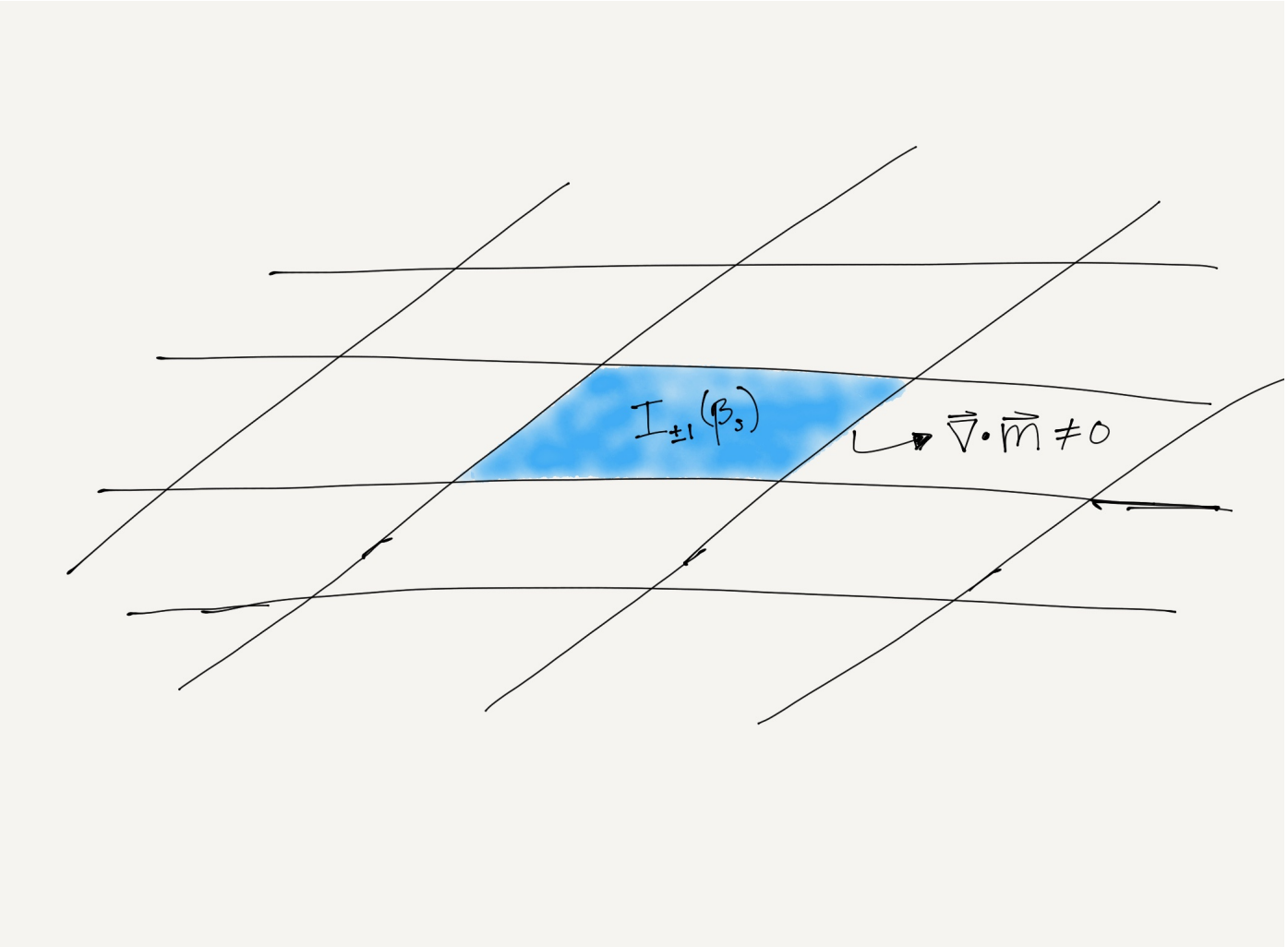

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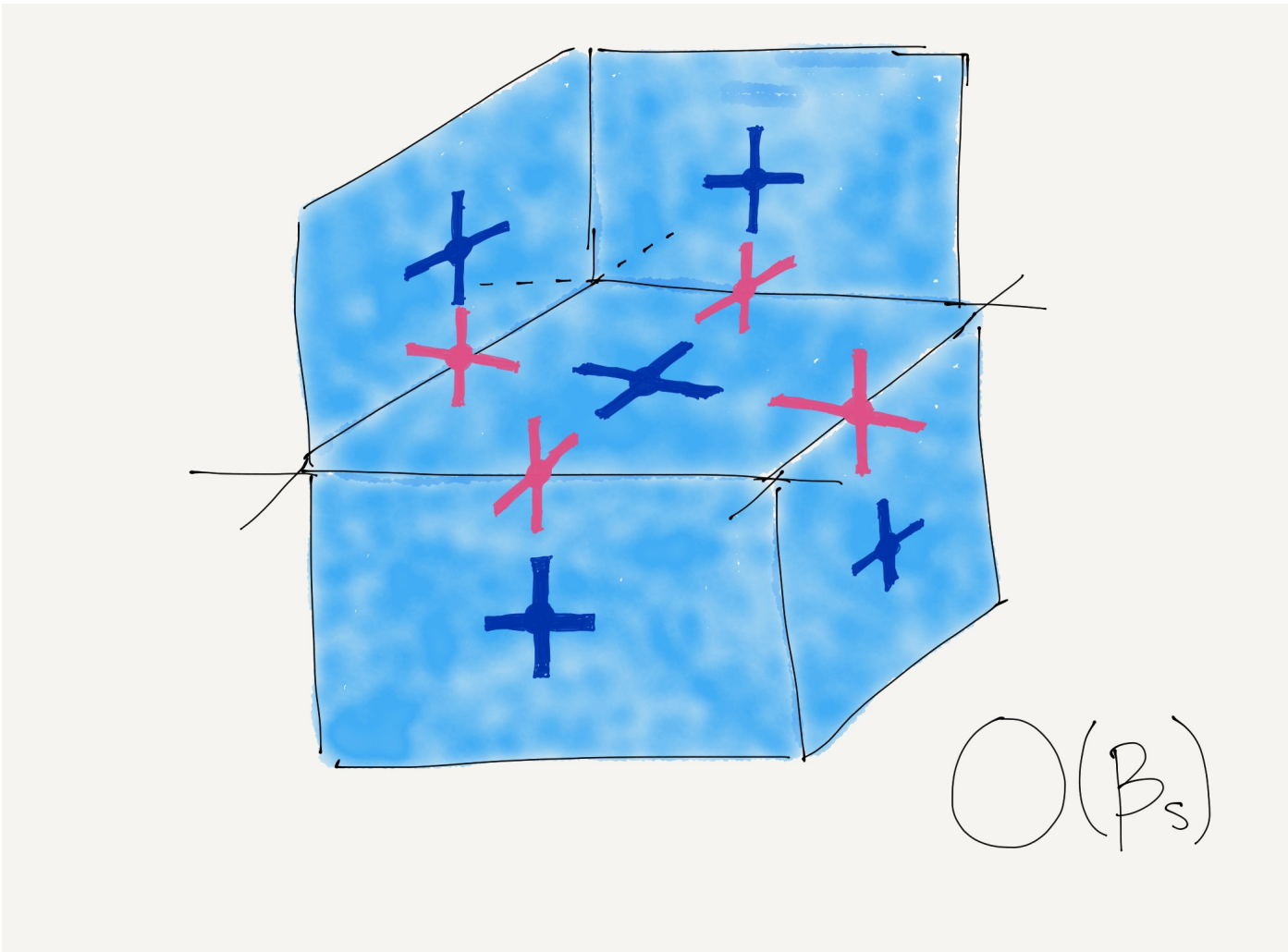

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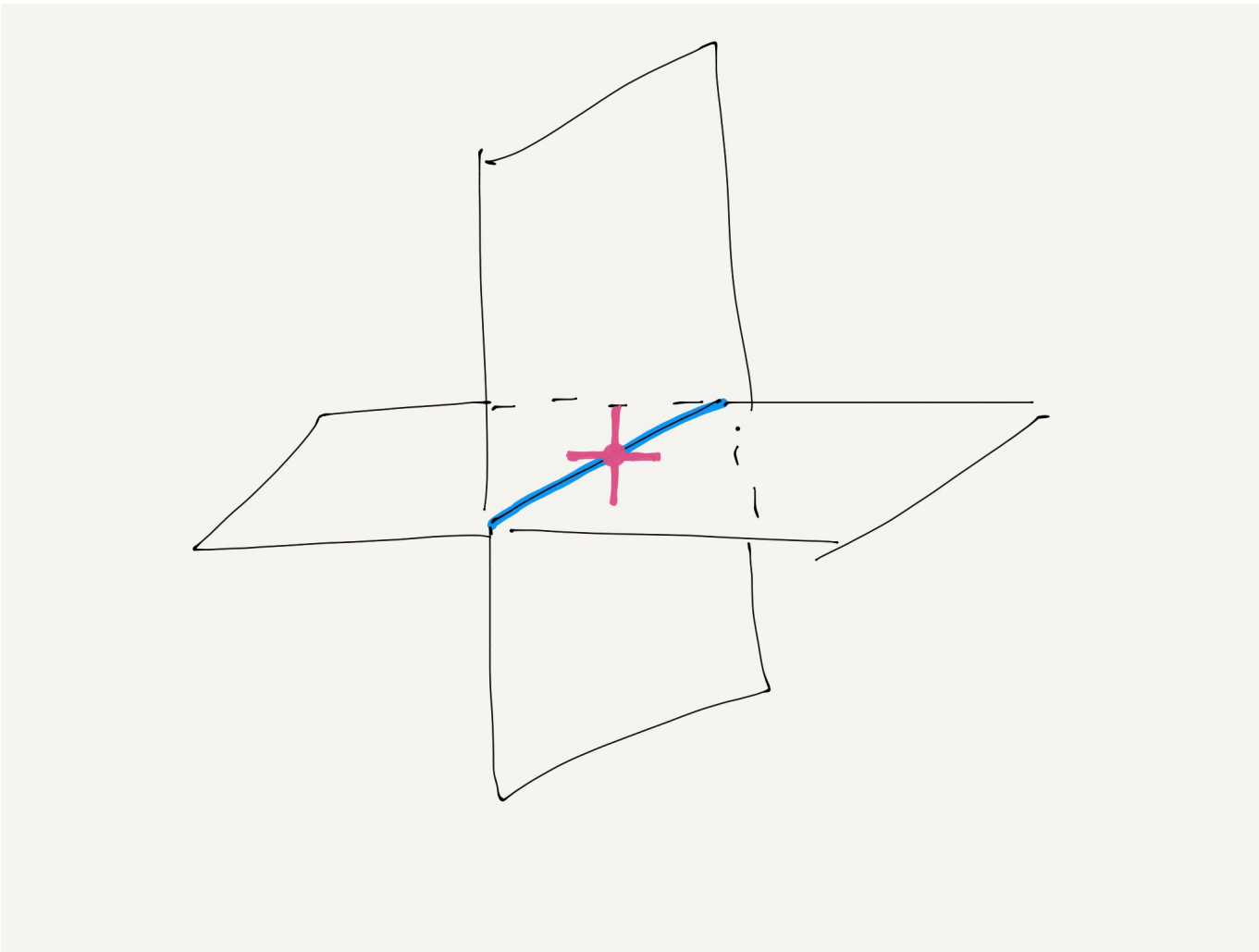

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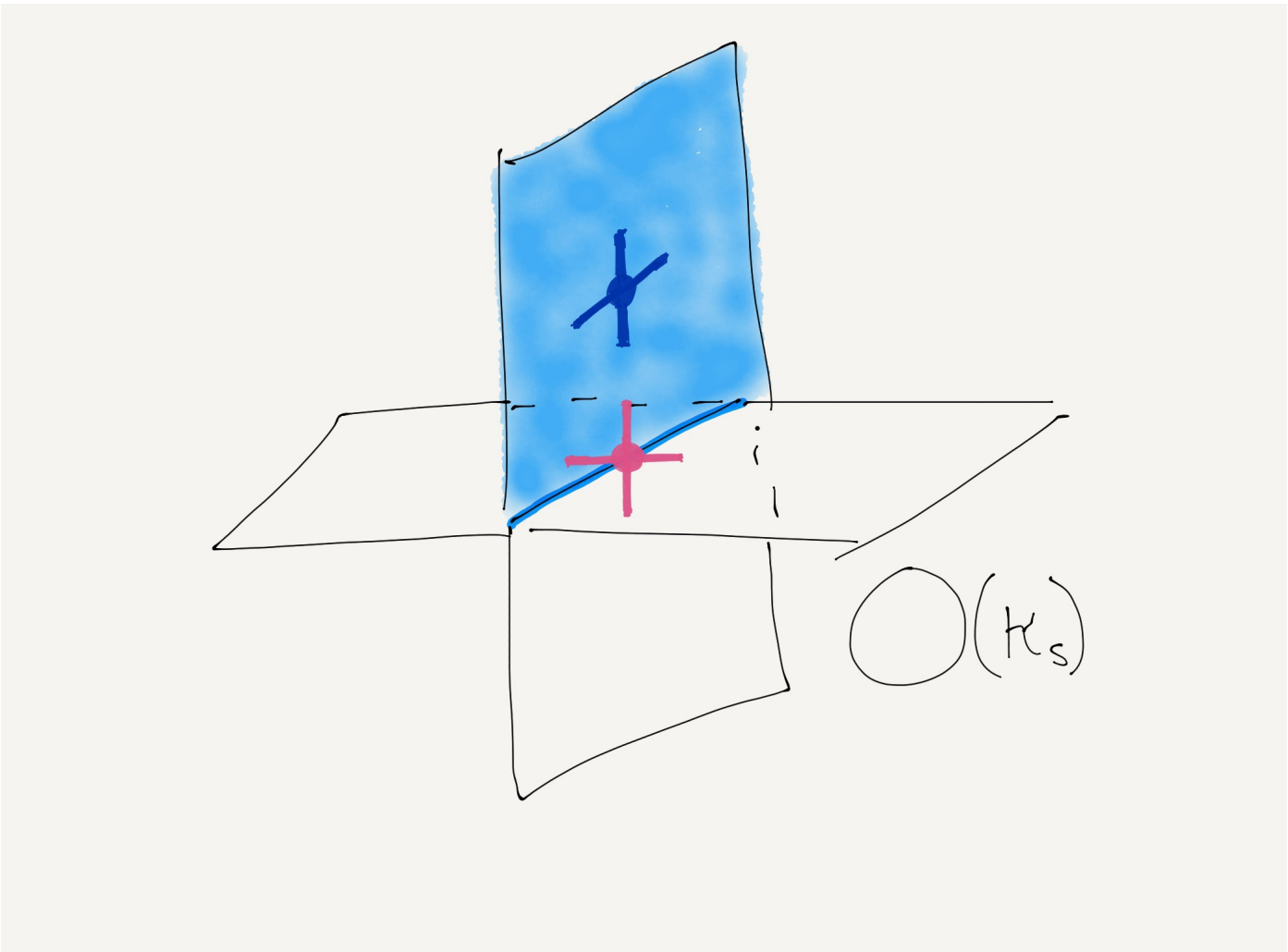

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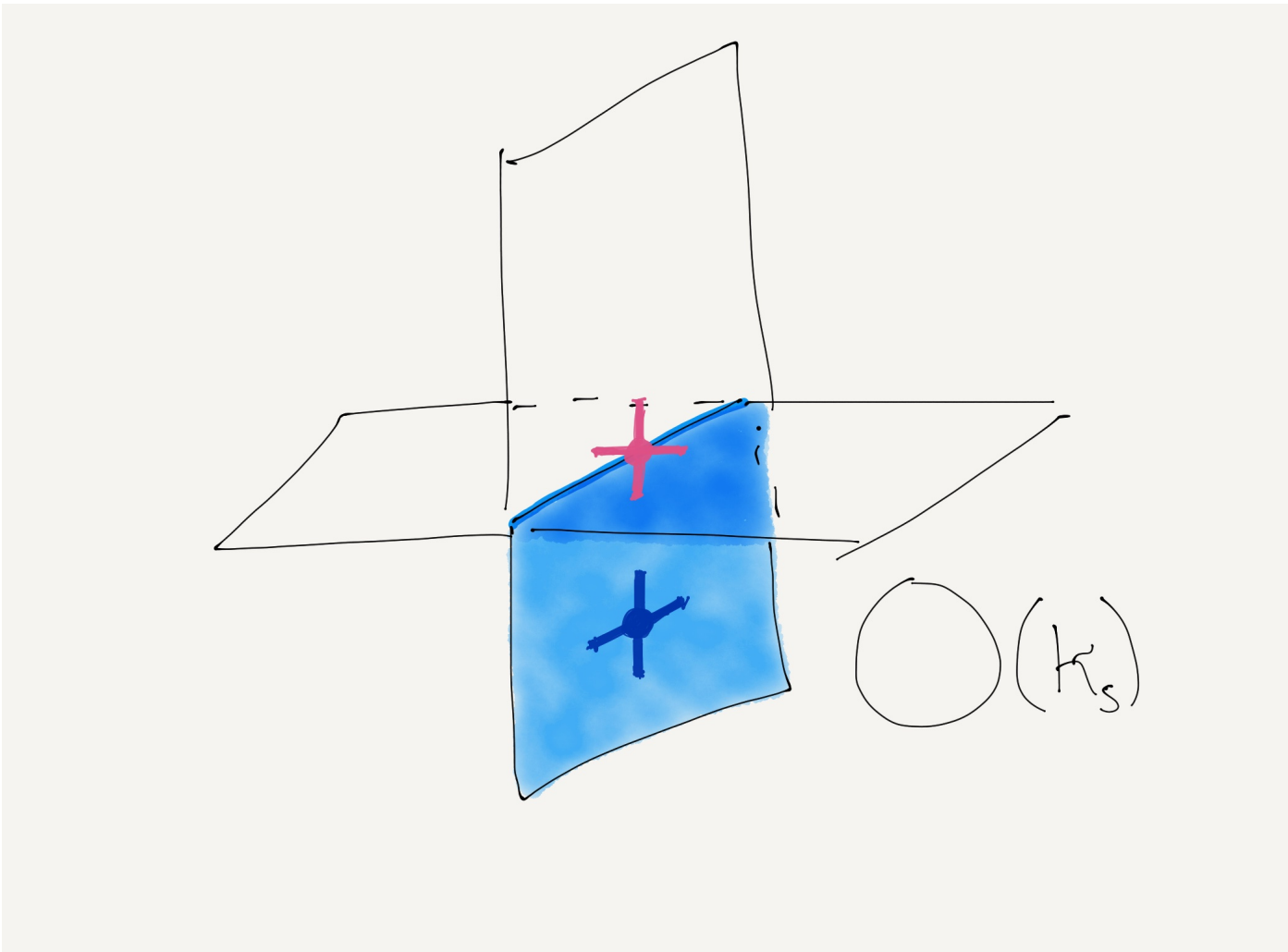

# Off-diagonal elements

$$H_{\text{off}} = -\underbrace{\frac{\beta_{\text{space}}}{2\tau}}_{W/2} \sum_x (U^+_{x,1} U^+_{x,2} U^-_{x,3} U^-_{x,4} + \text{h.c.})$$
$$-\underbrace{\frac{\kappa_{\text{space}}}{2\tau}}_{Z/2} \sum_x \sum_i (U^+_{x,i} + U^-_{x,i})$$

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# Final

$$H = \frac{U}{2}\sum_x\sum_i (L^z_{x,i})^2 + \frac{Y}{2}\sum_x (\vec{\nabla}\cdot\vec{L^z})^2_x$$
$$- \frac{W}{2}\sum_x (U^+_{x,1}U^+_{x,2}U^-_{x,3}U^-_{x,4} + \text{h.c.})$$
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Thank you!

# References I

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