Quantum links and beyond

David Berenstein, UCSB

Quantum Computing Methods for High Energy Physics, MIAPBP

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What we want

- Gauge theories in 3+1 on a quantum computer (e.g., QCD).
- Other field theories are also interesting!
- Many times very similar ingredients appear.
- Low dimensional models can be used to learn how to do quantum computations better.

Hamiltonian formulation of lattice gauge theory

Kogut-Susskind formulation

Discrete space, continuous time.

$A_0 = 0$

Group (Gauge) transformations act on left and right differently (two group actions)

$$U \to gUh^{-1}$$

$$\vec{v}_L = -U^{-1}\dot{U}$$

$$\vec{v}_R \sim \dot{U}U^{-1}$$

Velocity for Lagrangian can be chosen
right invariant
or left invariant:
they are valued in the adjoint and are
equivalent by similarity transformation
using U

Classical variables: unitaries (group elements) on links
+ canonical conjugates (electric fields)

Canonical conjugates generate gauge transformations on links: on both ends of the link (Left vs right multiplication).
They are the Lie generators of the group action.

$$\mathcal{H} = g_E \sum_{links} E^2 + g_\square \sum_{\square} Plaquette$$

When quantizing, turn everything into operators plus we need wave functions.

Link Boson problem

- Hilbert space for bosons is locally infinite dimensional.
- Even for a compact field (sigma model) we get locally $L^2(\mathscr{M})$: wave function of all possible values of local field.
- We obviously need to truncate somehow.
- May have symmetries: can they be realized in truncation?

We usually think in position representation

$$\psi(U) \in L^2(U)$$

- Plan A: Discretizing U (as a group) only works for U(1): the other non-abelian gauge groups do not have discrete subgroups that can become dense in a limit (can’t extrapolate to continuum). Equivalent to triangulating a manifold.

- Plan B: Can discretize the space of wave functions directly instead and keep gauge symmetries. This is where one gets away from standard Monte-Carlo (This approach has a lot of history). Route to harmonic analysis.

This has been known since forever, e.g.

T. Byrnes and Y. Yamamoto [quant-ph/0510027]

E.~Zohar and M.~Burrello, arXiv:1409.3085 [quant-ph]

David B. Kaplan and Jesse R. Stryker arXiv:1806.08797 [hep-lat]

Randy Lewis and R.M. Woloshyn arXiv:1905.09789 [hep-lat]

To keep group action we decompose Hilbert space into representations of gauge transformations

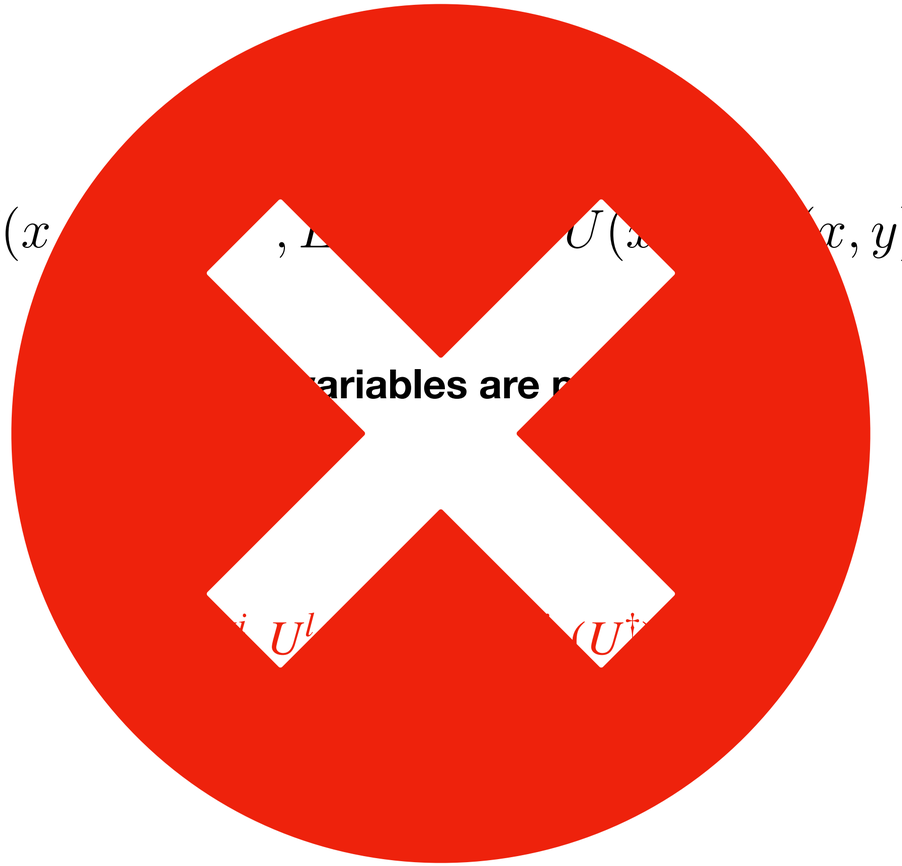
Peter-Weyl theorem

$$L^2(U) \simeq \oplus_R R \otimes \bar{R}$$

We need to sum over all irreps of R.
This is the electric basis.

Algebra problem

$$U^\dagger(x,y)U(x \quad , \quad U(x \quad (x,y)U^\dagger(x,y)$$



Left and right Lie algebras are the correct quantization of momenta

$$[E_L^\alpha, E_L^\beta] = i f_\gamma^{\alpha\beta} E_L^\gamma \quad , \quad [E_R^\alpha, E_R^\beta] = i f_\gamma^{\alpha\beta} E_R^\gamma \quad , \quad [E_L^\alpha, E_R^\beta] = 0$$

Representation of U under the left and right Lie algebra: generates group transformations properly: need to treat U and its adjoint as independent variables.

$$[E_L^\alpha, U] = \lambda_\alpha U \quad , \quad [E_R^\alpha, U] = -U\lambda_\alpha \quad , \quad [E_L^\alpha, U^\dagger] = -U^\dagger \lambda_\alpha \quad , \quad [E_R^\alpha, U^\dagger] = \lambda_\alpha U^\dagger$$

We call this the symplectic algebra: *IF WE KEEP BOTH,WE KEEP THE FULL* $L^2(G)$

The Hilbert space

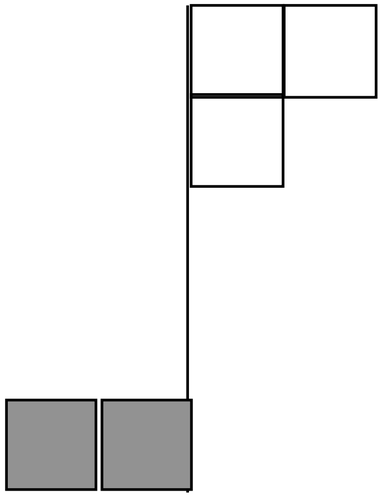
$$|0\rangle \equiv \psi(g) = 1$$

There is a ground state (trivial wave function)

Other wave functions are built by

Polynomials in components of $U^A{}_j, (U^\dagger)^i{}_B$

U adds boxes (electric flux), it's conjugate adds anti-boxes.
They can also remove antiboxes or boxes.



For U(N): Labeled by two Young diagrams whose lengths add lo less than or equal to N

Hilbert space is generated by the algebra acting on the ground state: we need interesting algebra realizations.

The kinetic (electric) term is the quadratic Cassimir of R.

Higher representations cost more energy (bigger Cassimir)

Can truncate to a finite Hilbert space for low energy effective field theory: need to put cutoffs in Cassimir.

Matrix elements of components of U, etc, can be hard to compute.

- Most of the next discussion is based on 2002.10028, 2201.02412, D.B., R. Brower, H. Kawai

- See also 2112.02090, H. Liu, S. Chandrasekharan.

GOALS

- FIND SIMPLE REALIZATIONS OF THE SYMPLECTIC ALGEBRA.
- CHECK THAT IN A PARTICULAR LIMIT ONE RECOVERS KOGUT-SUSSKIND HAMILTONIAN
- Are there any additional unwanted states?

Quantum links (D-theory, Rishons, …)

R. Brower, S. Chandrasekharan, U.J. Wiese, hep-th/9704106,
B.B. Beard, R.C. Brower, S. Chandrasekharan, D. Chen, A. Tsapalis et al.
hep-lat/9709120 ,
R. Brower, S. Chandrasekharan, S. Riederer,
U.J. Wiese, hep-lat/0309182

Main idea

Both E, U are composite.
We just satisfy the algebra relations we want to keep.

E Lie algebra, U bifundamental
(particular irrep of GxG: transforms correctly) .

Realize left algebra on fundamental fermions, or bosons (for example U(N))

$$\widehat{E}^i_{L,j} = (\vec{a}^\dagger \cdot \vec{a})^i_j = \sum_m (a^{\dagger m} a_m)^i_j$$

Flavor index: could have more than one such fermion.

Same trick works for right generators, with a different set of fermions b.

The object that transforms correctly under both transformations is composite:

$$U^i_j \to \widehat{U}^i_j = \vec{a}^i \cdot \vec{b}^\dagger_j = \sum_m (a^{\dagger m} b_m)^i_j$$

Original idea of (pick your name): D-theory, quantum links, Rishons,

Bar, Beard, Brower, Chandrasekharan, Chen, Schlittgen, Wiese ….. set of papers between 1997-2001

Models differ on how the flavor indices are treated (extra dimension, only one flavor, gauged…)

Notice sums over flavors look like a singlet under flavor symmetry in every object:
in some of our papers we gauged the flavor symmetry.
In some others, m is treated as an extra dimension.
If extra dimension, there are extra degrees of freedom (KK modes).

We can dispense with the fermions

$$E_L + E_R + U + U^\dagger \rightarrow Lie(U(2N))$$

Work with a single representation of U(2N):
which one?

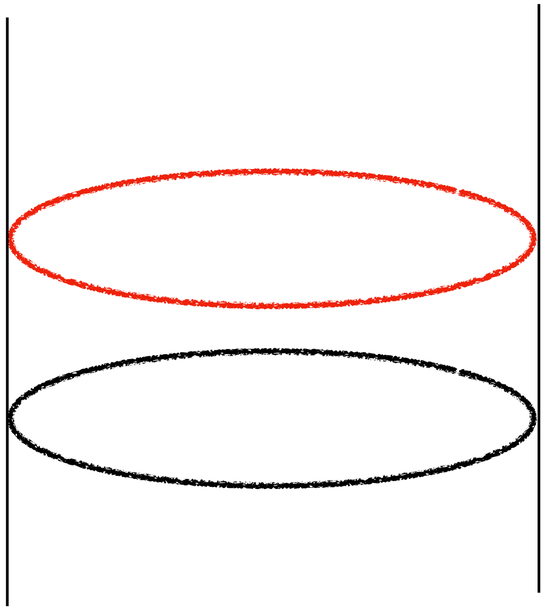
Simplest example: U(1)

With bifermions we get U(2).
Diagonal U(1) is central, so it acts as a c-number.

Quantization based upon the fermion intuition
gives us the Lie algebra of SU(2) on a fixed representation.
For bosons this would also work for U(1).

Different motivation

Classical variable is a coordinate on a circle.
The phase space associated to this setup is the tangent space of U(1): the cylinder



Momentum generates orbits at fixed p: copies of the circle

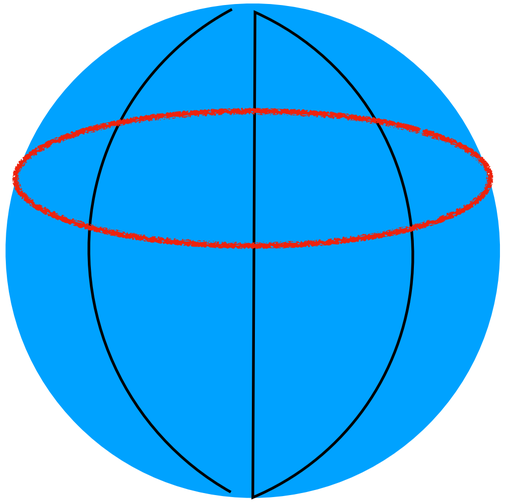
$$p \sim -i\partial_\theta$$

Infinite Hilbert space is due to infinite phase space volume

$$U = \exp(i\theta)$$

Is there another symplectic geometry with U(1) actions and finite volume?

YES: Sphere



$$p \sim L_z$$

We need

$$[p, U] = U$$

Suggests $U \to aL^+$

Need to pick correct normalization.

Torus doesn't work: generator of rotations is not a Hamiltonian function on torus. (Topological fact)

Does this actually work?
Can we recover Kogut-Susskind?

Hilbert space of cylinder

$$L^2(S^1) \sim \sum a_n \exp(in\theta)$$

Decomposition in charge is Fourier expansion

$$\exp(i\theta) * \exp(in\theta) = \exp(i(n+1)\theta)$$

What happens when we multiply by U

$$U|n\rangle = |n+1\rangle$$

On Angular momentum

$$aL^{+}\,|\,m\rangle = a\sqrt{(j-m)(j+m+1)}\,|\,m+1\rangle$$

Want to have same matrix elements as U at m=0.
(We need j integer spin: insures charge conjugation invariance).

$$a \to \frac{1}{\sqrt{j(j+1)}}$$

Corrections to matrix elements are of order $m^2/j^2 \sim E^2U/E^2_{max}$

Way to think about it

Truncated Kogut-Susskind + irrelevant operators in effective field theory

Recover Kogut-Susskind if we take $j \to \infty$

And keep the kinetic energy finite (low energy physics).
In gauged fermion flavor model
2j= number of flavors.

Hopefully: same universality class for reasonable j (2,3,4?).

Going to U(N)

First we need to double the algebra to U(2N)

Morphism of fermionic algebras: change raising for lowering operators.
Only works for fermions. Bosons will give a different answer.

$$a^j_m \leftrightarrow c^{j\dagger}_m \text{ and } a^{\dagger m}{}_j \leftrightarrow c^m_j$$

Some of the U generators become pure lowering or pure raising:

$$U^\dagger \propto a^\dagger \cdot b \equiv c \cdot b.$$

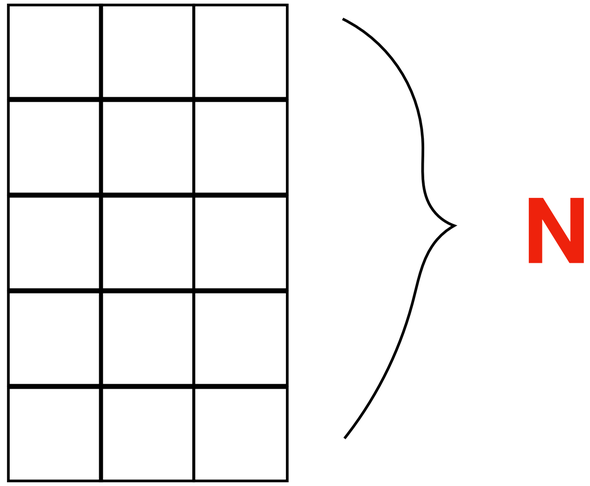
Canonical b,c vacuum

$$b\,|\,\Omega\rangle = c\,|\,\Omega\rangle = 0$$

Invariant under $SU(N) \times SU(N)$

So will be the highest weight state of U(2N) in the corresponding representation.

IRREPS OF SU(2N) with this property



Parametrized by a single integer parameter: the number of horizontal boxes.

Suggestive

$$S^2 \sim \mathbb{C}P^1 \equiv U(2)/U(1) \times U(1)$$

Is the compact phase space for U(1). The magnetic flux on the sphere is 2j.

There is a Grassmanian of complex N planes in 2N complex dimensions

$$Gr(N,2N) \sim U(2N)/U(N) \times U(N)$$

Shows up in a coherent state representation of action in an extra dimension

B. Schlittgen and U. J. Wiese, Phys. Rev., D63:085007, 2001

Special case U(2)

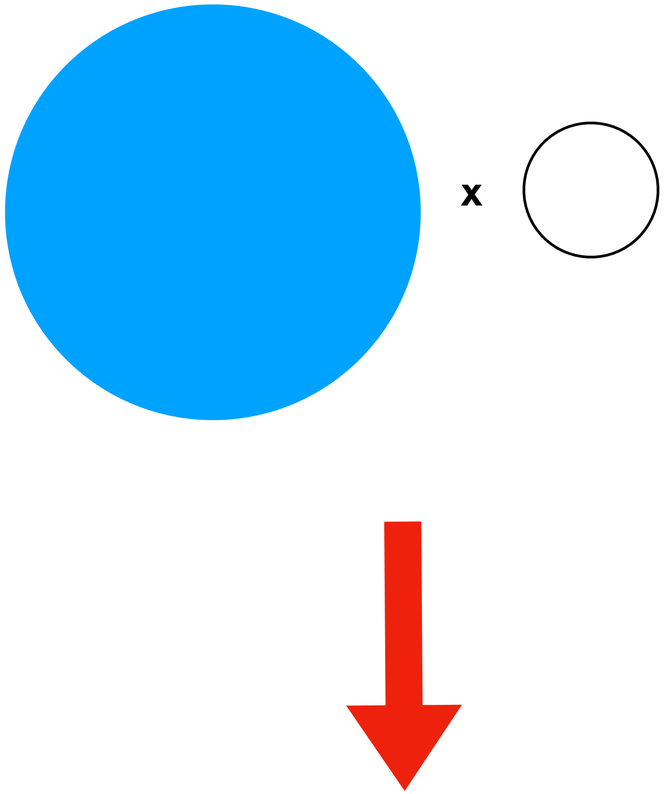
$$SU(4) \equiv SO(6)$$

These representations are symmetric traceless irreps. of SO(6).

This is actually spherical harmonics on a five sphere.

We are working at FIXED ℓ

Work in progress with S. Wang (23xx.????)

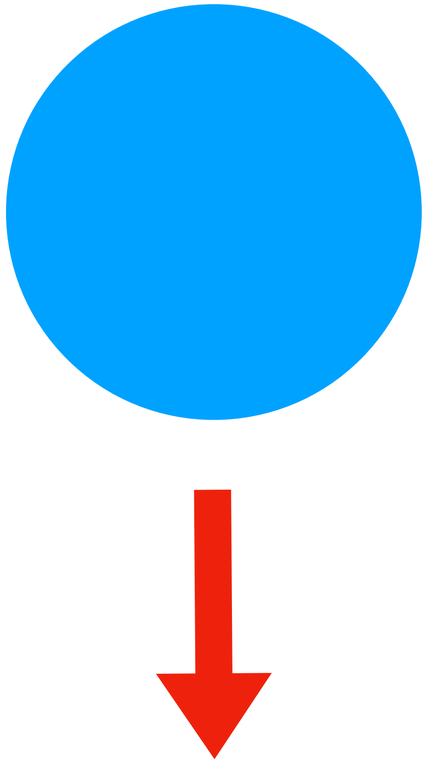


3-Sphere times circle
fibration over an interval. U(2) manifold
over a circle.

Can separate variables into
harmonics on sphere times
harmonic on circle, times “radial” direction.

Basically it shows that each representation
we want appears only once: no extra states.

Can play same trick on 4-sphere.



Three sphere
fiber over circle

Algebra: SO(5).
Showed up in
papers by U. Wiese.

We get the correct list
of representations with a cutoff.

Applications: sigma models

$$\sum E_i^2 + \lambda \sum_{<ij>} |(U_i - U_j)|^2$$

O(2) model
or U(N) model with qubits

Preserve the global symmetries under U(N)x U(N)

These are building blocks for local fields, not just gauge links.

Or gauge fields

Gauge systems: U lives on links, and we use plaquettes that are invariant under local symmetries.

Universality

If our goal is about universality of IR,
it means we do not need to get everything on the nose in the UV:
we just need to be in the basin of attraction of IR fixed point
(near phase transition).

For lattice QCD one can think of the extra terms as adding improvements
to the UV action: clovers, etc. So long as they are “irrelevant” one should be safe.

They can be useful numerically. In quantum computers,
they may reduce number of qubit gates to be executed.

Canonical example

Pure U(1) GT in 2+1

Link: simplest representation of U(2), one qubit.

Plaquette

$$Tr(\prod_{e} \sigma_e^+)$$

Expanded in Sigma chains for a plaquette with M edges, involves M qubit interactions.

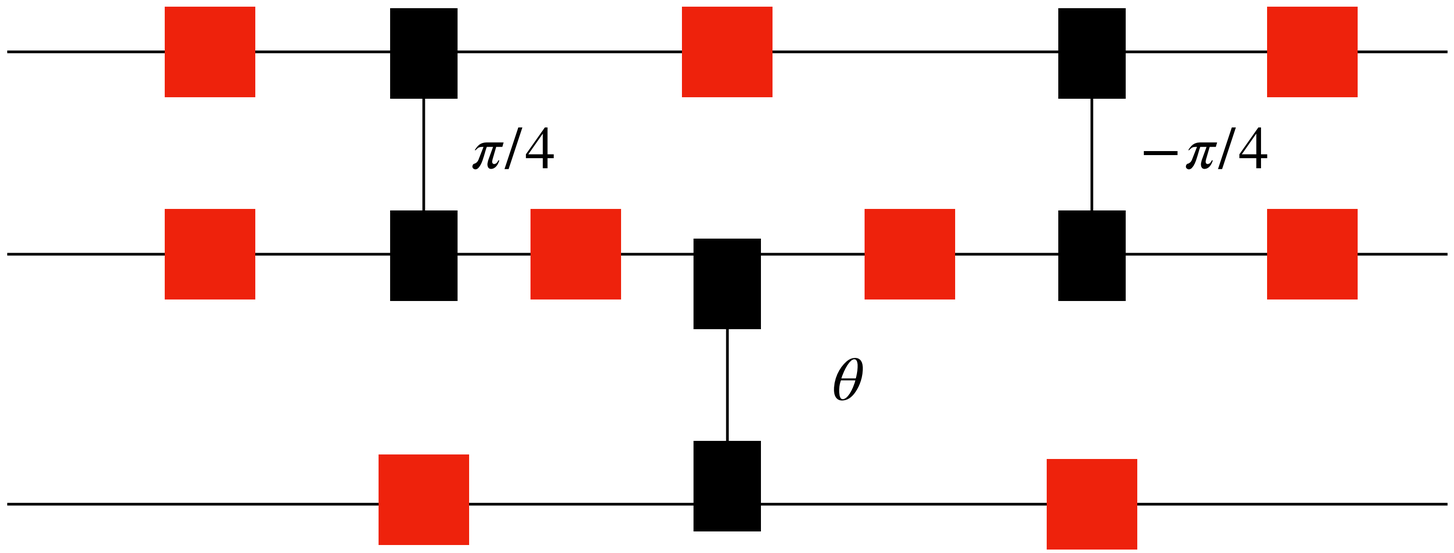
Want to go to smallest M: triangles.

$$Tr(U_{\triangle}) + cc \simeq XXX - XYY - YXY - YYX$$

These all commute with each other: can do pieces at a time for Trotterization.

Identity:

$$U_\theta(ZZZ) = \exp(i\pi/4(Z \otimes X \otimes 1))\exp(i\theta(1 \otimes Y \otimes Z))\exp(-i\pi/4(Z \otimes X \otimes 1))$$



Takes three two qubit ZZ gates.

Sometimes counted as 3 CNOT gates.

This is the Z2 plaquette!

For full plaquette: 8 two qubit gates (expensive): there are cancelations from ordering correctly.

On a square, 8 possible sigma chains that commute: 8 ZZZZ pieces.

A single ZZZZ gate requires 5 2-qubit gates

Corollary: Everyone should be working on triangular lattices.

Current problem of quantum computing: Running against the clock

Buy fish in fish market, with no refrigerator.

Need to cook the fish quickly, or it will spoil.

Mitigation

This is the main reason to do pre-processing (Ciavarella, Klco, Savage) and going to basis where gauge constraints are solved: both fewer qubits, hopefully fewer gates per link.

No such luck for sigma models. At least no plaquete.

On the other hand, plaquettes also show up in models of fractons and no residual gauge symmetry.

Higher representations are also more costly:

Trick: have small representation in extra dimension, and hope for coherence: interactions in extra dimension must be ferromagnetic.

Cross fingers that KK modes are sufficiently gapped in Trotterization: need to tune parameters correctly.

Conclusions

- Link variables can be composites of fermions. Suggests that one look at “compactifications of U(N) phase space”, eg. Grassmanians.
- Simple systems can be geometrized on spherical harmonics on sphere.
- Gauging flavor symmetry of the “rishons” gives a truncation without extra states.
- Links can also be used as building blocks for sigma models, etc.
- Depth cost of plaquette is one of the main obstacles to overcome.