Introduction to Effective Field Theory

Lecture 2: Hilbert Series

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Hilbert Series

Generating function for <u>numbers of invariants</u> with respect to group *G*

For $\begin{pmatrix} \text{bosonic} \\ \text{fermionic} \end{pmatrix}$ fields $\begin{pmatrix} \phi \\ \psi \end{pmatrix}$, invariants should $\begin{pmatrix} \text{symmetrized} \\ \text{antisymmetrized} \end{pmatrix}$ in the fields

Representations of the group *G*: $\phi \longrightarrow R_b(g) \,.\, \phi$

$$\psi \longrightarrow R_f(g) \,.\, \psi$$

Hilbert series for fields ϕ, ψ are functions of <u>complex variables</u> (spurions)

which can we denote by same symbols ϕ, ψ

They can be expressed as averages over the group *G*

bosonic field ϕ: $H(\phi) = \left\langle \dfrac{1}{\det[\mathbb{1} - \phi\, R_b(g)]} \right\rangle_G$

$H(\phi)$ can be expanded as a series in ϕ: $H(\phi) = \displaystyle\sum_{p=0}^{\infty} N_p\, \phi^p$

N_p is the <u>number of invariants</u> that can be formed from *p* factors of ϕ

fermionic field ψ: $H(\psi) = \left\langle \det\left[\mathbb{1} + \psi\, R_f(g)\right] \right\rangle_G$

Hilbert Series

Generating function for numbers of invariants with respect to group G

Representations of the group G: $\phi \longrightarrow R_b(g)\,.\,\phi$

$\psi \longrightarrow R_f(g)\,.\,\psi$

Hilbert series can be expressed as average over G of plethystic exponential

bosonic field ϕ: $$H_{R_b}(\phi) = \left\langle \frac{1}{\det[\mathbb{1} - \phi\, R_b(g)]} \right\rangle_G$$

$$= \left\langle \exp\left(\sum_{n=1}^{\infty} \frac{1}{n}\, \phi^n\, \mathrm{Tr}\left[R_b(g^n)\right] \right) \right\rangle_G$$

fermionic field ψ: $$H_{R_f}(\psi) = \left\langle \det\left[\mathbb{1} + \psi\, R_f(g)\right] \right\rangle_G$$

$$= \left\langle \exp\left(\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}\, \psi^n\, \mathrm{Tr}\left[R_f(g^n)\right] \right) \right\rangle_G$$

Plethystic exponentials are expressed in terms of characters $\mathrm{Tr}\left[R_b(g^n)\right]$, $\mathrm{Tr}\left[R_f(g^n)\right]$

Hilbert Series

Hilbert series for group *G* acting on direct sum $(\phi, \psi, \psi^\dagger)$

bosonic field ϕ, fermionic field ψ

Representations of the group *G*: $\phi \longrightarrow R_b(g)\,.\,\phi$

$$\psi \longrightarrow R_f(g)\,.\,\psi$$

$$\psi^\dagger \longrightarrow \psi^\dagger\,.\,R_f(g^\dagger)$$

Characters are additive:

$$\mathrm{Tr}\big[R\big(g^n\big)\big] = \mathrm{Tr}\big[R_b\big(g^n\big)\big] + \mathrm{Tr}\big[R_f\big(g^n\big)\big] + \mathrm{Tr}\big[R_f\big(g^\dagger\big)^n\big]$$

Plethystic exponentials are multiplicative:

$$H_R(\phi,\psi,\psi^\dagger) = \Bigg\langle \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\,\phi^n\,\mathrm{Tr}\big[R_b(g^n)\big]\right) \times \exp\left(\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\,\psi^n\,\mathrm{Tr}\big[R_f(g^n)\big]\right) \times \exp\left(\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\,\psi^{\dagger n}\,\mathrm{Tr}\big[R_f(g^{\dagger n})\big]\right)\Bigg\rangle_G$$

Hilbert Series

Hilbert series for direct product group $G = G_1 \times G_2$

Representations of the group *G*: $\phi \longrightarrow R_b(g_1 \times g_2)\,.\,\phi$

$$\psi \longrightarrow R_f(g_1 \times g_2)\,.\,\psi$$

Characters are multiplicative: $\mathrm{Tr}\left[R\left((g_1 \times g_2)^n\right)\right] = \mathrm{Tr}\left[R\left(g_1^n\right)\right]\,\mathrm{Tr}\left[R\left(g_2^n\right)\right]$

Plethystic exponentials

bosonic field ϕ: $H_{R_b}(\phi) = \left\langle \exp\left(\sum_{n=1}^{\infty} \frac{1}{n}\,\phi^n\,\mathrm{Tr}\left[R_b\left(g_1^n\right)\right]\,\mathrm{Tr}\left[R_b\left(g_2^n\right)\right] \right) \right\rangle_{G_1 \times G_2}$

fermionic field ψ: $H_{R_f}(\psi) = \left\langle \exp\left(\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}\,\phi^n\,\mathrm{Tr}\left[R_f\left(g_1^n\right)\right]\,\mathrm{Tr}\left[R_f\left(g_2^n\right)\right] \right) \right\rangle_{G_1 \times G_2}$

Hilbert Series

Generating function for numbers of invariants with respect to group *G*

Characters $\mathrm{Tr}\left[R(g^n)\right]$ are invariant under conjugation in *G*: $g \longrightarrow h\,g\,h^{-1}$

$$\mathrm{Tr}\left[R(g^n)\right] = \mathrm{Tr}\left[R\left(\left(hgh^{-1}\right)^n\right)\right]$$

If the group *G* is connected, conjugation $h\,g\,h^{-1}$ can be used to move any element *g* to a maximal abelian subgroup $U(1)^{r_G}$, where r_G is the rank of *G.*
The average over *G* can be replaced by an average over $U(1)^{r_G}$.

bosonic field ϕ: $H_{R_b}(\phi) = \left\langle \exp\left(\sum_{n=1}^{\infty} \frac{1}{n}\, \phi^n\, \mathrm{Tr}\left[R_b(g^n)\right] \right) \right\rangle_{U(1)^{r_G}}$

fermionic field ψ: $H_{R_f}(\psi) = \left\langle \exp\left(\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}\, \psi^n\, \mathrm{Tr}\left[R_f(g^n)\right] \right) \right\rangle_{U(1)^{r_G}}$

Hilbert Series

Hilbert series can be expressed as average over the group *G*

Average over a connected group *G*
can be expressed in terms of an integral over the G-invariant Haar measure.

Average over the maximal abelian subgroup $U(1)^{r_G}$, where r_G is the rank of *G*,
can be expressed as an integral over r_G real coordinates on $U(1)^{r_G}$.

$U(1)$ group $\phi \longrightarrow y^Q \phi$, where $y = e^{i\theta}$

measure: $d\mu_{U(1)} = \frac{1}{y} dy$

integral: $\int d\mu_{U(1)} = \frac{1}{2\pi i} \oint \frac{1}{y} dy = 1$

$SU(2)$ group

maximal abelian subgroup: $U(1)$ $\phi \longrightarrow y^{2I_3} \phi$, where $y = e^{i\theta}$

measure: $d\mu_{SU(2)} = \frac{1}{2y}(1 - y^2)(1 - 1/y^2)\, dy$

integral: $\int d\mu_{SU(2)} = \frac{1}{2\pi i} \oint \frac{1}{2y}(1 - y^2)(1 - 1/y^2)\, dy = 1$

Hilbert Series

Hilbert series can be expressed in terms of characters of the group *G*

Characters for representation *R*: $\chi_R(g^n) = \mathrm{Tr}\big[R(g^n)\big]$

If the group *G* is connected, conjugation $h\,g\,h^{-1}$ can be used to move any element *g* to a maximal abelian subgroup $U(1)^{r_G}$, where r_G is the rank of *G.*

The characters of group elements in the maximal abelian subgroup $U(1)^{r_G}$ are functions of the r_G real coordinates on $U(1)^{r_G}$.

$U(1)$ group: $\phi \longrightarrow y^Q\,\phi$

measure: $d\mu_{U(1)} = \frac{1}{y}dy$

Characters: $\chi_Q(y) = y^Q \quad \chi_Q(y^n) = y^{nQ}$

$SU(2)$ group: $\phi \longrightarrow y^{2I_3}\,\phi$

measure: $d\mu_{SU(2)} = \frac{1}{2y}(1-y^2)(1-1/y^2)\,dy$

characters: $\chi_1(y) = 1$ $\quad \chi_1(y^n) = 1$

$\chi_2(y) = \chi_{2*}(y) = y + 1/y \quad \chi_2(y^n) = \chi_{2*}(y^n) = y^n + 1/y^n$

$\chi_3(y) = y^2 + 1 + 1/y^2 \quad \chi_3(y^n) = y^{2n} + 1 + 1/y^{2n}$

Hilbert Series for constant fields

Space-time independent fields

Internal symmetry group *G* only: $U(1), SU(2), \ldots$

<u>Real bosonic field</u> ϕ

trivial symmetry group: $G = \{1\}$

Hilbert series: $H(\phi) = \exp\left(\Sigma_{n=1}^{\infty}(1/n)\phi^n\right)$

$$= \exp\Big(- \log(1-\phi)\Big)$$

$$= 1/(1-\phi)$$

expand in powers of ϕ:

$$H(\phi) = 1 + \phi + \phi^2 + \phi^3 + \phi^4 + \phi^5 + \ldots$$

$\Longrightarrow$ one invariant for each power *p*

Hilbert Series for constant fields

Space-time independent fields

Internal symmetry group *G* only: $U(1), SU(2), \ldots$

Fermionic field $\psi, \psi^\dagger$

$U(1)$ symmetry: $\psi \longrightarrow y\psi, \psi^\dagger \longrightarrow (1/y)\psi^\dagger$

Hilbert series:

$$H(\psi, \psi^\dagger) = \left\langle \exp\left(\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \psi^{\dagger n} \left(\frac{1}{y}\right)^n \right) \exp\left(\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \psi^n y^n \right) \right\rangle_{U(1)}$$

$$= \frac{1}{2\pi i} \oint \frac{1}{y} \exp\left(\log\left(1 + \psi^\dagger \frac{dy}{y}\right)\right) \exp\left(\log\left(1 + \psi\right)\right)$$

$$= \frac{1}{2\pi i} \oint \frac{dy}{y} \left(1 + \psi^\dagger \frac{1}{y}\right)\left(1 + \psi\, y\right)$$

$$= 1 + \psi^\dagger \psi$$

$\Longrightarrow$ only invariant is $\psi^\dagger \psi$

Hilbert Series for constant fields

Complex bosonic field ϕ, ϕ^*

$U(1)$ symmetry: $\phi \longrightarrow y\phi,\ \phi^* \longrightarrow (1/y)\phi^*$

Hilbert series:

$$H(\phi,\phi^*) = \left\langle \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\phi^{*n}\left(\frac{1}{y}\right)^n\right)\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\phi^n y^n\right)\right\rangle_{U(1)}$$

$$= \frac{1}{2\pi i}\oint\frac{dy}{y}\,\exp\left(-\log\left(1-\phi^*\frac{1}{y}\right)\right)\exp\left(-\log\left(1-\phi\, y\right)\right)$$

$$= \frac{1}{2\pi i}\oint\frac{dy}{y}\,\frac{1}{1-\phi^*/y}\frac{1}{1-\phi\, y}$$

$$= \frac{1}{2\pi i}\oint\frac{dy}{y}\,\sum_{n=0}^{\infty}\phi^{*n}\frac{1}{y^n}\sum_{n'=0}^{\infty}\phi^{n'}y^{n'}$$

$$= \frac{1}{2\pi i}\oint\frac{dy}{y}\,\sum_{n=0}^{\infty}\phi^{*n}\frac{1}{y^n}\phi^n y^n$$

$$= \frac{1}{1-\phi^*\phi}$$

expand in powers of $\phi^*\phi$:

$$H(\phi,\phi^*) = 1 + \phi^*\phi + (\phi^*\phi)^2 + (\phi^*\phi)^3 + \dots$$

$\Longrightarrow$ one invariant for each power *p*

Hilbert Series for fields

Internal symmetry group *G*
Lorentz symmetry group $SO(3,1)$

Hilbert series can be expressed as average of plethystic exponential over the group $G \times SO(3,1)$

bosonic field: $\phi \longrightarrow R_b(g_1 \times g_2) \, . \, \phi$

$$H_{R_b}(\phi) = \left\langle \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \phi^n \, \mathrm{Tr}\left[R_b(g_1^n)\right] \, \mathrm{Tr}\left[R_b(g_2^n)\right] \right) \right\rangle_{G\times SO(3,1)}$$

fermionic field: $\psi \longrightarrow R_f(g_1 \times g_2) \, . \, \psi$

$$H_{R_f}(\psi) = \left\langle \exp\left(\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \psi^n \, \mathrm{Tr}\left[R_f(g_1^n)\right] \, \mathrm{Tr}\left[R_f(g_2^n)\right] \right) \right\rangle_{G\times SO(3,1)}$$

Hilbert Series for fields

For functions invariant under conjugation $g \longrightarrow h\, g\, h^{-1}$,
average over connected group *G* is equal to average over maximal abelian subgroup

Lorentz group: $SO(3,1)$

generators of $SO(3,1)$ are also generators of $SU(2)_L \times SU(2)_R$

average over $SU(2)_L \times SU(2)_R$ can be expressed as

average over subgroup $U(1)_L \times U(1)_R$

integral over two real coordinates on $U(1)_L \times U(1)_R$

$SO(3,1)$ can be obtained by analytic continuation of parameters of $SU(2)_L \times SU(2)_R$

average over $SO(3,1)$ can be expressed as

integral over two real coordinates on $U(1)_L \times U(1)_R$

measure: $$d\mu_{SO(3,1)} = \frac{1}{2x_1}(1 - x_1^2)(1 - 1/x_1^2)\, dx_1 \times \frac{1}{2x_2}(1 - x_2^2)(1 - 1/x_2^2)\, dx_2$$

integral: $$\int d\mu_{SO(3,1)} = \left(\frac{dx}{2\pi i} \oint \frac{1}{2x}(1 - x^2)(1 - 1/x^2)\, dx \right)^2 = 1$$

average of plethystic exponential over $SO(3,1)$ can be expressed as integral over x_1, x_2
evaluated by contour integration

Hilbert Series for real scalar field

Hilbert series involves integrals of functions of characters of the group $SO(3,1)$

Characters for representation *R of* $SO(3,1)$: $\chi_R(g^n) = \mathrm{Tr}\left[R(g^n)\right]$

characters of $SO(3,1)$ are more complicated
because Lorentz transformations mix fields with their derivatives

real scalar field $\phi(x)$

Lorentz transformation acts on infinite-dimensional vector space whose components are ϕ and all its derivatives

representation of $SO(3,1)$ is <u>reducible</u> because $\partial^2\phi = \partial_\mu\partial^\mu\phi$ transforms like a scalar

$$\begin{pmatrix} \phi \\ \partial_\mu\phi \\ \partial_\mu\partial_\nu\phi \\ \partial_\mu\partial_\nu\partial_\lambda\phi \\ \vdots \end{pmatrix}$$

<u>Irreducible</u> representation of $SO(3,1)$:

$$\begin{pmatrix} \phi \\ \partial_\mu\phi \\ \left[\partial_\mu\partial_\nu - \frac{1}{4}g_{\mu\nu}\partial^2\right]\phi \\ \left[\partial_\mu\partial_\nu\partial_\lambda - \frac{1}{6}\left(g_{\mu\nu}\partial_\lambda + g_{\nu\lambda}\partial_\mu + g_{\lambda\mu}\partial_\nu\right)\partial^2\right]\phi \\ \vdots \end{pmatrix}$$

Hilbert Series for real scalar field

Hilbert series involves integrals of functions of characters of the group $SO(3,1)$

real scalar field $\phi(x)$

irreducible representation of $SO(3,1)$

$$\begin{pmatrix} \phi \\ \partial_\mu \phi \\ \left[\partial_\mu \partial_\nu - \frac{1}{4} g_{\mu\nu} \partial^2\right] \phi \\ \left[\partial_\mu \partial_\nu \partial_\lambda - \frac{1}{6}\left(g_{\mu\nu}\partial_\lambda + g_{\nu\lambda}\partial_\mu + g_{\lambda\mu}\partial_\nu\right)\partial^2\right]\phi \\ \vdots \end{pmatrix}$$

Characters for representation *R of* $SO(3,1)$: $\chi_R(g^n) = \mathrm{Tr}\left[R(g^n)\right]$

character for irreducible representation

can be expressed as function of complex variable ∂ (spurion)

and function of coordinates x_1, x_2 on $U(1)_L \times U(1)_R$ manifold

$$\chi(\partial, x_1, x_2) = P(\partial, x_1, x_2)\,(1 - \partial^2)\,\partial$$

$$P(\partial, x_1, x_2) = \left(1 - x_1 x_2 \partial\right)\left(1 - \frac{x_1}{x_2}\partial\right)\left(1 - \frac{x_2}{x_1}\partial\right)\left(1 - \frac{1}{x_1 x_2}\partial\right)$$

takes into account Equation Of Motion redundancies

Hilbert Series for real scalar field

Integration By Parts redundancies

can be taken into account by inserting a factor $1/(P(\partial, x_1, x_2)$

into integral for average over $SO(3,1)$

$$P(\partial, x_1, x_2) = \left(1 - x_1\, x_2\, \partial\right)\left(1 - \frac{x_1}{x_2}\,\partial\right)\left(1 - \frac{x_2}{x_1}\,\partial\right)\left(1 - \frac{1}{x_1 x_2}\,\partial\right)$$

$SO(3,1)$ factor in Hilbert series:

$$\left\langle \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\,\phi^n\,\mathrm{Tr}\left[R_b(h^n)\right]\right)\right\rangle_{SO(3,1)} = \int d\mu_{SO(3,1)}\frac{1}{P(\partial, x_1, x_2)}\,\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\,\phi^n\,\chi(\partial, x_1^n, x_2^n)\right)$$

measure for integration over $U(1)_L \times U(1)_R$ manifold:

$$d\mu_{SO(3,1)} = \frac{1}{2x_1}(1 - x_1^2)(1 - 1/x_1^2)\,dx_1 \times \frac{1}{2x_2}(1 - x_2^2)(1 - 1/x_2^2)\,dx_2$$

correction term in Hilbert series

for operators with $d \leq 4$ (spacetime dimension):

real scalar field: $\phi(x)$

$$\Delta H = \phi\,\partial^2 - \partial^4$$

Hilbert Series for real scalar field

Hilbert series for real scalar field $\phi(x)$:

$$H(\phi, \partial) = \int d\mu_{SO(3,1)} \, \frac{1}{P(\partial, x_1, x_2)} \, \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \, \phi^n \, \chi(\partial, x_1^n, x_2^n) \right) + \Delta H$$

correction term: $\Delta H = \phi \, \partial^2 - \partial^4$

measure for integration over $U(1) \times U(1)$ manifold:

$$d\mu_{SO(3,1)} = \frac{1}{2x_1}(1 - x_1^2)(1 - 1/x_1^2) \, dx_1 \times \frac{1}{2x_2}(1 - x_2^2)(1 - 1/x_2^2) \, dx_2$$

factor $1/P(\partial, x_1, x_2)$ takes into account Integration By Parts redundancies

$$P(\partial, x_1, x_2) = \left(1 - x_1 \, x_2 \, \partial\right)\left(1 - \frac{x_1}{x_2}\,\partial\right)\left(1 - \frac{x_2}{x_1}\,\partial\right)\left(1 - \frac{1}{x_1 x_2}\,\partial\right)$$

Characters for irreducible representation of g^n on ϕ and its derivatives:

$$\chi(\partial, x_1^n, x_2^n) = P(\partial, x_1^n, x_2^n)\,(1 - \partial^2)\,\partial$$

$$P(\partial, x_1^n, x_2^n) = \left(1 - x_1^n \, x_2^n \, \partial\right)\left(1 - \frac{x_1^n}{x_2^n}\,\partial\right)\left(1 - \frac{x_2^n}{x_1^n}\,\partial\right)\left(1 - \frac{1}{x_1^n x_2^n}\,\partial\right)$$

takes into account Equation Of Motion redundancies

Hilbert Series for real scalar field

Hilbert series for real scalar field $\phi(x)$:

$$H(\phi,\partial) = \left(\phi\,\partial^2 - \partial^4\right) + \oint \frac{dx_1}{2\pi i}\frac{(1-x_1^2)(1-1/x_1^2)}{2x_1}\oint \frac{dx_1}{2\pi i}\frac{(1-x_2^2)(1-1/x_2^2)}{2x_2}$$
$$\times \frac{1}{P(\partial,x_1,x_2)}\;\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\,\phi^n\,\chi(\partial,x_1^n,x_2^n)\right)$$
$$P(\partial,x_1,x_2) = \left(1 - x_1x_2\,\partial\right)\left(1-(x_1/x_2)\,\partial\right)\left(1-(x_2/x_1)\,\partial\right)\left(1-(1/(x_1x_2))\,\partial\right)$$
$$\chi(\partial,x_1^n,x_2^n) = P(\partial,x_1^n,x_2^n)\,(1-\partial^2)\,\partial$$

- expand plethystic exponential and factor $1/P(\partial,x_1,x_2)$ in powers of ϕ and ∂
- evaluate contour intervals over x_1 and x_2

Expansion in powers of ϕ and ∂:

$$H(\phi,\partial) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} N_{pq}\,\phi^p\,\partial^q$$

coefficient N_{pq} is number of invariants with *p* powers of ϕ and *q* powers of ∂

Hilbert Series for real scalar field

Hilbert series for real scalar field $\phi(x)$:

expansion in powers of ϕ and ∂:

$$H(\phi, \partial) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} N_{pq}\, \phi^p\, \partial^q$$

Scaling dimension for scalar field $\phi(x)$: 1

derivative ∂_μ: 1

Expansion according to scaling dimension:

$$\begin{aligned} H(\phi, \partial) = 1 &+ \phi + \phi^2 + \phi^3 + \phi^4 + \phi^5 + \phi^6 + \phi^7 \\ &+(\phi^8 + \phi^4\partial^4) \\ &+(\phi^9 + \phi^5\partial^4) \\ &+(\phi^{10} + \phi^6\partial^4 + \phi^4\partial^6) \\ &+(\phi^{11} + \phi^7\partial^4 + \phi^5\partial^6) \\ &+(\phi^{12} + \phi^8\partial^4 + 2\phi^6\partial^6 + \phi^4\partial^8) + \dots \end{aligned}$$

1 invariant with scaling dimensions 1, 2, 3, 4, 5, 6, 7

2 invariants with scaling dimensions 8, 9

3 invariants with scaling dimensions 10, 11

5 invariants with scaling dimension 12

…

Hilbert Series for QPhD

Quantum PhotoDynamics

Low-energy Effective Field Theory for QED

describing photons with energies up to order m_e

Electromagnetic gauge field: $A_\mu(x)$ — scaling dimension: *d = 1*

Derivative: ∂_μ — *d = 1*

Electromagnetic field strength: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ — *d = 2*

Symmetries: U(1) gauge invariance $\Longrightarrow$ $\mathscr{L}$ depends on A_μ only through $F_{\mu\nu}$

Lorentz invariance $\Longrightarrow$ $\mathscr{L}$ is a Lorentz scalar

Charge conjugation *C* $\Longrightarrow$ $\mathscr{L}$ is even function of $F_{\mu\nu}$

Parity *P*

Permutation Identity: $\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$

d = 4 Lagrangian: $\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

Equation Of Motion: $\partial_\mu F^{\mu\nu} = 0$

Hilbert Series for QPhD

Low-energy Effective Field Theory for QED

Effective Lagrangian $\mathscr{L}$ is a function of field strength $F_{\mu\nu}$ and derivative ∂_μ
with all possible terms consistent with symmetries

Hilbert series $H(F, \partial)$ is function of complex variables *F* and ∂
can be expressed as average over $SO(3,1)$ of plethystic exponential

$$H(F,\partial) = \left\langle \frac{1}{P(\partial, g)} \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} F^N \chi_F(\partial, g) \right) \right\rangle_{SO(3,1)}$$

Character for irreducible representation of Lorentz transformations
of $F_{\mu\nu}$ and its derivatives: $\chi_F(g, \partial) = \mathrm{Tr}\left[R_F(g^n)\right]$

Express as integral over $U(1)_L \times U(1)_R$:

$$H(F,\partial) = \int d\mu_{SO(3,1)} \frac{1}{P(\partial, x_1, x_2)} \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} F^n \chi_F(\partial, x_1^n, x_2^n) \right)$$

expansion in powers of *F* and ∂:

$$H(F,\partial) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} N_{pq} F^p \partial^q$$

N_{pq} is number of terms with *p* powers of $F_{\mu\nu}$ and *q* powers of ∂_μ
after eliminating IBP redundancies and EOM redundancies

Scaling dimension: $d = 2p + q$

Hilbert Series for QPhD

Determine character of $F_{\mu\nu}$ from conformal field theory

chiral components of $F_{\mu\nu}$: $F_{L\mu\nu} = F_{\mu\nu} + i\,\epsilon_{\mu\nu\lambda\sigma}F^{\lambda\sigma}$

$$F_{R\mu\nu} = F_{\mu\nu} - i\,\epsilon_{\mu\nu\lambda\sigma}F^{\lambda\sigma}$$

$$\chi_F(\partial, x_1, x_2) = \left[\left(x_1x_2 + \frac{1}{x_1x_2} + \frac{x_1}{x_2} + \frac{x_2}{x_1} + 2\right) - 2\left(x_1 + \frac{1}{x_1} + x_2 + \frac{1}{x_2}\right)\partial + 2\,\partial^2\right]\partial^2\,P(\partial, x_1, x_2)$$

$$P(\partial, x_1, x_2) = \left(1 - x_1\,x_2\,\partial\right)\left(1 - \frac{x_1}{x_2}\,\partial\right)\left(1 - \frac{x_2}{x_1}\,\partial\right)\left(1 - \frac{1}{x_1x_2}\,\partial\right)$$

Hilbert Series:

$$H(F,\partial) = \int d\mu_{SO(3,1)}\frac{1}{P(\partial, x_1, x_2)}\,\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\,F^n\,\chi_F(\partial, x_1^n, x_2^n)\right)$$

$$d\mu_{SO(3,1)} = \frac{1}{2x_1}(1 - x_1^2)(1 - 1/x_1^2)\,dx_1 \times \frac{1}{2x_2}(1 - x_2^2)(1 - 1/x_2^2)\,dx_2$$

Hilbert series

Expand in powers of F and ∂

and then evaluate contour integrals over x_1, x_2

$$H(F,\partial) = 1 + F^2 + 2F^4 + 3F^4\partial^2 + (2F^6 + 3F^4\partial^4) + \ldots$$

scaling dimension 6: 0 terms

8: 2 terms

10: 3 terms

12: 5 terms