"Effective Field Theory and Lattice Field Theory" School - Lattice QCD: Assignments

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1 Exercise 1

Let ∂_μ and ∂^*_μ be the forward and backward lattice derivatives. Show that the generalized Leibniz rules

$$\partial_\mu \left[f(x)g(x)\right] = (\partial_\mu f(x))g(x) + f(x)(\partial_\mu g(x)) + a(\partial_\mu f(x))(\partial_\mu g(x))\,, \tag{1}$$

$$\partial^*_\mu \left[f(x)g(x)\right] = (\partial^*_\mu f(x))g(x) + f(x)(\partial^*_\mu g(x)) - a(\partial^*_\mu f(x))(\partial^*_\mu g(x))\,, \tag{2}$$

hols for any lattice functions g and g.

2 Exercise 2

Verify the identities

$$[\partial_\mu, \partial_\nu] = [\partial_\mu, \partial^*_\nu] = \left[\partial^*_\mu, \partial^*_\nu\right] = 0\,, \tag{3}$$

$$a^4 \sum_x f(x)\partial_\mu g(x) = -a^4 \sum_x (\partial^*_\mu f)(x)g(x)\,. \tag{4}$$

3 Exercise 3

Consider the lattice action for a real scalar field ϕ

$$S = \frac{1}{2}a^4 \sum_x \phi(x)\left[-\partial^*_\mu\partial_\mu + m^2\right]\phi(x)\,. \tag{5}$$

Add an irrelevant term (that vanishes in the classical continuum limit) to the lattice action to improve at the classical level the cutoff effects from O(a^2) to O(a^4).

4 Exercise 4

- Calculate the poles in $p_0 = \pm i\omega(\mathbf{p})$, $\mathbf{p} = (p_1, p_2, p_3)$ of the propagator for a scalar field

$$\widetilde{G}(p) = \frac{1}{\hat{p}^2 + m^2}\,, \quad \hat{p}^2 = \sum_{\mu=1}^{4} \hat{p}_\mu \hat{p}_\mu\,, \quad \hat{p}_\mu = \frac{2}{a}\sin\left(\frac{ap_\mu}{2}\right)\,. \tag{6}$$

- Plot in the first Brillouin zone $\omega(\mathbf{p})$ as a function of p_1 with $\mathbf{p} = (p_1, 0, 0)$, $a = 1$ amd $m = 0.1$.
- Compare the plot with $\omega(\mathbf{p})$ in the continuum with $\mathbf{p} = (p_1, 0, 0)$.

5 Exercise 5

Derive the relation between the lattice spacing a and the bare coupling g using the 1-loop and 2-loop expression for the beta function $\beta(g_0) = -\beta_0 g^3 - \beta_1 g^5$.

6 Exercise 6

Show that the generating functional

$$W[\theta, \overline{\theta}] = \int \left(\prod_{i=1}^{N} d\eta_i \; d\overline{\eta}_i \right) \exp\left\{ \overline{\eta}_k M_{kl} \eta_l + \overline{\theta}_i \eta_i + \overline{\eta}_i \theta_i \right\} , \tag{7}$$

can be written as

$$W[\theta, \overline{\theta}] = \det[M] \exp\left\{ -\overline{\theta}_k (M^{-1})_{kl} \theta_l \right\} , \tag{8}$$

where $\eta_i, \overline{\eta}_i, \theta_i, \overline{\theta}_i \quad i = 1, \ldots, N$ are $4N$ Grassmann variables anticommuting with each other. (Repeated indices in the exponentials are summed over.)

7 Exercise 7

Consider the free-quark theory on a an infinite volume lattice. Calculate the values of the poles in the energy plane (as a function of the spatial momentum $\mathbf{p}$ and the mass m) with and without the Wilson term. Plot the corresponding dispersion relations.

8 Exercise 8

Consider the free-quark theory on a lattice of size $T \times L^3$ with Wilson fermions. Impose the boundary conditions

$$\psi(x + T\hat{4}) = -\psi(x) \, , \quad \psi(x + L\hat{k}) = \psi(x) \, , \quad k = 1, 2, 3 \, , \tag{9}$$

$$\overline{\psi}(x + T\hat{4}) = -\overline{\psi}(x) \, , \quad \overline{\psi}(x + L\hat{k}) = \overline{\psi}(x) \, , \quad k = 1, 2, 3 \, , \tag{10}$$

on the quark fields and write down the quark propagator for this case in momentum space.

9 Exercise 9

Show that the fermion Wilson action is invariant under parity $\mathcal{P}$, charge conjugation $\mathcal{C}$ and is γ_5-Hermitian.

10 Exercise 10

Show how the fermion bilinears transform under parity $\mathcal{P}$ and charge conjugation $\mathcal{C}$.

11 Exercise 11

Perform the contractions and express the π^0 2-point function in terms of the quark propagators.

12 Exercise 12

The nucleon-nucleon 2-point function data shared via email are divided in directories: each directory contains the 2-point function for a given gauge ensemble. The name of the directory contains information about the lattice size and the κ values for the $N_f = 2$ light sector (κ_{ud}) and for the strange quark (κ_s). Additioanl information about the 16, 20 and 28 ensembles can be found in "https://www.jldg.org/ildg-data/CPPACS+JLQCDconfig.html". For the large volumes ensembles consult "http://www.jldg.org/ildg-data/PACSCSconfig.html".
Perform the following analysis using bootstrap (or jackknife) to evaluate statistical uncertainties

- Calculate the euclidean time dependence of each correlator and plot them.
- Calculate the effective mass of each correlator as a function of Euclidean time and plot them.
- Perform a constant fit where the effective mass plateaus and plot the extracted mass as a function of the pion mass and pion mass squared (pion masses can be found on the links above)
- Plot the extracted mass from the previous analysis as a function of the lattice spacing squared (values of the lattice spacings can be found in the links above).