Problems with Fermions

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Methods of Effective Field Theory and Lattice Field Theory,
Bad Honnef Physics School, Germany, July 9-21, 2023

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Forward or backward derivatives

- Show that forward or backward derivatives for fermions instead of symmetric ones break discrete symmetries, i.e. charge conjugation, and parity or time reflection.

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The discrete Fourier transform is given by:

$$\begin{aligned} \psi_n &= \frac{1}{V_d}\sum_{k\in\tilde{\Lambda}} \mathrm{e}^{+iak\cdot n}\tilde{\psi}_k \ , \quad \bar{\psi}_n = \frac{1}{V_d}\sum_{k\in\tilde{\Lambda}} \mathrm{e}^{-iak\cdot n}\tilde{\bar{\psi}}_k \ , \\ \tilde{\psi}_k &= a^d\sum_{k\in\tilde{\Lambda}} \mathrm{e}^{-iak\cdot n}\psi_n \ , \quad \tilde{\bar{\psi}}_k = a^d\sum_{k\in\tilde{\Lambda}} \mathrm{e}^{+iak\cdot n}\bar{\psi}_n \ . \end{aligned} \tag{1}$$

- Confirm the orthogonality of the basis functions $a^d\mathrm{e}^{\mathrm{i}ak\cdot n}$ and determine the corresponding normalization constant of

$$\sum_{n\in\Lambda} a^d\mathrm{e}^{\mathrm{i}a(k-l)\cdot n} \tag{2}$$

- Transform $D_{\mathrm{W}}[U](n,m)$ in the free theory ($U_{\mu,n}=1$) to momentum space.

Lattice fermions in $d = 2$

- Predict the taste degeneracy patterns for $d = 2$. How much doubling is there?
- Which two symmetry operations of $d = 4$ have become identical in $d = 2$?
- Compute the spectrum on a small $d = 2$ lattice for naive or Wilson fermions.

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Symmetries of KS fermions

- Verify one or both of the Clifford algebras for KS fermions.
- Demonstrate the shift symmetry for KS fermions.
- How are the spin and taste structure of single-site operators related? Can you show this without an explicit calculation?

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γ_5 or ϵ hermiticity

- Show γ_5 resp. ϵ hermiticity for Wilson, KS fermions, and overlap fermions with the Wilson kernel. Use coordinate space for the interacting theory.
- Show that eigenvalues are real or contribute in complex conjugate pairs (for both operators).
- Show the reality of the determinant (for both operators).

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Eigenvalue counting

- How many eigenvalues does the one-flavor Wilson Dirac operator have?
- Explicitly and analytically calculate the eigenvalues λ_p of the free (i.e. non-interacting) Wilson Dirac operator for a given box of $V_4/a^4 = N^4$ sites with periodic boundary conditions in all directions.
- Which degeneracy patterns exist among the eigenvalues in the free theory?
- How many eigenvectors with the same λ_p exist (for fixed p), and how are they related?
- How much of these degeneracy patterns survive in the interacting theory?

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Lattice dispersion relations

- Derive the dispersion relation for a complex scalar field.
- Derive the dispersion relation for free Wilson fermions in momentum space, and expand it up to the leading $\mathcal{O}(a)$ correction. In the case of general $r \neq 1$ you have to choose the appropriate root. Argue how to find the right one.
- Derive the dispersion relation for KS fermions.
- Plot all three dispersion relations for a short small spatial momenta and zero or non-zero masses, and compare to the continuum.

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Lattice Noether currents

To derive the Noether theorem on the lattice, the symmetries are gauged as usual. Transform until you arrive at forward derivatives of the infinitesimal parameter.

- Derive both conserved point-split currents for KS fermions.
- Derive the conserved point-split vector current for Wilson fermions.
- Show that the Noether current for Wilson fermions has a non-zero divergence even for $m_0 = 0$.

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