Introduction to Effective Field Theory

Lecture 3: Nonrelativistic EFTs

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Quantum Field Theory

QFT: essential formalism for

- combining quantum mechanics with special relativity
- describing annihilation and creation of particles

QFT: natural formalism for

- describing quantum mechanics of identical bosons
  and identical fermions

QFT has many useful applications in nonrelativistic physics

- few-body physics and many-body physics
- low-energy hadrons, ultracold atoms, …

Nonrelativistic Quantum Field Theory

Nonrelativistic fermions or bosons

quantum field operator: $\psi(\vec{r}, t)$ annihilates particle

$\psi^\dagger(\vec{r}, t)$ creates particle

Free particle

Lagrangian: $\mathscr{L} = \frac{1}{2}\left(\psi^\dagger i \frac{\partial}{\partial t}\psi + \mathrm{h\,.\,c\,.}\right) - \mathscr{H}_{\text{free}}$

Hamiltonian: $\mathscr{H}_{\text{free}} = \dfrac{1}{2m}\nabla\psi^\dagger \cdot \nabla\psi$

Equation of motion:

$$\left(i\hbar\frac{\partial}{\partial t}\psi + \frac{1}{2m}\nabla^2\right)\psi = 0$$

solutions: $\psi(\vec{r}, t) = \exp(-iEt + i\vec{p}\cdot\vec{r})$ with $E = \dfrac{p^2}{2m}$

parameter *m* is the kinetic mass

Interacting particles

Lagrangian density: $\mathscr{L} = \frac{1}{2}\left(\psi^\dagger i\hbar \frac{\partial}{\partial t}\psi + \mathrm{h.c.}\right)\psi - \left(\mathscr{H}_{\text{free}} + \mathscr{H}_{\text{int}}\right)$

Hamiltonian density: $\mathscr{H}_{\text{free}} = \dfrac{1}{2m}\nabla\psi^\dagger \cdot \nabla\psi$

Interaction Hamiltonian density: $\mathscr{H}_{\text{int}}$

Nonlocal interactions through potential $V(r)$

$$\mathscr{H}_{\text{int}}(\vec{r}) = \int d^3r'\, \psi^\dagger\psi(\vec{r}')\, V(|\vec{r}' - \vec{r}|)\, \psi^\dagger\psi(\vec{r})$$

Local interactions

$\mathscr{H}_{\text{int}}$ is a function of ψ, $\psi^\dagger$, and their derivatives

$\mathscr{H}_{\text{int}} = g\,\psi^\dagger\psi\,\psi^\dagger\psi + \ldots$

only valid at energies too low to probe structure of particle
or shape of inter particle potential

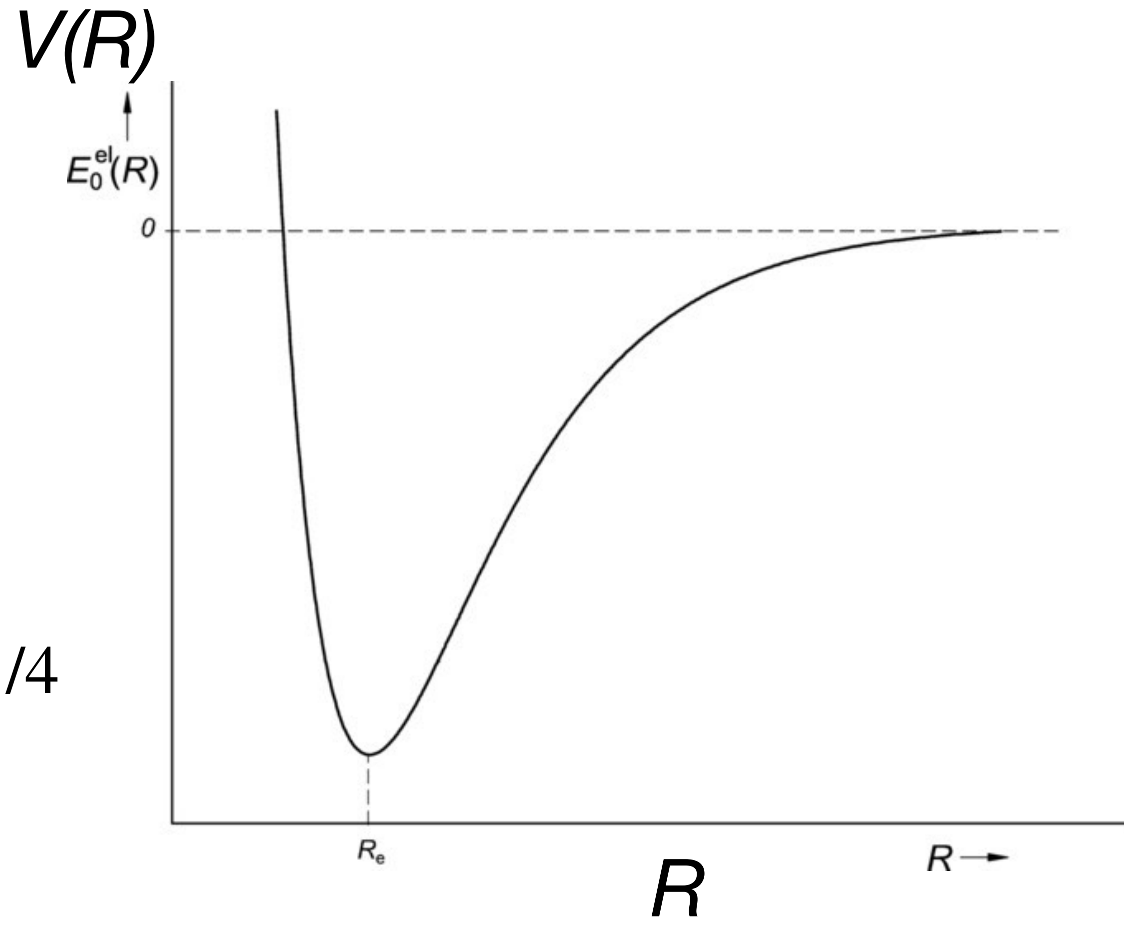
QFT is a low-energy effective field theory!

Interatomic potential $V(R)$

- short-range repulsion
- minimum: $R_0 \approx$ few angstroms
- long-range van der Waals tail:

$$V(R) \longrightarrow -C_6/R^6$$

van der Waals length: $R_6 = (mC_6/\hbar^2)^{1/4}$



Energy is *E* too low to probe shape of interatomic potential

when $E \ll \hbar^2/mR_6^2$

for alkali atoms (below *H* in periodic table):

$R_6 \approx$ dozens of angstroms

$\longrightarrow$ $E \ll$ microKelvin

Local low-energy EFT applies only to ultracold atoms!

Nonrelativistic Low-energy EFT

Nonrelativistic particles with short-range interactions

- identify high-momentum scale Λ

  particles with momenta > Λ will not be described explicitly in EFT
  (but their effects will be taken into account through parameters of EFT)

- introduce quantum fields that annihilate/create particles

- identify action of symmetries on fields

- construct most general Lagrangian for fields consistent with symmetries

- establish power counting for operators in the Lagrangian

  organizes EFT into low-momentum expansion in powers of $1/\Lambda$

Nonrelativistic Low-energy EFT

Nonrelativistic particles with short-range interactions

- low-energy hadrons (electrically neutral?)
- ultracold atoms

Identify high-momentum scale Λ

- Λ ~ 1/range of interactions

Introduce quantum fields that annihilate/create each type of particle

- $\psi_i(\vec{r}, t)$: bosonic field for bosons
  fermionic field for fermions

Identify actions of symmetries on fields

- $U(1)$ phase symmetry for each field: $\psi_i(\vec{r}, t) \longrightarrow e^{i\, m_i\, \theta}\, \psi_i(\vec{r}, t)$
  implies conservation of particle numbers

- translational invariance: $\psi_i(\vec{r}, t) \longrightarrow \psi_i(\vec{r} + \vec{a}, t)$
  implies conservation of momentum

- time translational invariance: $\psi_i(\vec{r}, t) \longrightarrow \psi_i(\vec{r}, t + \epsilon)$
  implies conservation of energy

Nonrelativistic Low-energy EFT

Identify actions of symmetries on fields

- $U(1)$ phase symmetry
- translational invariance
- time translational invariance

- rotational symmetry: $\psi_i(\vec{r},t) \longrightarrow \psi_i(O\,.\,\vec{r},t)$, O is orthogonal matrix

  implies conservation of angular momentum

- Galilean symmetry: $\psi_i(\vec{r},t) \longrightarrow \exp\Big(i\,m_i\big(\vec{v}\cdot\vec{r}-\frac{1}{2}v^2t\big)\Big)\,\psi_i(\vec{r}-\vec{v}t,t)$

  invariance under boost to frame moving with velocity $\vec{v}$

  phase depends on kinetic mass m_i

  implies conservation of kinetic mass

- parity: $\psi(\vec{r},t) \longrightarrow +\,\psi(-\vec{r},t)$ or $\psi(\vec{r},t) \longrightarrow -\,\psi(-\vec{r},t)$

- time reversal: $\psi(\vec{r},t) \longrightarrow +\,\psi^*(\vec{r},-\,t)$

Nonrelativistic Quantum Field Theory

- construct most general Lagrangian for fields consistent with symmetries

Galilean symmetry provides strong constraints:

$$\psi_i(\vec{r},t) \longrightarrow \exp\Big(i\, m_i\big(\vec{v}\cdot\vec{r} - \tfrac{1}{2}v^2 t\big)\Big)\, \psi_i(\vec{r}-\vec{v}t,t)$$

$\left(i\dfrac{\partial}{\partial t} + \dfrac{1}{2m_i}\nabla^2\right)\psi_i$ transforms like ψ_i

$\left(i\dfrac{\partial}{\partial t} + \dfrac{1}{2(m_1+m2)}\nabla^2\right)\psi_1\psi_2$ transforms like $\psi_1\psi_2$

$\psi_1 \overleftrightarrow{\nabla} \psi_2 = \psi_1(\nabla\psi_2) - (\nabla\psi_1)\psi_2$ transforms like $\psi_1\psi_2$

$\psi_1 \overleftrightarrow{\nabla}^i \overleftrightarrow{\nabla}^j \psi_2$ transforms like $\psi_1\psi_2$

Phases must cancel between annihilation fields ψ_i and creation fields $\psi_j^\dagger$

$\Longrightarrow$ conservation of kinetic mass: $\displaystyle\sum_i m_i = \sum_j m_j$

Nonrelativistic Quantum Field Theory

Special relativity $\Longrightarrow$ rest mass = kinetic mass

Einstein: $E(p) = \sqrt{m^2 + p^2}$

small momentum: $E(p) = m + \dfrac{p^2}{2m} + \ldots$

rest mass kinetic mass

Galilean invariance

$\Longrightarrow$ conservation of kinetic mass: $\sum_i m_i = \sum_j m_j$

- pionless EFT for nucleons

  only explicit particles are nucleons and their number is conserved
  (effects of pions are taken into account through parameters in Lagrangian)

- ultracold atoms

  particles are atoms and their number is conserved

- charm mesons and pions

  D^+, D^0, D^{*+}, D^{*0} π^+, π^0, π^-

  $D^* \longleftrightarrow D\,\pi$ transitions violate conservation of mass by less than (5%)m_π

Nonrelativistic Quantum Field Theory

Determine scaling dimensions from free theory:

$$S_{\text{free}} = \int dt \int d^3r\, \mathscr{L}_{\text{free}}$$

$$\mathscr{L}_{\text{free}} = \tfrac{1}{2}\left(\psi^\dagger i \frac{\partial}{\partial t}\psi + \mathrm{h.c.}\right) + \frac{1}{2m}\nabla\psi^\dagger \cdot \nabla\psi$$

∇ has scaling dimension $d = 1$

In nonrelativistic physics,

mass *m* is just a conversion factor between momentum-squared and energy

$\Longrightarrow$ $\partial/\partial t$ has scaling dimension $d = 2$

action S_{free} is dimensionless $\Longrightarrow$ $\mathscr{L}_{\text{free}}$ has scaling dimension $d = 5$

$\Longrightarrow$ ψ has scaling dimension 3/2

other terms in Lagrangian

$d = 3$: $\psi^*\psi$

$d = 6$: $\psi^*\psi^*\psi\psi$

$d = 7$: $\psi^*\left(i\partial/\partial t + \nabla^2/2m\right)^2\psi$ (redundant from EOM)

$d = 8$: $\psi^*\overleftrightarrow{\nabla}\psi^* \cdot \psi\overleftrightarrow{\nabla}\psi$

$d = 9$: $\psi^*\psi^*\psi^*\psi\psi\psi$

Nonrelativistic Quantum Field Theory

Lagrangian for EFT

$$\mathscr{L}_{\text{EFT}} = \mathscr{L}_{\text{free}} - \mu\,\psi^\dagger\psi + \sum_{d\ge6}\sum_{n} c_{d,n}\mathscr{O}_{d,n}$$

μ is chemical potential

sum over scaling dimensions: $d = 6,8,9,10,...$

For each d, sum over independent operators with that scaling dimension (after removing redundancies from EOM and IBP)

Leading order interaction terms only: $d = 6$

- 1 bosonic field ψ

$$\mathscr{L}_{\text{EFT}} = \tfrac{1}{2}\Big[\psi^\dagger\Big(i\frac{\partial}{\partial t} + \frac{1}{2m}\nabla^2 + \mu\Big)\psi + \mathrm{h.c.}\Big] - \frac{2\pi a}{m}\psi^\dagger\psi^\dagger\psi\psi$$

boson with S-wave scattering length *a*

- 1 fermionic field ψ

free theory (interaction term is 0)

- 2 fermionic fields ψ_1, ψ_2

$$\mathscr{L}_{\text{EFT}} = \tfrac{1}{2}\sum_{i=1}^{2}\Big[\psi_i^\dagger\Big(i\frac{\partial}{\partial t} + \frac{1}{2m}\nabla^2 + \mu\Big)\psi_i + \mathrm{h.c.}\Big] - \frac{4\pi a}{m}\psi_1^\dagger\psi_2^\dagger\psi_2\psi_1$$

two fermions with S-wave scattering length *a*

Weakly Interacting Bose Gas

EFT for single nonrelativistic boson field

$$\mathscr{L}_{\text{free}} = \tfrac{1}{2}\Big[\psi^\dagger\Big(i\frac{\partial}{\partial t} + \frac{1}{2m}\nabla^2\Big)\psi + \text{h.c.}\Big]$$

$$\mathscr{L}_{\text{EFT}} = \mathscr{L}_{\text{free}} + \mu\,\psi^\dagger\psi - \frac{2\pi a}{m}\psi^\dagger\psi^\dagger\psi\psi$$

mass *m*, chemical potential μ, scattering length *a*

number density operator: $\psi^\dagger\psi$

Bose gas: specified number density *n,* zero temperature

adjust μ so $\langle\psi^\dagger\psi\rangle = n$

T = 0 ⟹ Bose-Einstein condensation

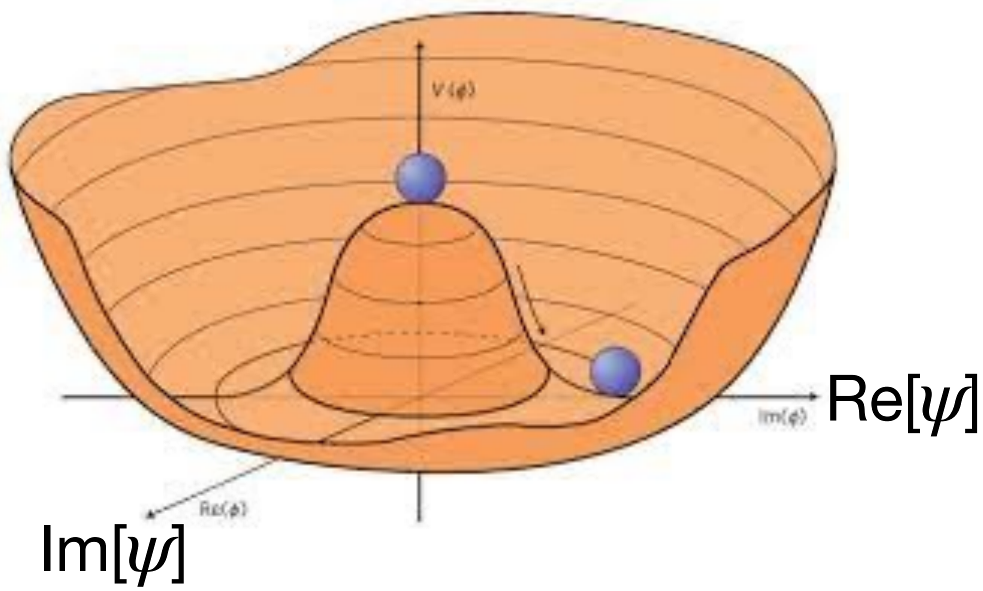
field ψ has nonzero vacuum expectation value: $n = \psi^*\psi$

Potential energy density:

$$\mathscr{V}(\psi^*\psi) = -\,\mu\,(\psi^*\psi) + \frac{2\pi a}{m}(\psi^*\psi)^2$$

minimize with respect to $\psi^*\psi$: $-\mu + \dfrac{4\pi a}{m}(\psi^*\psi) = 0$

chemical potential: $\mu = \dfrac{4\pi\,a}{m}\psi^*\psi = \dfrac{4\pi\,a}{m}n$



Spontaneous breaking of *U(1)* symmetry: $\psi(x) \longrightarrow e^{i\theta}\psi(x)$

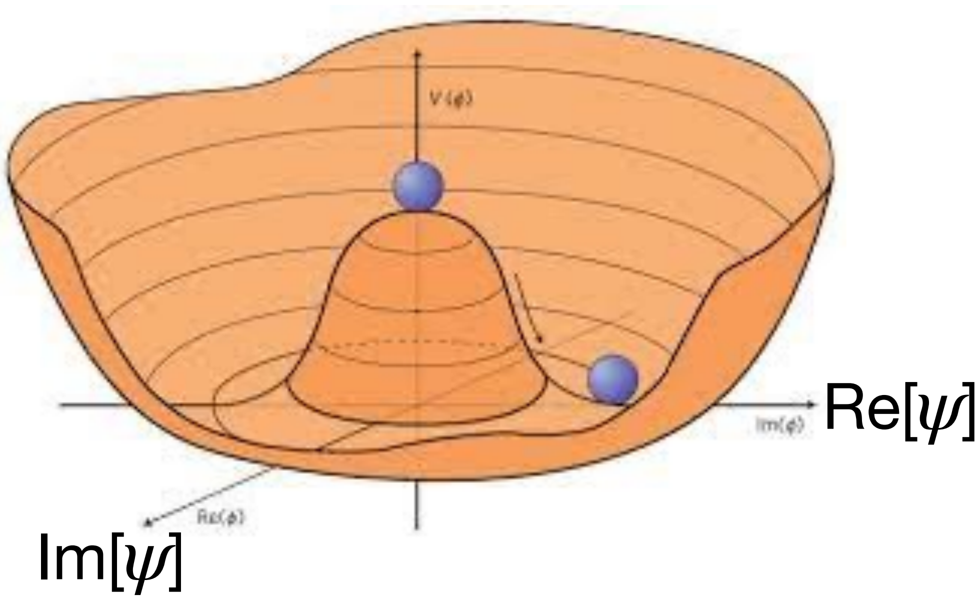
Weakly Interacting Bose Gas

Bose gas with number density *n, T* = *0*

boson field ψ has nonzero vacuum expectation value

chemical potential: $\mu = \frac{4\pi\, a}{m}\psi^*\psi = \frac{4\pi\, a}{m} n$

number density: $n = \psi^*\psi = \frac{m}{4\pi\, a}\,\mu$



Pressure of the Bose gas

homogeneous system: $P = -\,\mathscr{F}$, $\mathscr{F}$ = free energy density

contribution for Bose-Einstein condensate: $P = -\,\mathscr{V}(\psi^*\psi)$

$$P(\mu) = \mu\left(\frac{m\,\mu}{4\pi\, a}\right) - \frac{2\pi a}{m}\left(\frac{m\,\mu}{4\pi\, a}\right)^2 = \frac{m}{8\pi\, a}\,\mu^2$$

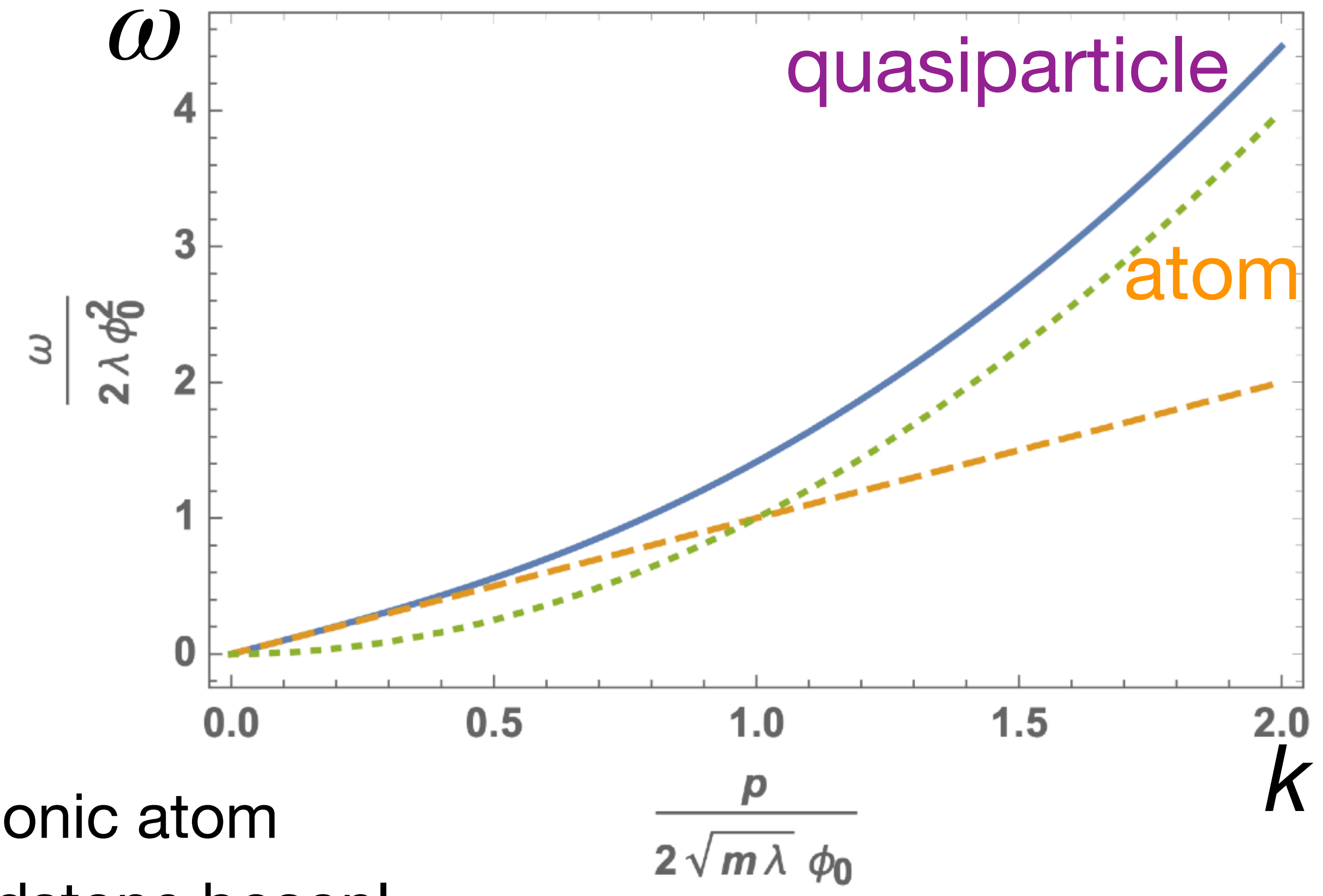
OR $P(n) = \frac{2\pi\, a}{m}\, n^2$

Weakly Interacting Bose Gas

Quasiparticles in the Bose gas

Bogoliubov dispersion relation: $\omega(k) = \dfrac{k\sqrt{k^2 + k_B^2}}{2m}$, $\quad k_B = \sqrt{16\pi a n}$

See Exercise



as $k \to \infty$, $\omega(k) \longrightarrow k^2/(2m)$ bosonic atom

as $k \to 0$, $\omega(k) \longrightarrow k\, k_B/(2m)$ Goldstone boson!

speed: $v_B = k_B/(2m) = \sqrt{4\pi a n/m^2}$

Dispersion relation for Goldstone mode
can be derived more easily using EFT for Goldstone boson

Effective field theory for Goldstone boson

from integrating out Bogoliubov momentum scale $k_B = \sqrt{16\pi a n}$

- identify high momentum scale of EFT: $\Lambda = k_B$

- identify field for Goldstone mode

  as phase of atom field ψ : $\psi(\vec{r},t) = \sqrt{n}\,\exp\big(i\phi(\vec{r},t)\big)$

- phase symmetry

  atom field: $\psi(\vec{r},t) \longrightarrow \exp(i\,\theta)\,\psi(\vec{r}-\vec{v}t,t)$

  Goldstone field: $\phi(\vec{r},t) \longrightarrow \phi(\vec{r},t) + \theta$

- Galilean symmetry

  atom field: $\psi(\vec{r},t) \longrightarrow \exp\Big(i\,m\big(\vec{v}\cdot\vec{r} - \tfrac{1}{2}v^2 t\big)\Big)\,\psi(\vec{r}-\vec{v}t,t)$

  Goldstone field: $\phi(\vec{r},t) \longrightarrow \phi(\vec{r}-\vec{v}t,t) + m\big(\vec{v}\cdot\vec{r} - \tfrac{1}{2}v^2 t\big)$

  $\Longrightarrow \quad \dot{\phi} + \dfrac{1}{2m}\nabla\phi\cdot\nabla\phi$ is invariant

Weakly Interacting Bose Gas

Effective field theory for Goldstone boson

$\mathscr{L}_{\text{EFT}}$ is scalar function of ϕ and its derivatives: $\dot{\phi}$, $\nabla\phi$, $\ddot{\phi}$, $\nabla^i\nabla^j\phi$, ...

To describe Goldstone boson with velocity v_B,
terms with fewest derivatives should reduce to the form

$$\mathscr{L}_{\text{free}} = \frac{1}{2}Z\dot{\phi}^2 - \frac{1}{2}Z v_B^2 \nabla\phi\cdot\nabla\phi$$

Equation of motion:

$$Z\ddot{\phi} - Z v_B^2 \nabla^2\phi = 0$$

Solution: $\phi(\vec{r},t) = \phi_0\,\exp(-i\omega t + i\vec{k}\cdot\vec{r}),\ \omega^2 - v_B^2 k^2 = 0$

Determine power counting:

$$S_{\text{free}} = \int dt \int d^3r\,\mathscr{L}_{\text{free}}$$

∇ has scaling dimension $d = 1$

$\Longrightarrow$ $\partial/\partial t$ has scaling dimension $d = 1$

action S_{free} is dimensionless $\Longrightarrow$ $\mathscr{L}_{\text{free}}$ has scaling dimension $d = 4$

$\Longrightarrow$ ϕ has scaling dimension 1

Weakly Interacting Bose Gas

Effective field theory for Goldstone boson

$\mathscr{L}_{\text{EFT}}$ is scalar function of ϕ and its derivatives: $\dot{\phi},\ \nabla\phi,\ \ddot{\phi},\ \nabla^i\nabla^j\phi, \ldots$

- phase symmetry

  Goldstone field: $\phi(\vec{r},t) \longrightarrow \phi(\vec{r},t) + \theta$

  $\mathscr{L}_{\text{EFT}}$ cannot depend on ϕ

- Galilean symmetry

  Goldstone field: $\phi(\vec{r},t) \longrightarrow \phi(\vec{r}-\vec{v}t,t) + m\left(\vec{v}\cdot\vec{r} - \frac{1}{2}v^2 t\right)$

  $\Longrightarrow$ $\mathscr{L}_{\text{EFT}}$ can depend on ϕ only in the combination $\dot{\phi} + \dfrac{1}{2m}\nabla\phi\cdot\nabla\phi$

- to determine dependence of $\mathscr{L}_{\text{EFT}}$ on chemical potential μ

  consider atom theory as a gauge theory with gauge fields $A_0(\vec{r},t)$, $\overrightarrow{A}(\vec{r},t)$

  in the special case: $A_0(\vec{r},t) = \mu$, $\overrightarrow{A}(\vec{r},t) = 0$

Weakly Interacting Bose Gas

Effective field theory for Goldstone boson

Gauge theory for atoms

$$\mathscr{L} = \tfrac{1}{2}\left[\psi^\dagger i D_0 \psi + \mathrm{h.c.}\right] - \frac{1}{2m}\overrightarrow{D}\psi^\dagger \cdot \overrightarrow{D}\psi - \frac{2\pi a}{m}\psi^\dagger\psi^\dagger\psi\psi$$

covariant derivatives: $D_0 = \partial/\partial t - iA_0(\vec{r},t)$, $\overrightarrow{D} = \nabla - i\overrightarrow{A}(\vec{r},t)$

- Gauge transformations

$$\psi(\vec{r},t) \longrightarrow \exp\big(i\theta(\vec{r},t)\big)\,\psi(\vec{r},t) \implies \phi(\vec{r},t) \longrightarrow \phi(\vec{r},t) + \theta(\vec{r},t)$$

$$A_0(\vec{r},t) \longrightarrow A_0(\vec{r},t) + \dot{\theta}(\vec{r},t)$$

$$\overrightarrow{A}(\vec{r},t) \longrightarrow \overrightarrow{A}(\vec{r},t) + \nabla\theta(\vec{r},t)$$

So $\dot{\phi} - A_0$ is gauge invariant

- Recall $\dot{\phi} + \frac{1}{2m}\nabla\phi\cdot\nabla\phi$ is Galilean invariant

  so $\dot{\phi} - A_0 + \frac{1}{2m}\overrightarrow{D}\phi\cdot\overrightarrow{D}\phi$ is Galilean invariant and gauge invariant

- Special case: $A_0(\vec{r},t) = \mu$, $\overrightarrow{A}(\vec{r},t) = 0$

  Lagrangian can only depend on μ through the combination $\dot{\phi} - \mu + \frac{1}{2m}\nabla\phi\cdot\nabla\phi$

Weakly Interacting Bose Gas

Effective field theory for Goldstone boson

$\mathscr{L}_{\text{EFT}}$ must be a function of $X = \mu - \dot{\phi} - \dfrac{1}{2m}\nabla\phi\cdot\nabla\phi$

$\mathscr{L}_{\text{EFT}} = P(X)$ + terms with 4 or more derivatives of ϕ

- drop terms with scaling dimension 8 and higher

$\mathscr{L}_{\text{EFT}} = P(X)$

- *e*xpand *X* around μ:

$$\mathscr{L}_{\text{EFT}} = P(\mu) - P'(\mu)\big(\dot{\phi} + \frac{1}{2m}\nabla\phi\cdot\nabla\phi\big) + \tfrac{1}{2}P''(\mu)\big(\dot{\phi} + \frac{1}{2m}\nabla\phi\cdot\nabla\phi\big)^2 + \ldots$$

- drop total derivative term $\dot{\phi}$
- drop terms with scaling dimension higher than 4

$$\mathscr{L}_{\text{EFT}} = P(\mu) - \frac{P'(\mu)}{2m}\nabla\phi\cdot\nabla\phi \ + \tfrac{1}{2}P''(\mu)\big(\dot{\phi}\big)^2 + \ldots$$

Equation of motion:

$$-\frac{P'(\mu)}{2m}\nabla^2\phi + \tfrac{1}{2}P''(\mu)\ddot{\phi} + \ldots = 0$$

Weakly Interacting Bose Gas

Effective field theory for Goldstone boson

Equation of motion:

$$-\frac{P'(\mu)}{2m}\nabla^2\phi+\tfrac{1}{2}P''(\mu)\ddot{\phi}+\ldots=0$$

Solution: $\phi(\vec{r},t)=\phi_0\,\exp(-i\omega t+i\vec{k}\cdot\vec{r}),$

dispersion relation: $mP''(\mu)\,\omega^2-P'(\mu)\,k^2=0$

Recall expression for pressure:

$$P(\mu)=\frac{m}{8\pi\,a}\,\mu^2\implies P'(\mu)=\frac{m}{4\pi\,a}\,\mu,\,P''(\mu)=\frac{m}{4\pi\,a}$$

Dispersion relation: $\omega^2-(\mu/m)\,k^2=0\implies$

Goldstone boson speed: $v_B=\sqrt{\mu/m}=\sqrt{4\pi an/m^2}$

agrees with Bogoliubov dispersion relation!