Effective Field Theories for Heavy Quarks

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L2: Heavy-light systems, HQET

Heavy-light systems

$Q\bar{q}$ bound state , $m_Q >> \Lambda_{\rm QCD}$, $\alpha_{\rm s}(m_Q) << 1$

- The heavy quark is essentially at rest in the rest frame of the heavy meson
- If the heavy meson moves with velocity v, $v^2 = 1$, the heavy quark will also move with velocity v up to fluctuations $\sim \Lambda_{\rm QCD}/m_Q$, namely its momentum can be written as $p = m_Q v + k$, $k \sim \Lambda_{\rm QCD} \ll m_Q$
- Naïvely, we can write $Q(x) \sim e^{-im_Q vx} p_+ h_v^{(Q)}(x)$, $p_\pm = (1 \pm \not{v})/2$ projects onto the particle (antiparticle) subspace, and $h_v^{(Q)}(x)$ contains the momentum fluctuations $k \sim \Lambda_{\rm QCD}$

Heavy-light systems

If this is done for all heavy quarks in the heavy quark part of the QCD lagrangian

$$\sum_{Q=c,b} \mathcal{L}_Q \longrightarrow \sum_{Q=c,b} \mathcal{L}^{(Q)}_{HQET} = \sum_{Q=c,b} \bar{h}^{(Q)}_v iv.Dh^{(Q)}_v$$

- It enjoys an accidental $SU(2N_Q)$ spin-flavor symmetry
  - $m_{b\bar{q}} - m_b \sim m_{c\bar{q}} - m_c \quad , \quad m_{bqq} - m_b \sim m_{cqq} - m_c$

$$H = (Ql)\ ,\ l = \bar{q}\,, qq \quad , \quad M_H = m_Q + \Lambda_l' + \mathcal{O}\left(\frac{1}{m_Q}\right)$$

  - For each Q, we have spin doublets $(S, S+1)$. $m_B \sim m_{B^*}$, $m_D \sim m_{D^*}$, $m_{\Sigma_c} \sim m^*_{\Sigma_c}$, $m_{\Sigma_b} \sim m^*_{\Sigma_b}$

$$M_{H^*} - M_H = \frac{\Lambda_l^2}{m_Q} + \mathcal{O}\left(\frac{1}{m_Q^2}\right)$$

Heavy-light systems

- How well does it work?

$$1 \underset{\text{Th}}{\simeq} \frac{M_{\Lambda_b} - M_B}{M_{\Lambda_c} - M_D} \underset{\text{Exp}}{=} 0.82 \quad , \quad 1 \underset{\text{Th}}{\simeq} \frac{M_{B^*}^2 - M_B^2}{M_{D^*}^2 - M_D^2} \underset{\text{Exp}}{=} 0.88$$

- One expects the naïve picture to be modified by
  - Power corrections: $\sim 1/m_Q$ suppressed terms

$$\Lambda_{\rm QCD}/m_c \sim 0.3 \quad , \quad \Lambda_{\rm QCD}/m_b \sim 0.08$$

  - Radiative corrections: $\sim \alpha_{\rm s}(m_Q)$ suppressed terms

$$\alpha_{\rm s}(m_c) \sim 0.46 \quad , \quad \alpha_{\rm s}(m_b) \sim 0.26$$

Power Corrections

Suppose A_μ contains small momenta ($\ll m_Q$) only

$$Q(x) = e^{-im_Q vx} p_+ h_v^{(Q)}(x) + H_v^{(Q)}(x)$$

- $p_+ H_v^{(Q)}(x)$ contains the momenta which are NOT in $e^{-im_Q vx} p_+ h_v^{(Q)}(x)$
- $p_- H_v^{(Q)}(x)$ contains all momenta
- $h_v^{(Q)}(x)$ contains small momenta ($\ll m_Q$) only

Power Corrections

$$\mathcal{L}_Q = \left(\bar{h}_v^{(Q)} p_+ e^{im_Q vx} + \bar{H}_v^{(Q)}\right)(i\not{D} - m_Q)\left(e^{-im_Q vx} p_+ h_v^{(Q)} + H_v^{(Q)}\right)$$
$$= \bar{h}_v^{(Q)} iv.D h_v^{(Q)} \quad + \quad \bar{h}_v^{(Q)} p_+ e^{im_Q vx} i\not{D} p_- H_v^{(Q)}$$
$$+\bar{H}_v^{(Q)} p_- i\not{D} e^{-im_Q vx} p_+ h_v^{(Q)}(x) \quad + \quad \bar{H}_v^{(Q)}(i\not{D} - m_Q) H_v^{(Q)}$$

- Momentum conservation:
  - Only $p_- H_v^{(Q)}$ of momenta $\sim e^{-im_Q vx} p_+ h_v^{(Q)}$
  - These modes do not couple to $p_+ H_v^{(Q)}$
- $iD_\mu e^{-im_Q vx} = e^{-im_Q vx}(iD_\mu + m_Q v_\mu)$
- $p_\pm \gamma_\mu p_\pm = \pm v_\mu p_\pm$
- $\not{v} p_\pm = \pm p_\pm$

Power Corrections

Integrating out $p_- H_v^{(Q)}$

$$\rightarrow \bar{h}_v^{(Q)} iv.D h_v^{(Q)} - \bar{h}_v^{(Q)} p_+ e^{i m_Q v x} i\not{D} p_- \frac{1}{-iv.D - m_Q} p_- i\not{D} e^{-i m_Q v x} p_+ h_v^{(Q)}$$

$$= \bar{h}_v^{(Q)} iv.D h_v^{(Q)} \quad - \quad \bar{h}_v^{(Q)} p_+ i\not{D} p_- \frac{1}{-iv.D - 2m_Q} p_- i\not{D} p_+ h_v^{(Q)}$$

$$= \bar{h}_v^{(Q)} iv.D h_v^{(Q)} \quad - \quad \bar{h}_v^{(Q)} p_+ i\not{D} p_- \left(\sum_{n=0}^{\infty} \frac{(-iv.D)^n}{(-2m_Q)^{n+1}} \right) p_- i\not{D} p_+ h_v^{(Q)}$$

- Up to local field redefinitions. Standard form: eliminate all $v.D$ except for the leading term.
- Note that the $n = 0$ correction contains the leading order of the NRQCD lagrangian for the paricle ($v = (1, \mathbf{0})$, $p_+ i\not{D} p_- i\not{D} p_+ \; = p_+ \left(\mathbf{D}^2 - \frac{1}{4}[\gamma^i, \gamma^j][D_i, D_j] \right) p_+$).

Radiative Corrections

A_μ also contains large momenta ($\sim m_Q$) $\longrightarrow$ $\alpha_{\rm s}(m_Q)$ corrections expected.

$$\mathcal{L}_Q = \sum_{n=0}^{\infty} \frac{c_n(m_Q,\mu) O_n(\mu)}{m_Q^n}$$

$$O_n(\mu) = \bar{h}_v^{(Q)} p_+ f_n(v_\mu, D_\mu, F_{\mu\nu}, \gamma_\mu) p_+ h_v^{(Q)}$$

$$c_n(m_Q,\mu) = c_n(\alpha_{\rm s}(m_Q), \ln(m_Q/\mu))$$

- Formally Lorentz invariant if v_μ is transformed
- But v_μ is fixed and hence the constraints of actual Lorentz symmetry are not implemented (yet).

Power + Radiative Corrections

- For $v = (1, \mathbf{0})$, we have $(B^k = -\epsilon^{kij} F_{ij}/2,\ E^k = F_{0k})$

$$\mathcal{L} = \bar{h}_v \Big\{ iD^0 + c_2 \frac{\mathbf{D}^2}{2m} + c_4 \frac{\mathbf{D}^4}{8m^3} + c_F\, g \frac{\boldsymbol{\sigma}\cdot\mathbf{B}}{2m} + c_D\, g \frac{[\mathbf{D}\cdot\mathbf{E}]}{8m^2}$$

$$+ i c_S\, g \frac{\boldsymbol{\sigma}\cdot(\mathbf{D}\times\mathbf{E} - \mathbf{E}\times\mathbf{D})}{8m^2} + \mathcal{O}(1/m^3) \Big\} h_v,$$

- v arbitrary is recovered by $D^0 \to v.D$, $\mathbf{D} \to D^\mu - v^\mu v.D$ and $\boldsymbol{\sigma} \to i p_+ \gamma^\nu p_- \gamma^\rho p_+ \epsilon_{\mu\nu\rho\sigma} v^\sigma / 2$

Matching

The $c_n(m_Q, \mu)$ are obtained by imposing that the Green functions (or matrix elements) of QCD and HQET coincide at a given order in $\alpha_s(m_Q)$, $p_+ Q(x) = Z_h^{1/2}(m_Q/\mu) e^{-im_Q vx} p_+ h_v^{(Q)}(x)$.

E.g.

$$\int d^4x e^{ipx} \langle T\{p_+ Q(x) \bar{Q}(0) p_+\}\rangle = Z_h(m_Q/\mu) \int d^4x e^{ikx} \langle T\{h_v^{(Q)}(x) \bar{h}_v^{(Q)}(0)\}$$

$p = m_Q v + k$, $k \ll m_Q$

- $c_n(m_Q, \mu)$ (and $Z_h(m_Q/\mu)$) contain contributions of QCD at the scale m_Q only
- The IR behavior is not important for the matching calculation, but must be treated in the same footing in QCD and HQET

Matching

- Early approaches: expand on k and introduce a finite gluon mass to regulate the IR
- Most efficient way: expand on k, and use DR to regulate both the UV and IR, and $\overline{\mathrm{MS}}$ to subtract the $1/\epsilon$ poles (Manohar 1997)
- E.g. Consider a one-loop matching calculation
  - The QCD calculation will have the form

$$\left[\frac{A}{\epsilon_{UV}} + \frac{B}{\epsilon_{IR}} + (A+B)\log\frac{\mu}{m} + D\right]$$

A, B, D, polynomials in k^2

Matching

- E.g. Consider a one-loop matching calculation (Cont.)
  - The HQET loop calculation will give zero (only scaleless integrals, $1/\epsilon_{UV} = 1/\epsilon_{IR} = 1/\epsilon$

$$A_{\rm eff}\left(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}}\right)$$

  - E.g.

$$\int \frac{d^d\,k}{(2\pi)^d}\frac{1}{k^4} = \int \frac{d^d\,k}{(2\pi)^d}\left[\frac{1}{k^2\,(k^2+m^2)} + \frac{m^2}{k^4\,(k^2+m^2)}\right] = \frac{1}{16\pi^2}\left[\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}}\right] = 0$$

  - The $1/\epsilon_{UV}$ are subtracted in the $\overline{\rm MS}$ scheme both in QCD and HQET

Matching

- E.g. Consider a one-loop matching calculation (Cont.)
  - Then the matching condition reads

$$\frac{B}{\epsilon_{IR}} + (A+B)\log\frac{\mu}{m} + D = -A_{\rm eff}\frac{1}{\epsilon_{IR}} + c_n$$

  - QCD and HQET have the same IR $\longrightarrow B = -A_{\rm eff}$

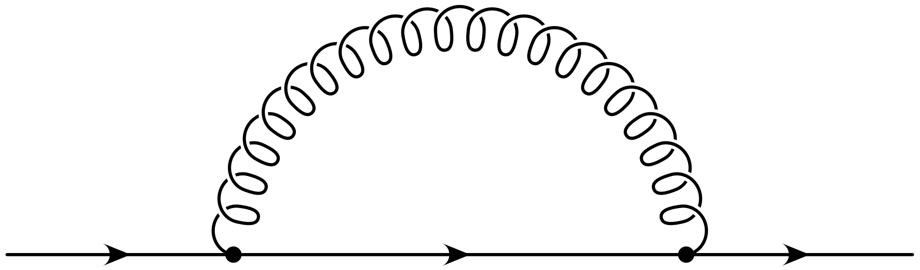
$$(A+B)\log\frac{\mu}{m} + D = c_n$$

  - If we use $\overline{\rm MS}$ for the IR divergences of QCD the c_n can be read off the QCD calculation directly (!)

Matching

E.g. Calculation of $Z_h(m_Q, \mu)$

- We only need the QCD calculation of ($D = 4 - \epsilon$, $C_f = (N_c^2 - 1)/2N_c$)



$$
\begin{aligned}
-i\Sigma(p) &= -iC_f \frac{\alpha_s}{4\pi} \left(A\left(p^2\right) m + B\left(p^2\right) \not{p} \right) \\
A\left(p^2\right) &= \int_0^1 dx\, \Gamma(\epsilon/2)(4-\epsilon)\left[m^2 x - p^2 x(1-x)\right]^{-\epsilon/2} \\
B\left(p^2\right) &= -\int_0^1 dx\, \Gamma(\epsilon/2)(2-\epsilon)(1-x)\left[m^2 x - p^2 x(1-x)\right]^{-\epsilon/2}
\end{aligned}
$$

Matching

E.g. Calculation of $Z_h(m_Q/\mu)$ (Cont.)

- $Z_h = 1 + \delta Z_h$

$$\delta Z_h^{bare} = -C_f \frac{\alpha_s}{4\pi}\left[B\left(m^2\right) + 2m^2 \left(\frac{\partial A}{\partial p^2} + \frac{\partial B}{\partial p^2}\right)_{p^2=m^2}\right]$$

$$= C_f \frac{\alpha_s}{\pi}\left[\frac{3}{2\hat{\epsilon}} + 1 - \frac{3}{2}\log\frac{m}{\mu}\right]$$

$$2/\hat{\epsilon} = 2/\epsilon - \gamma_E + \ln 4\pi, \quad \left(\frac{3}{2\epsilon} = \frac{1}{2\epsilon_{UV}} + \frac{1}{\epsilon_{IR}}\right)$$

-

$$\delta Z_h(m_Q/\mu) = C_f \frac{\alpha_s}{\pi}\left[1 - \frac{3}{2}\log\frac{m_Q}{\mu}\right]$$

Matching

The c_n in the lagrangian are most easily obtained by matching the on-shell scattering off a background field in the background field gauge in QCD and HQET

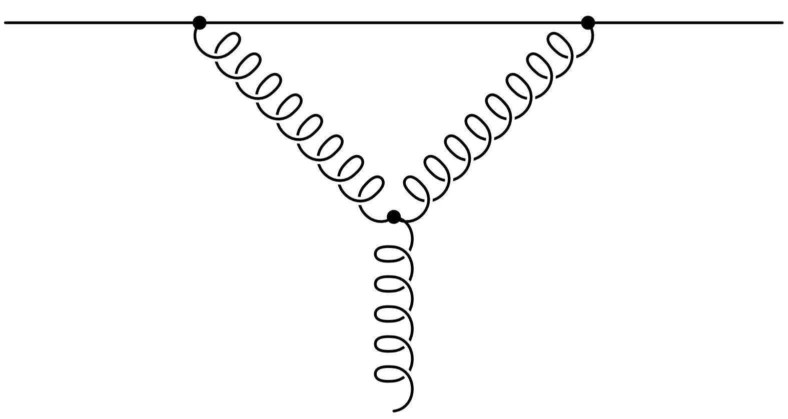
$$\langle \mathbf{p}|\mathbf{p}'\rangle|_{QCD} = Z_h \frac{E(\mathbf{p})E(\mathbf{p}')}{m^2}\langle \mathbf{p}|\mathbf{p}'\rangle|_{HQET}$$

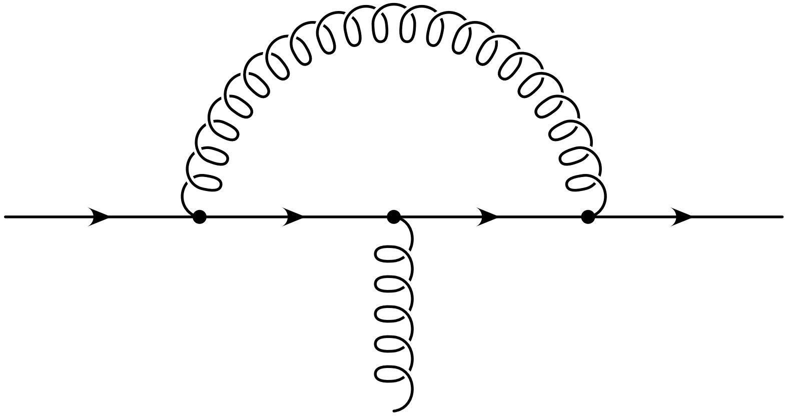
$$E(\mathbf{p}) = \sqrt{m^2 + \mathbf{p}^2}$$

- Z_h is matching coefficient of the heavy quark field (calculated before at one loop)
- The $E(\mathbf{p})/m$ factors correct for the different normalization of states in QCD and HQET (non-relativistic)

Matching

- QCD diagrams needed for the one-loop matching coefficients of HQET power corrections





Matching

$$
\begin{aligned}
c_2 &= c_4 = 1 \\
c_F &= 1 + \frac{\alpha_s}{\pi}\left[\frac{1}{2}C_f + \left(\frac{1}{2} - \frac{1}{2}\log\frac{m}{\mu}\right)C_A\right] \\
c_D &= 1 + \frac{\alpha_s}{\pi}\left[\left(\frac{8}{3}\log\frac{m}{\mu}\right)C_f + \left(\frac{1}{2} + \frac{2}{3}\log\frac{m}{\mu}\right)C_A\right] \\
c_S &= 1 + \frac{\alpha_s}{\pi}\left[C_f + \left(1 - \log\frac{m}{\mu}\right)C_A\right] \quad , \quad C_A = N_c
\end{aligned}
$$

$$
\Bigg[\quad \mathcal{L} = \bar{h}_v\Big\{iD^0 + c_2\,\frac{\mathbf{D}^2}{2m} + c_4\,\frac{\mathbf{D}^4}{8m^3} + c_F\,g\frac{\boldsymbol{\sigma}\cdot\mathbf{B}}{2m} + c_D\,g\frac{[\mathbf{D}\cdot\mathbf{E}]}{8m^2}
$$

$$
+ic_S\,g\frac{\boldsymbol{\sigma}\cdot(\mathbf{D}\times\mathbf{E}-\mathbf{E}\times\mathbf{D})}{8m^2} + \mathcal{O}(1/m^3)\Big\}h_v \quad \Bigg]
$$

Lorentz Symmetry

A practical way to implement it is the so called Reparameterization Invariance (Luke, Manohar, 1992)

- The splitting $p = mv + k$ is not unique
- $p = mv + k = mv' + k', \; v' = v + \delta v, \; m\delta v \ll m, \; k' = k - m\delta v$
- The HQET lagrangian must be the same for v and for v'
- The relation between the heavy quark fields reads

$$h_{v'}(x) = e^{im\delta v}\Lambda(v', \frac{p}{m})\Lambda^{-1}(v, \frac{p}{m})h_v(x)$$

$\Lambda(v, v')$ is the Lorentz transformation that takes v' into v

$$\Lambda(v, v') = \frac{1 + \not{v}\not{v}'}{\sqrt{2(1 + v.v')}}$$

Lorentz Symmetry

Imposing invariance ($p/m \to (v+iD/m)/|v+iD/m|$) and using local field redefinitions to bring the HQET lagrangian back to its standard form one gets:

$$c_2 = c_4 = 1 \quad , \quad c_s = 2c_F - 1$$

- It is possible to write the HQET lagrangian in terms of manifestly reparameterization invariant operators
- Local field redefinitions have to be used to bring it to the standard form

Renormalization Group

- The $c_n(\mu)$ contain $\ln(m/\mu)$, which for $\Lambda_{\rm QCD} \lesssim \mu \ll m$ may be large
- Use the fact that physical observables cannot depend on μ to write down RG equations
- The solution of the RG equations sums up terms of the kind $\alpha_{\rm s}^k(m)\ln^k(m/\mu)$
- The cancellation of μ must occur at each order in $1/m$ in the amplitudes $\Longrightarrow$ the RG equations have a triangular form

$$\mu\frac{d}{d\mu}c_n(\mu) = \gamma_n\left(\alpha_{\rm s}(\mu), c_k(\mu)\right) \quad , \quad k \le n$$

Renormalization Group

- E.g. $n = 1$, we only have c_F

$$\gamma_F = \gamma_F\left(\alpha_s(\mu), c_F(\mu)\right) = \tilde{\gamma}_F(\alpha_s(\mu))c_F(\mu)$$

  - The last equality is due to spin symmetry (no mixing with the kinetic term)
  - Exercise 2: calculate $\tilde{\gamma}_F(\alpha_s(\mu))$ at one loop
- E.g. $n = 2$, we only have $c_D(\mu)$ (c_S is given in terms of c_F)

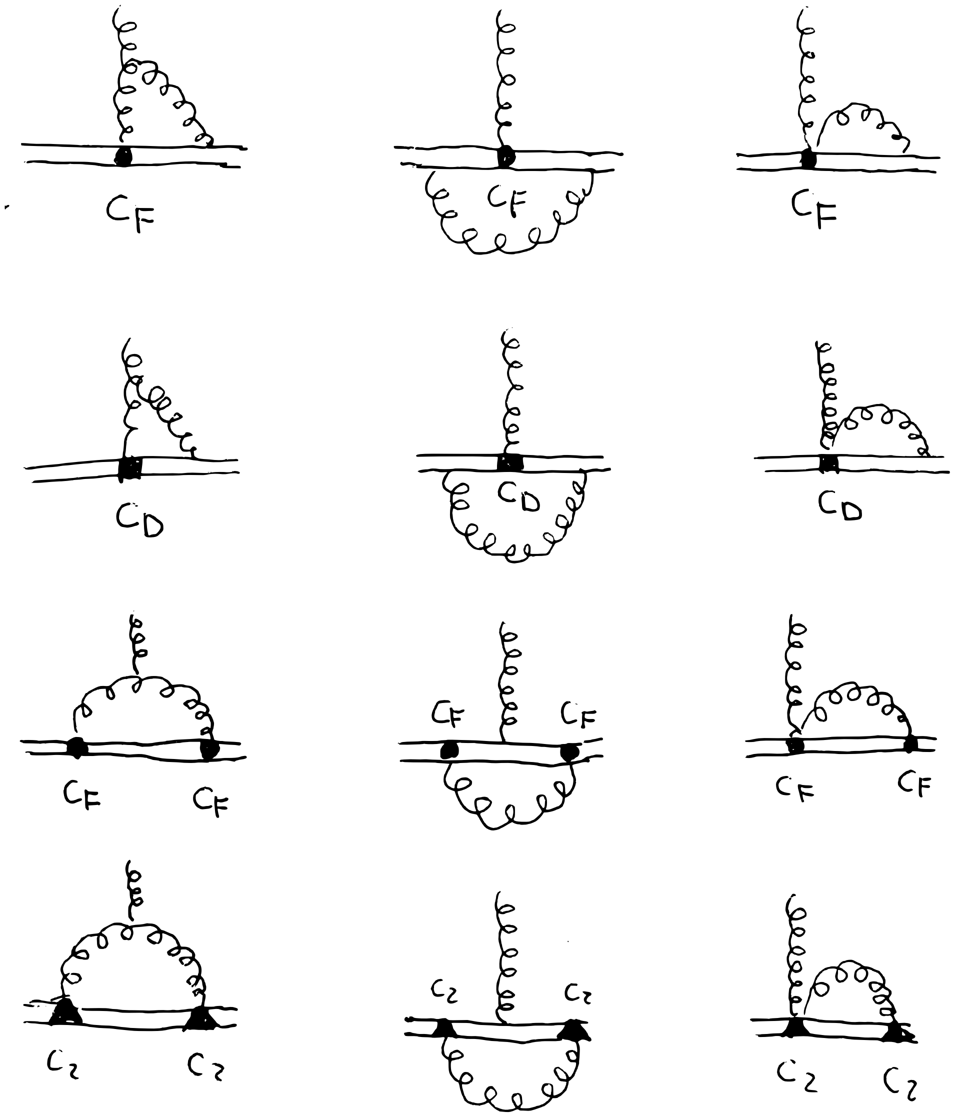
$$\begin{aligned}\gamma_D &= \gamma_D\left(\alpha_s(\mu), c_F(\mu), c_D(\mu)\right) \\ &= \tilde{\gamma}_{D2}(\alpha_s(\mu))c_2^2 + \tilde{\gamma}_{DF}(\alpha_s(\mu))c_F^2(\mu) + \tilde{\gamma}_D(\alpha_s(\mu))c_D(\mu)\end{aligned}$$

  - Spin symmetry forbids terms linear in $c_F(\mu)$ and $c_S(\mu)$

Renormalization Group

$\mu \dfrac{dc_F}{d\mu}$:

$\mu \dfrac{dc_D}{d\mu}$:



Renormalization Group

- $\mu \frac{dc_F}{d\mu}$:

$$\tilde{\gamma}_F = \frac{C_A \alpha_s}{2\pi} \qquad (C_A = N_c) \qquad \text{Bauer, Manohar, 97}$$

- $\mu \frac{dc_D}{d\mu}$:

$$\tilde{\gamma}_D = \frac{13 C_A \alpha_s}{12\pi} \quad , \quad \tilde{\gamma}_{DF} = -\frac{C_A \alpha_s}{12\pi} \quad , \quad \tilde{\gamma}_{D2} = -\frac{\alpha_s}{13\pi}\left(5C_A + 8C_f\right)$$

- No contributions from heavy-light operators and pure gluonic operators

Renormalization Group

The solution of the RG equations at one loop reads

$$c_F(\mu) = z^{-C_A}$$

$$c_D(\mu) = z^{-2C_A} + \left(\frac{20}{13} + \frac{32}{13}\frac{C_F}{C_A}\right)\left[1 - z^{-13C_A/6}\right]$$

$$z = \left[\frac{\alpha_s(\mu)}{\alpha_s(m)}\right]^{1/b_0} \quad , \quad b_0 = 11C_A/3 - 4T_F n_f/3$$

- Application:

$$\frac{m_{B^*} - m_B}{m_{D^*} - m_D} = \left[\frac{\alpha_s(m_D)}{\alpha_s(m_B)}\right]^{-C_A/b_0} \frac{m_D}{m_B} + \mathcal{O}\left(\frac{\Lambda_{\rm QCD}}{m_Q}, \alpha_{\rm s}(m_Q)\right)$$

Renormalization Group

- Does it help?

$$\frac{M_{B^*}^2 - M_B^2}{M_{D^*}^2 - M_D^2} \underset{\text{Exp}}{=} 0.88$$

  - Tree level (LO)

$$\frac{M_{B^*}^2 - M_B^2}{M_{D^*}^2 - M_D^2} \underset{\text{LO}}{\simeq} 1$$

  - One loop matching (NLO)

$$\frac{M_{B^*}^2 - M_B^2}{M_{D^*}^2 - M_D^2} \underset{\text{NLO}}{\simeq} 0.74$$

  - RG improved (LL)

$$\frac{M_{B^*}^2 - M_B^2}{M_{D^*}^2 - M_D^2} \underset{\text{LL}}{\simeq} 0.82$$

Renormalization Group

Remarks:

- The anomalous dimensions in DR can be obtained from the $1/\epsilon$ UV poles
- The integrals must be IR finite $\longrightarrow$ off-shell Green functions
- The following trick helps calculating Feynman integrals involving heavy quark propagators:

$$\int_0^\infty d\lambda \frac{1}{(a\lambda + b)^2} = \frac{1}{ab}$$

$a \longrightarrow$ denominator of a heavy quark propagator

$b \longrightarrow$ denominator of a gluon or light quark propagator

Heavy-to-Light currents

- Leading result (tree level, no power correction):
  $J_\Gamma = \bar{q}\Gamma Q \to \bar{q}\Gamma p_+ h_v e^{-imvx}$
- Power correction (tree level):

$$J_\Gamma \to \bar{q}\Gamma e^{-imvx}\left(1 + \frac{1}{iv.D + 2m_Q} p_- i\not{D}\right) p_+ h_v$$

  expanded in $1/m_Q$
- Radiative correction (one loop, no power correction, $m_q = 0$),
  $\Gamma = \gamma^\mu$, ($\gamma^5\gamma^\mu$ analogous)

$$J_\Gamma \to (C_1(\mu)J_1 + C_2(\mu)J_2)\, e^{-imvx} \quad , \quad J_1 = \bar{q}\gamma^\mu h_v \quad , \quad J_2 = \bar{q}v^\mu h_v$$

Heavy-to-Light currents

- Matching on-shell matrix elements, expanding on $k/m, k'/m$ and using DR $\overline{\mathrm{MS}}$ for both UV and IR divergences the $C_i(\mu)$ can be directly read off the QCD calculation:

$$C_1(\mu) = 1 + \frac{3C_f\alpha_s}{4\pi}\left(\ln\frac{m}{\mu} - \frac{4}{3}\right) \quad , \quad C_2(\mu) = \frac{C_f\alpha_s}{2\pi}$$

- RG improvement (one loop, no power correction):

$$\mu\frac{d}{d\mu}C_1(\mu) = \gamma_1\left(\alpha_s(\mu), C_1(\mu)\right) = \tilde{\gamma}_1\left(\alpha_s(\mu)\right)C_1(\mu)$$

  - In this case the anomalous dimension can be read off the matching coefficient

$$\tilde{\gamma}_1 = -\frac{3C_f\alpha_s}{4\pi}$$

Heavy-to-Light currents

- At leading-log (LL) we have

$$C_1(\mu) = \left[\frac{\alpha_s(\mu)}{\alpha_s(m)}\right]^{\frac{3C_f}{2b_0}} \quad , \quad C_2(\mu) = 0$$

- Radiative+Power corrections
  - At a given order in $1/m$, write down all possible currents compatible with the quantum numbers of the QCD current
  - Take into account RI constraints
  - The matching calculation still reduces to a QCD calculation expanded in powers of k/m, k'/m
  - For the RG equations, take into account $1/m$ contributions both from the current and from the lagrangian

Heavy-to-Heavy currents

- Leading result (tree level, no power correction):
  $J_\Gamma = \bar{Q}'\Gamma Q \to \bar{h}_{v'} p'_+ \Gamma p_+ h_v e^{-imvx+im'v'x}$
- Power corrections (tree level):

$$J_\Gamma \quad \to \quad \bar{h}_{v'} p'_+ \left(1 + i\not{D} p'_- \frac{1}{iv'.D + 2m'}\right) \Gamma e^{-imvx+im'v'x}$$
$$\left(1 + \frac{1}{iv.D + 2m} p_- i\not{D}\right) p_+ h_v$$

  expanded in $1/m_Q$
- Radiative corrections. Let us consider two cases($m = m_b$, $m' = m_c$) : $m \sim m'$ or $m \gg m'$

Heavy-to-Heavy currents

- $m \sim m'$

$$\sum_{Q=c,b} \mathcal{L}_Q \longrightarrow \sum_{Q=c,b} \mathcal{L}_{HQET}^{(Q)}$$

  - Same matching procedure, but integrals in QCD will depend on two scales m and m' (more difficult)

- $m \gg m'$. A two step matching is convenient:

$$\sum_{Q=c,b} \mathcal{L}_Q \longrightarrow \mathcal{L}_c + \mathcal{L}_{HQET}^{(b)} \longrightarrow \sum_{Q=c,b} \mathcal{L}_{HQET}^{(Q)}$$

  - In the first step m' is expanded as well as k and k', only one-scale integral $\sim$ heavy-to-light case (simpler)
  - In the second step only k and k' are expanded $\sim$ calculation in HQET with a massive “light” quark

Heavy-to-Heavy currents

- First step:

$$\bar{c}\gamma^\mu b \longrightarrow \left(C_1(m_b,\mu)J_1 + C_2(m_b,\mu)J_2\right) e^{-im_b v x}$$

$$J_1 = \bar{c}\gamma^\mu h_v^{(b)} \quad , \quad J_2 = \bar{c} v^\mu h_v^{(b)}$$

  - At LO in m_c/m_b the C_i are the same as for the heavy-to-light case

- Second step:

$$J_i \longrightarrow e^{im_c v' x} \sum_{j=1}^{3} C_{ij}(m_c,\mu,v.v') J'_j$$

$$J'_j = \bar{h}_{v'}^{(c)} \Gamma_j h_v^{(b)} \quad , \quad \Gamma_1 = \gamma^\mu \; , \; \Gamma_2 = v^\mu \; , \; \Gamma_3 = v'^\mu$$

Heavy-to-Heavy currents

LL calculation:

- Tree level matching in each step (trivial)
- RG evolution in each step
  - First step, identical to the heavy-to-light case ($m_c \neq 0$ does not modify the UV behavior)
  - Second step,

$$\mu \frac{d}{d\mu} C_{11}(\mu) = \frac{\gamma^{hh}(w)\alpha_s(\mu)}{2\pi} C_{11}(\mu) \quad , \quad w = v.v'$$

$$\gamma^{hh}(w) = 4C_f\,(wr(w) - 1)\ , \quad r(w) = \frac{1}{\sqrt{w^2-1}} \ln\left(w + \sqrt{w^2-1}\right)$$

  - Velocity dependent anomalous dimension (!)

Heavy-to-Heavy currents

LL calculation (Cont.):

$$\bar{c}\gamma^\mu b \longrightarrow e^{im_c v' x} e^{-im_b v x} \left[\frac{\alpha_s(m_b)}{\alpha_s(m_c)}\right]^{-\frac{3C_F}{2b_0[n_f=4]}} \left[\frac{\alpha_s(m_c)}{\alpha_s(\mu)}\right]^{\frac{\gamma^{hh}(v.v')}{2b_0[n_f=3]}} \bar{h}_{v'}^{(c)}\gamma^\mu h_v^{(b)}$$

Decay constants

$|H(p)\rangle$, heavy pseudoscalar state in QCD

$$\langle 0|j_5^\mu(0)|H(p)\rangle = f_H p^\mu \quad , \quad j_5(x) = \bar{q}\gamma^\mu\gamma^5 Q(x)$$

Consider

$$\lim_{p^2\to m_H^2} \int d^4x e^{-ipx} \langle 0|T\{j_5^{\mu\dagger}(x), j_5^\nu(0)\}|0\rangle \sim \frac{f_H^2 p^\mu p^\nu}{p^2 - m_H^2}$$

This equals in tree level HQET ($p = mv + k$)

$$\lim_{v.k\to\bar{\Lambda}} \int d^4x e^{-ikx} \langle 0|T\{j_1^{5\mu\dagger}(x), j_1^{5\nu}(0)\}|0\rangle \sim \frac{\Lambda^3 v^\mu v^\nu}{v.k - \bar{\Lambda}}$$

$m_H = m + \bar{\Lambda}$, $\quad \Lambda$, $\bar{\Lambda} \sim \Lambda_{\rm QCD}$, one obtains $f_H \sim 1/\sqrt{m_H}$

Decay constants

At LL we have

$$\frac{f_B}{f_D} = \left[\frac{\alpha_s(m_b)}{\alpha_s(m_c)}\right]^{-\frac{3C_F}{2b_0[n_f=4]}} \sqrt{\frac{m_D}{m_B}}$$

- How well does it work?

$$0.90 \underset{\text{Lattice}}{\simeq} \frac{f_B}{f_D} \underset{\text{LO}}{\simeq} 0.59 \underset{\text{NLO}}{\simeq} 0.69 \underset{\text{LL}}{\simeq} 0.67$$

Matrix Elements

- m scaling of QCD hadronic states ($|H(p)\rangle = K|\mathcal{H}(v)\rangle$, $|\mathcal{H}(v)\rangle$, mass independent)

$$f_H m_H v^\mu = \langle 0|j_5^\mu(0)|H(p)\rangle = K\ \langle 0|j_1^{5\mu}(0)|\mathcal{H}(v)\rangle$$

  Hence, $K \sim \sqrt{m_H}$

- Covariant representation
  - Heavy-to-light currents ($|X(p')\rangle$ light degrees of freedom, $p' \ll m$)

$$\langle X(p')|\bar{q}\Gamma Q(0)|H(p)\rangle \sim \sqrt{m_H}\langle X(p')|\bar{q}\Gamma h_v(0)|\mathcal{H}(v)\rangle$$

Matrix Elements

$$\langle X(p')|\bar{q}\Gamma h_v(0)|\mathcal{H}(v)\rangle = tr\{A(q',v)\Gamma\mathcal{M}(v)\}$$

- A, Dirac matrix consistent with the symmetries of $X(p')$
- $\mathcal{M}(v)$ must be consistent with the symmetries of HQET

  E.g. For lowest spin multiplet (pseudoscalar, vector)

$$\mathcal{M}(v) = p_+ \begin{pmatrix} -\gamma^5 \\ \not{\epsilon} \end{pmatrix} \quad , \quad p_+ = \frac{1+\not{v}}{2}$$

$\epsilon.v = 0$, $\mathcal{M}(v)p_+ = 0$, e.g. for $v = (1,0,0,0)$,
$\epsilon^\mu_\pm = \frac{1}{\sqrt{2}}(0,1,\pm i,0)$, $\epsilon^\mu_3 = (0,0,0,1)$

- E.g. Vector meson decay constant ($X(p') = 0$, $\Gamma = \gamma^\mu$)

  $\Longrightarrow \quad A = 1$

Matrix Elements

$$f_{H^*}\epsilon^\mu = \langle 0|\bar{q}\gamma^\mu Q(0)|H^*(p)\rangle \sim \sqrt{m_H}\langle 0|\bar{q}\gamma^\mu h_v(0)|\mathcal{H}^*(v)\rangle$$

$$\langle 0|\bar{q}\gamma^\mu h_v(0)|\mathcal{H}^*(v)\rangle = tr\{1.\gamma^\mu p_+ \not{\epsilon}\} = 2\epsilon^\mu$$

Hence $f_{H^*} \sim \sqrt{m_H}$, and repeting the above for the pseudoscalar one obtains

$$f_{H^*} = -f_H m_H$$

E.g. $X(p') = \pi(p')$, $\Gamma = \gamma^\mu(1-\gamma^5)$
$\longrightarrow A(p') = A_1(p'.v)\gamma^5 + A_2(p'.v)\gamma^5 \not{p}'$

- The same two form factors describe the decay of B and D to $\pi l \nu$ when the pion is soft (!)
- In QCD there are two independent form factors for each decay

Matrix Elements

- Covariant representation (Cont.)
  - Heavy-to-heavy currents ($X(p') = H'(p')$, $p' = mv'$)

$$\langle H'(p')|\bar{Q}'\Gamma Q(0)|H(p)\rangle \sim \sqrt{m_H\, m_{H'}}\langle \mathcal{H}(v')|\bar{h}_{v'}\Gamma h_v(0)|\mathcal{H}(v)\rangle$$

$$\langle \mathcal{H}(v')|\bar{h}_{v'}\Gamma h_v(0)|\mathcal{H}(v)\rangle = tr\{A(v',v)\bar{\mathcal{M}}(v')\Gamma\mathcal{M}(v)\}$$

$A(v',v)$, Dirac matrix that transforms like a scalar $\Longrightarrow$

$$A(v',v) = A_0(v'.v).1 + A_1(v'.v)\not{v} + A_2(v'.v)\not{v}' + A_3(v'.v)\not{v}\not{v}'$$

but it is sandwitched

$$\begin{aligned} p_-\ A(v',v)p'_- &= p_-(A_0(v'.v) - A_1(v'.v) - A_2(v'.v) + A_3(v'.v))p'_- \\ &\equiv -\xi(v.v')p_-\ p'_- \end{aligned}$$

Matrix Elements

- Heavy-to-heavy semi-leptonic decays depend on a single form factor, $\xi(\omega)$, $\omega = v.v'$, the Isgur-Wise function (!)
- B to D, D^* decays depend on 6 form factors in QCD

$$
\begin{aligned}
\langle D(v')|\,\bar{c}\gamma^\mu b\,|B(v)\rangle &= h_+(\omega)(v+v')^\mu + h_-(\omega)(v-v')^\mu \\
\langle D^*(v')|\,\bar{c}\gamma^\mu b\,|B(v)\rangle &= i h_v(\omega)\epsilon^{\mu\nu\alpha\beta}\epsilon^*_\nu v'_\alpha v_\beta \\
\langle D^*(v')|\,\bar{c}\gamma^\mu\gamma^5 b\,|B(v)\rangle &= h_{A_1}(\omega)\epsilon^{*\mu} - \left[h_{A_2}(\omega)v^\mu + h_{A_3}(\omega)v'^\mu\right]\epsilon^*.v
\end{aligned}
$$

- At leading order in $1/m$ we have

$$
h_+(\omega) = h_v(\omega) = h_{A_1}(\omega) = h_{A_3}(\omega) = \xi(\omega) \quad , \quad h_-(\omega) = h_{A_2}(\omega) = 0
$$

- Note that the LL corrections have been displayed before
- Power and radiative corrections bring into the game more HQET operators, which can be dealt with analogously.

Matrix Elements

- Note that current conservation implies (V=space volume)

$$\langle B(v)|\int d^3\mathbf{x}\,\bar{h}_v^{(b)}\,\gamma^0 h_v^{(b)}(x)\,|B(v)\rangle = \langle B(v)|B(v)\rangle = 2v^0(2\pi)^3\delta(m\mathbf{v}=0) = 2v^0$$

$$\|$$

$$V\,\langle B(v)|\,\bar{h}_v^{(b)}\,\gamma^0 h_v^{(b)}(0)\,|B(v)\rangle = V\xi(1)2v^0 \quad \Longrightarrow \quad \xi(1)=1$$

- Measuring $B\to Dl\bar{\nu}$ at zero recoil $(v=v')$ has no hadronic uncertainties up to $1/m$
- Luke's theorem (Luke, 90): no hadronic uncertainties at zero recoil up to $1/m^2$
- Very useful for a precise determination of V_{cb}

Bibliography (Reviews)

- Grinstein, 91
- Neubert, 93
- Mannel, 96