Effective Field Theories for Heavy Quarks

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L3: Heavy-heavy systems, NRQCD, pNRQCD

Heavy-heavy systems

$Q\bar{Q}$ bound state = heavy quarkonium , $m_Q >> \Lambda_{QCD}$, $\alpha_s(m_Q) << 1$

- The heavy quarks in the rest frame of the heavy quarkonium are not at rest anymore, but move slowly $\mathrm{v} << 1$.

  Achtung: v must not be mistaken with v, the velocity of the meson
- In the heavy quarkonium rest frame, the typical momentum of the heavy quarks $p = m_Q v + k$ ($v = (1, \mathbf{0})$) is such that $k^0 \sim m_Q \mathrm{v}^2$, $\mathbf{k} \sim m_Q \mathrm{v}$
- Note that the only difference with HQET is that in HQET $k^0 \sim \mathbf{k} \ll m_Q$, whereas here we have $k^0 \sim m_Q \mathrm{v}^2 \ll \mathbf{k} \sim m_Q \mathrm{v} \ll m_Q$.

Heavy-heavy systems

We face a multi-scale problem:

- $m_Q >> m_Q\mathrm{v} >> m_Q\mathrm{v}^2$
- $m_Q >> \Lambda_{QCD}$

Naïve derivation of NRQCD: make $SU(3)$ gauge invariant the lagrangian of a non-relativistic particle,

$$\mathcal{L}_Q + \mathcal{L}_{\bar{Q}} \longrightarrow \mathcal{L}_{NRQCD}^{Q\bar{Q}} =$$

$$\psi^\dagger \left(iD_0 + \frac{1}{2m_Q}\mathbf{D}^2 \right) \psi + \chi^\dagger \left(-iD_0 + \frac{1}{2m_Q}\mathbf{D}^2 \right) \chi$$

$\psi \sim h_v$ in HQET, Pauli spinor field that annihilates non-relativistic quarks; $\chi \sim h_{-v}$ in HQET, Pauli spinor field that creates non-relativistic anti-quarks. Caswell, Lepage, 86

Heavy-heavy systems

$$\mathcal{L}^{Q\bar{Q}}_{NRQCD} = \psi^\dagger \left(iD_0 + \frac{1}{2m_Q}\mathbf{D}^2 \right) \psi + \chi^\dagger \left(-iD_0 + \frac{1}{2m_Q}\mathbf{D}^2 \right) \chi$$

- Note that the heavy flavor symmetry of HQET is lost, but the $SU(2)$ spin symmetry for each heavy flavor remains $\Longrightarrow$ The spectrum ($^{2S+1}L_J$) is organized in spin multiplets
  - Doublets for $L = 0$, e.g. ($^3S_1$, $^1S_0$)

$$(J/\psi\,,\,\eta_c) \quad , \quad (\Upsilon\,,\,\eta_b)$$

  - Quadruplets for $L \neq 0$, e.g. ($^3P_0$, $^3P_1$, $^3P_2$, $^1P_1$)

$$(\chi_{c0}\,,\,\chi_{c1}\,,\,\chi_{c2}\,,\,h_c) \quad , \quad (\chi_{b0}\,,\,\chi_{b1}\,,\,\chi_{b2}\,,\,h_b)$$

Heavy-heavy systems

- How well does it work?

$$m_{J/\psi} - m_{\eta_c} \simeq 123\,\mathrm{MeV} \ll m_{\chi_{c1}} - m_{J/\psi} \simeq 514\,\mathrm{MeV}$$

$$m_{\Upsilon} - m_{\eta_b} \simeq 61\,\mathrm{MeV} \ll m_{\chi_{b1}} - m_{\Upsilon} \simeq 413\,\mathrm{MeV}$$

- No heavy flavor symmetry

$$2m_D - m_{J/\psi} \simeq 643\,\mathrm{MeV} \nsim 2m_B - m_{\Upsilon} \simeq 1098\,\mathrm{MeV}$$

- One expects the naïve picture to be modified by
  - Power corrections: $\sim 1/m_Q$ suppressed terms (including operators with four heavy fields)
  - Radiative corrections: $\sim \alpha_{\mathrm{s}}(m_Q)$ suppressed terms

Power Corrections

Suppose A_μ contains small momenta ($\ll m_Q$) only

$$Q(x) = e^{-im_Q vx} p_+ h_v^{(Q)}(x) + e^{im_Q vx} p_- h_{-v}^{(Q)}(x) + H_v^{(Q)}(x)$$

- $p_+ H_v^{(Q)}(x)$ contains the momenta which are NOT in $e^{-im_Q vx} p_+ h_v^{(Q)}(x)$
- $p_- H_v^{(Q)}(x)$ contains the momenta which are NOT in $e^{im_Q vx} p_- h_{-v}^{(Q)}(x)$
- $h_v^{(Q)}(x)$, $h_{-v}^{(Q)}(x)$ contain small momenta ($\ll m_Q$) only

Power Corrections

$$
\begin{aligned}
\mathcal{L}_Q &= \left(\bar{h}_v^{(Q)} p_+ e^{im_Q vx} + \bar{h}_{-v}^{(Q)} p_- e^{-im_Q vx} + \bar{H}_v^{(Q)}\right)(i\not{D} - m_Q) \\
&\times \left(e^{-im_Q vx} p_+ h_v^{(Q)} + e^{im_Q vx} p_- h_{-v}^{(Q)}(x) + H_v^{(Q)}\right) \\
&= \bar{h}_v^{(Q)} iv.D h_v^{(Q)} \quad + \quad \bar{h}_{-v}^{(Q)} \left(-iv.D\right) h_{-v}^{(Q)} \\
&+ \bar{h}_v^{(Q)} p_+ e^{im_Q vx} i\not{D} p_- H_v^{(Q)} + \bar{H}_v^{(Q)} p_- i\not{D} e^{-im_Q vx} p_+ h_v^{(Q)}(x) \\
&+ \bar{h}_{-v}^{(Q)} p_- e^{-im_Q vx} i\not{D} p_+ H_v^{(Q)} + \bar{H}_v^{(Q)} p_+ i\not{D} e^{im_Q vx} p_- h_{-v}^{(Q)}(x) \\
&+ \bar{H}_v^{(Q)} \left(i\not{D} - m_Q\right) H_v^{(Q)}
\end{aligned}
$$

- Momentum conservation:
  - Only $p_- H_v^{(Q)}$ of momenta $\sim m_Q v$ couple to $e^{-im_Q vx} p_+ h_v^{(Q)}$
  - These modes do not couple to $p_+ H_v^{(Q)}$
  - Only $p_+ H_v^{(Q)}$ of momenta $\sim -m_Q v$ couple to $e^{im_Q vx} p_- h_{-v}^{(Q)}$
  - These modes do not couple to $p_- H_v^{(Q)}$

Power Corrections

- Hence, the quark and antiquark contributions totally decouple (!)

$$\mathcal{L}_{NRQCD}^{Q\bar{Q}} = \mathcal{L}_{HQET}^{(Q)} + \mathcal{L}_{HQET}^{(\bar{Q})}$$

including power corrections $\left(\mathcal{L}_{HQET}^{(\bar{Q})} = \mathcal{L}_{HQET}^{(Q)}|_{v\to -v}\right)$

  - $h_v^{(Q)}(x) = \psi(x)$, annihilates NR quarks ($v = (1, \mathbf{0})$)
  - $h_{-v}^{(Q)}(x) = \chi(x)$, creates NR antiquarks ($v = (1, \mathbf{0})$)

- Note that $\mathcal{L}_{NRQCD}^{Q\bar{Q}}$ has a global $U(1)^2$ symmetry $\longrightarrow$ the number of NR quarks and the number of NR antiquarks are separately conserved

- Same result if the flavor of Q and $\bar{Q}$ differ

Radiative Corrections

A_μ also contains large momenta ($\sim m_Q$) $\longrightarrow$ $\alpha_{\rm s}(m_Q)$ corrections expected.

- Single NR quark sector: the same as in HQET
- Single NR antiquark sector: the same as in HQET
- One NR quark and one NR antiquark sector (does not exist in HQET):

$$\mathcal{L}_{\psi\chi} = \frac{f_1(^1S_0)}{m_Q^2} O_1(^1S_0) + \frac{f_1(^3S_1)}{m_Q^2} O_1(^3S_1) + \\ + \frac{f_8(^1S_0)}{m_Q^2} O_8(^1S_0) + \frac{f_8(^3S_1)}{m_Q^2} O_8(^3S_1) + \mathcal{O}(\frac{1}{m_Q^4})$$

Radiative Corrections

- One NR quark and one NR antiquark sector (Cont.)

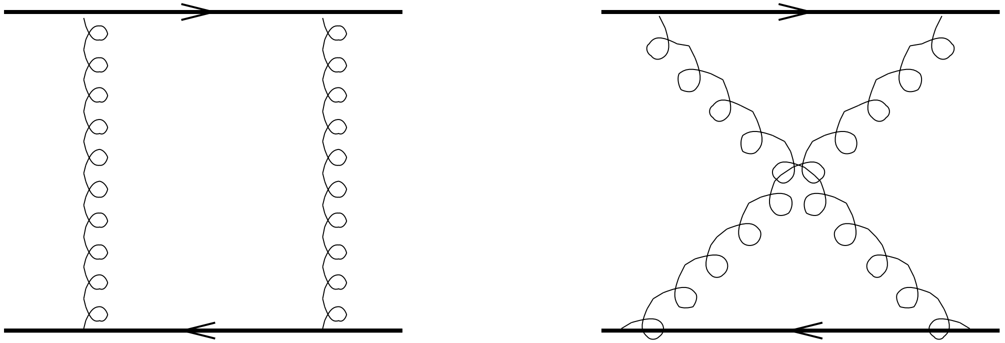
$$O_1(^1S_0) = \psi^\dagger \chi \, \chi^\dagger \psi \quad , \quad O_1(^3S_1) = \psi^\dagger \boldsymbol{\sigma} \chi \, \chi^\dagger \boldsymbol{\sigma} \psi,$$
$$O_8(^1S_0) = \psi^\dagger \mathrm{T}^a \chi \, \chi^\dagger \mathrm{T}^a \psi \quad , \quad O_8(^3S_1) = \psi^\dagger \mathrm{T}^a \boldsymbol{\sigma} \chi \, \chi^\dagger \mathrm{T}^a \boldsymbol{\sigma} \psi.$$

  - The fs are short distance matching coefficients which depend on m_Q and μ (factorization scale)
  - The fs can also be obtained from a calculation in QCD in DR $\overline{\mathrm{MS}}$ for both IR and UV divergences
  - The fs contain imaginary parts

Bodwin, Braaten, Lepage, 94

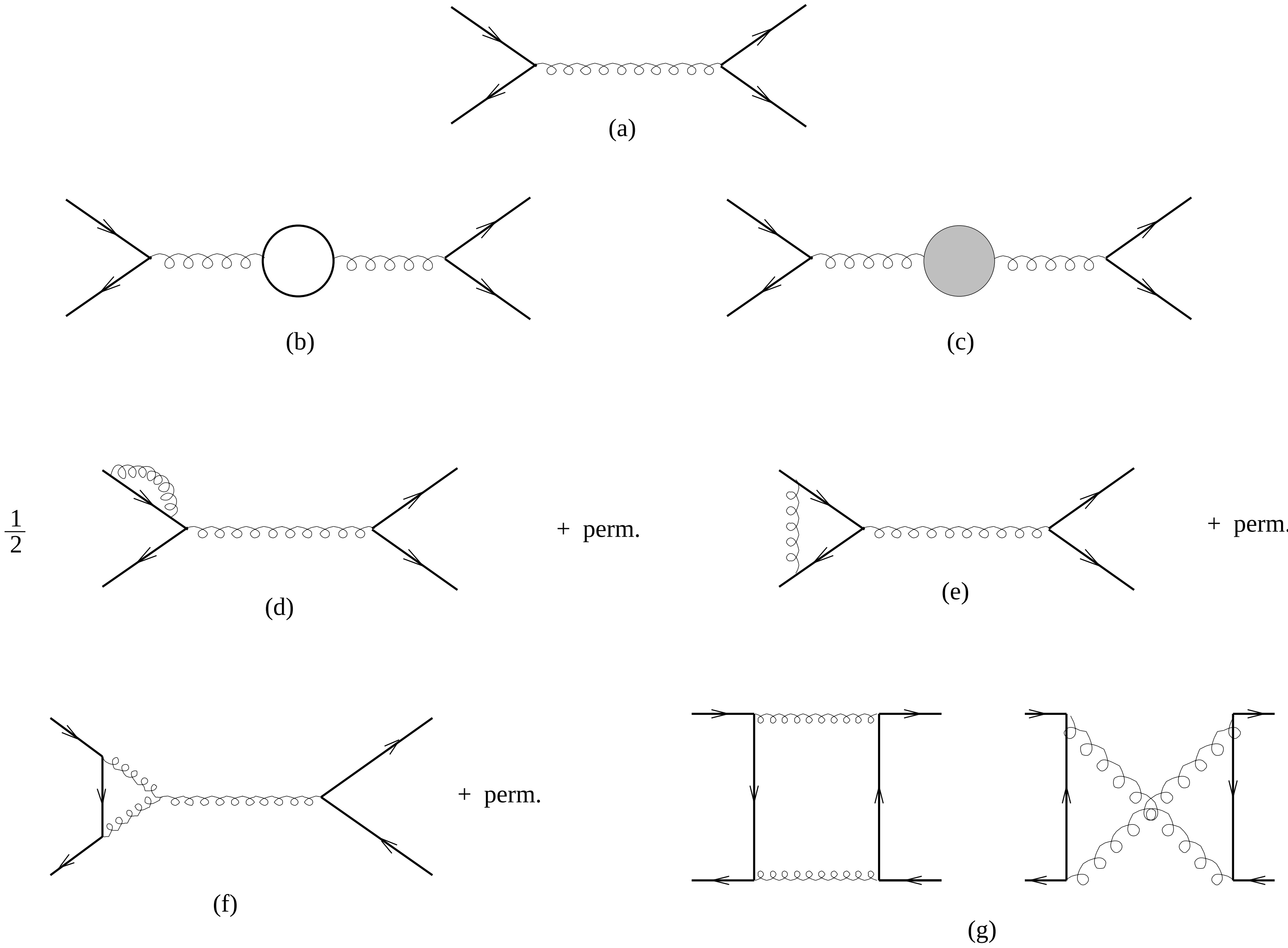
NRQCD(Cont.)

- QCD diagrams for the NRQCD 4-fermion operators
  - Always



NRQCD(Cont.)

- If Q and $\bar{Q}$ have the same flavor



Radiative Corrections

- Inclusive decays

$$\Gamma(\chi_Q(nJS) \to LH) = \frac{2}{m_Q^2}\Bigg(\mathrm{Im}\, f_1({}^{2S+1}P_J) \times$$

$$\times \frac{\langle \chi_Q(nJS)|O_1({}^{2S+1}P_J)|\chi_Q(nJS)\rangle}{m_Q^2}$$

$$+\mathrm{Im}\, f_8({}^{2S+1}S_S)\langle \chi_Q(nJS)|O_8({}^1S_0)|\chi_Q(nJS)\rangle\Bigg)$$

Radiative Corrections

- For P-wave decays higher dimensional operators are needed

$$\delta\mathcal{L}_{\chi\psi} = \frac{f_1({}^1P_1)}{m_Q^4}\,\mathcal{O}_1({}^1P_1) + \frac{f_1({}^3P_0)}{m_Q^4}\,\mathcal{O}_1({}^3P_0) + \frac{f_1({}^3P_1)}{m_Q^4}\,\mathcal{O}_1({}^3P_1) + \frac{f_1({}^3P_2)}{m_Q^4}\,\mathcal{O}_1({}^3P_2) + \ldots$$

$$\begin{aligned}
\mathcal{O}_1({}^1P_1) &= \psi^\dagger(-\tfrac{i}{2}\overleftrightarrow{\mathbf{D}})\chi\cdot\chi^\dagger(-\tfrac{i}{2}\overleftrightarrow{\mathbf{D}})\psi,\\
\mathcal{O}_1({}^3P_0) &= \frac{1}{3}\,\psi^\dagger(-\tfrac{i}{2}\overleftrightarrow{\mathbf{D}}\cdot\boldsymbol{\sigma})\chi\,\chi^\dagger(-\tfrac{i}{2}\overleftrightarrow{\mathbf{D}}\cdot\boldsymbol{\sigma})\psi\\
\mathcal{O}_1({}^3P_1) &= \frac{1}{2}\,\psi^\dagger(-\tfrac{i}{2}\overleftrightarrow{\mathbf{D}}\times\boldsymbol{\sigma})\chi\cdot\chi^\dagger(-\tfrac{i}{2}\overleftrightarrow{\mathbf{D}}\times\boldsymbol{\sigma})\psi\\
\mathcal{O}_1({}^3P_2) &= \psi^\dagger(-\tfrac{i}{2}\overleftrightarrow{D}{}^{(i}\sigma^{j)})\chi\,\chi^\dagger(-\tfrac{i}{2}\overleftrightarrow{D}{}^{(i}\sigma^{j)})\psi
\end{aligned}$$

Radiative Corrections

- Im$(f_1({}^{2S+1}P_J)$ at $\mathcal{O}(\alpha_s^3)$ are factorization scale dependent (Barbieri, Caffo, Gatto, Remiddi, 80)

$$
\begin{aligned}
\mathrm{Im}\, f_1({}^1P_1) &= \frac{4(C_A^2-4)C_F}{3}\left(\frac{C_A}{2}-C_F\right)^2 \alpha_s^3\left(-\frac{7}{3}+\frac{7}{48}\pi^2-\log\frac{\mu}{2m}\right)\\
\mathrm{Im}\, f_1({}^3P_0) &= 3C_F\left(\frac{C_A}{2}-C_F\right)\pi\alpha_s(m)^2\\
&\times\left\{1+\frac{\alpha_s}{\pi}\left[\left(-\frac{7}{3}+\frac{\pi^2}{4}\right)C_F+\left(\frac{427}{81}-\frac{\pi^2}{144}\right)C_A+\frac{4}{27}n_f\left(-\frac{29}{6}-\log\frac{\mu}{2m}\right)\right]\right\}\\
\mathrm{Im}\, f_1({}^3P_1) &= \frac{C_F}{2}\left(\frac{C_A}{2}-C_F\right)\alpha_s^3\left[\left(\frac{587}{27}-\frac{317}{144}\pi^2\right)C_A+\frac{8}{9}n_f\left(-\frac{4}{3}-\log\frac{\mu}{2m}\right)\right]\\
\mathrm{Im}\, f_1({}^3P_2) &= \frac{4}{5}C_F\left(\frac{C_A}{2}-C_F\right)\pi\alpha_s(m)^2\\
&\times\left\{1+\frac{\alpha_s}{\pi}\left[-4C_F+\left(\frac{2185}{216}-\frac{337}{384}\pi^2+\frac{5}{3}\log 2\right)C_A+\frac{5}{9}n_f\left(-\frac{29}{15}-\log\frac{\mu}{2m}\right)\right]\right\}
\end{aligned}
$$

Radiative Corrections

- The scale dependence cancels against the one of soft gluon corrections to the color octet operator

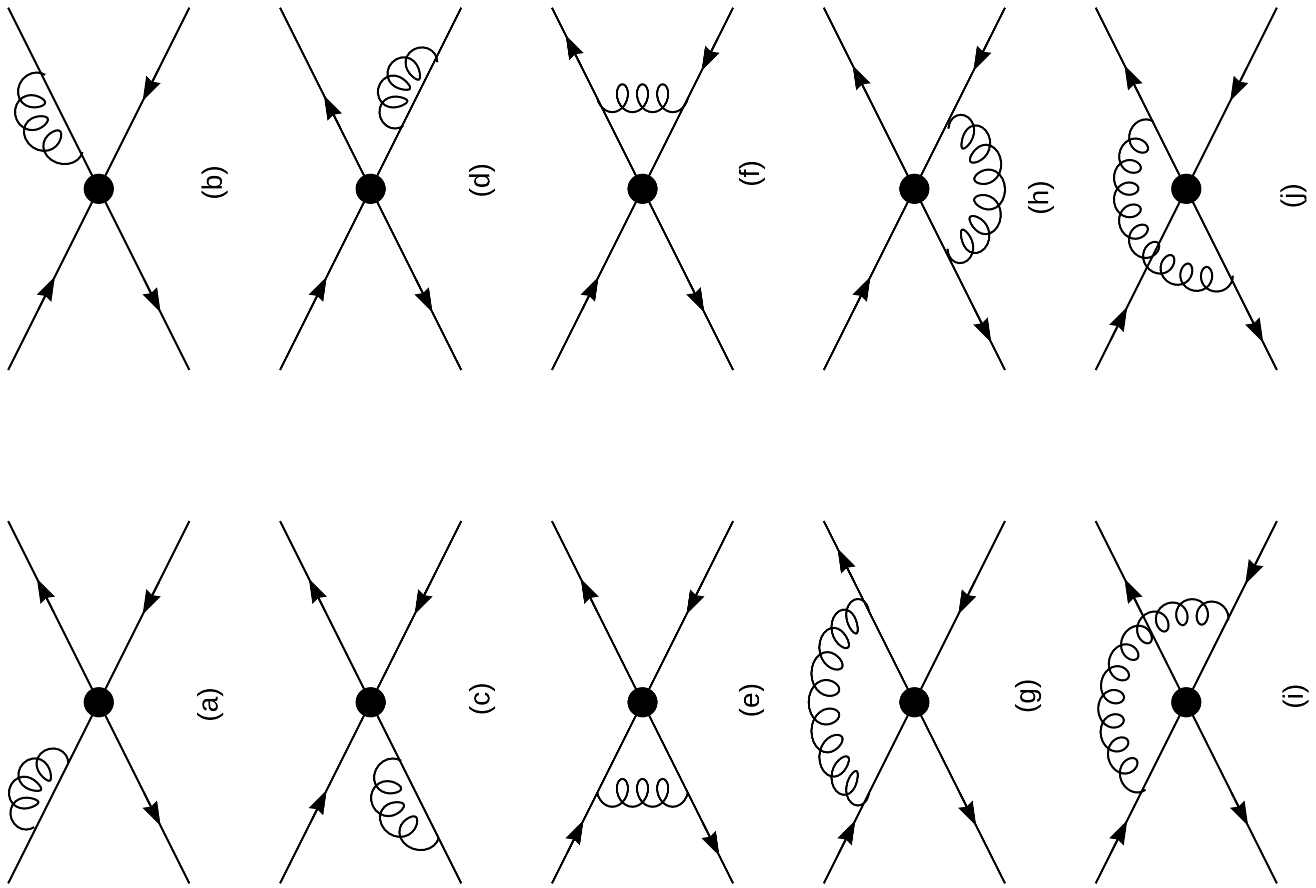


Figure 11

Calculations in NRQCD

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    - Mode decomposition

Calculations in NRQCD

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- The scales $m_Q\mathrm{v}$, $m_Q\mathrm{v}^2$ and Λ_{QCD} are not disentangled $\Longrightarrow$ The counting is not homogeneous
  - Solutions:
    - Integrate out energy scales $\sim m_Q v$ $\longrightarrow$ pNRQCD, A. Pineda and J. Soto, Nucl. Phys. Proc. Suppl. **64**, 428 (1998)
    - Mode decomposition $\longrightarrow$ vNRQCD, M. E. Luke, A. V. Manohar and I. Z. Rothstein, Phys. Rev. D **61**, 074025 (2000)

pNRQCD

Keep only fields of energies $\sim m_Q \mathrm{v}^2$, i. e. integrate out soft and potential gluons and soft quarks $\Longrightarrow$ potentials arise as a matching coefficients

- Tree level

$$L_{pNRQCD} = \int d^3\mathbf{x} \left[\psi^\dagger \left(iD_0 + \frac{1}{2m_Q}\mathbf{D}^2 \right) \psi + \chi^\dagger \left(-iD_0 + \frac{1}{2m_Q}\mathbf{D}^2 \right) \chi \right]$$
$$+ \int d^3\mathbf{x}_1 \int d^3\mathbf{x}_2 \frac{\alpha_s}{|\mathbf{x}_1 - \mathbf{x}_2|} tr\left(\psi^\dagger(\mathbf{x}_1) T^a \psi(\mathbf{x}_1) \chi^\dagger(\mathbf{x}_2) T^a \chi(\mathbf{x}_2) \right)$$

  - D_0, $\mathbf{D}$ contain US gluons only ($E \sim m_Q \mathrm{v}^2$)
- Projecting onto $\int d^3\mathbf{x}_1 \int d^3\mathbf{x}_2 \psi^\dagger(\mathbf{x}_1) \Psi(\mathbf{x}_1, \mathbf{x}_2, t) \chi(\mathbf{x}_2) |0\rangle$

pNRQCD

$$S_{pNRQCD} = \int d^3\mathbf{x}_1\, d^3\mathbf{x}_2\, dt\, \mathrm{tr}\Bigg(\Psi^\dagger(\mathbf{x}_1, \mathbf{x}_2, t)$$

$$\left\{ iD_0 + \frac{\mathbf{D}^2_{\mathbf{x}_1}}{2m} + \frac{\mathbf{D}^2_{\mathbf{x}_2}}{2m} \right\} \Psi(\mathbf{x}_1, \mathbf{x}_2, t) \Bigg)$$

$$+ \frac{\alpha_s}{|\mathbf{x}_1 - \mathbf{x}_2|} \mathrm{tr}\Bigg(T^a \Psi(\mathbf{x}_1, \mathbf{x}_2, t) T^a \Psi^\dagger(\mathbf{x}_1, \mathbf{x}_2, t) \Bigg)$$

Upon making the local field redefinition,

$$\Psi(\mathbf{x}_1, \mathbf{x}_2, t) = P\left[e^{ig\int_{\mathbf{x}_2}^{\mathbf{x}_1} \mathbf{A}\, d\mathbf{x}}\right] S(\mathbf{x}, \mathbf{X}, t)$$

$$+ P\left[e^{ig\int_{\mathbf{X}}^{\mathbf{x}_1} \mathbf{A}\, d\mathbf{x}}\right] O(\mathbf{x}, \mathbf{X}, t) P\left[e^{ig\int_{\mathbf{x}_2}^{\mathbf{X}} \mathbf{A}\, d\mathbf{x}}\right]$$

$$\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2, \;\; \mathbf{X} = (\mathbf{x}_1 + \mathbf{x}_2)/2$$

pNRQCD

Under gauge transformations,

$$\Psi(\mathbf{x}_1,\mathbf{x}_2,t) \rightarrow g(\mathbf{x}_1,t)\Psi(\mathbf{x}_1,\mathbf{x}_2,t)g^{-1}(\mathbf{x}_2,t)$$

$$S(\mathbf{x},\mathbf{X},t) \rightarrow S(\mathbf{x},\mathbf{X},t)$$

$$O(\mathbf{x},\mathbf{X},t) \rightarrow g(\mathbf{X},t)O(\mathbf{x},\mathbf{X},t)g^{-1}(\mathbf{X},t)$$

Upon multipole expanding (up to O(x^2)),

$$\mathcal{L}_{pNRQCD} = \int d^3\mathbf{x}\, tr\Bigg\{ S^\dagger\left\{ i\partial_0 - \frac{\mathbf{p}^2}{m} + \frac{C_f\alpha_s}{|\mathbf{x}|}\right\}S + O^\dagger\left\{ iD_0 - \frac{\mathbf{p}^2}{m} - \frac{1}{2N_c}\frac{\alpha_s}{|\mathbf{x}|}\right\}O$$

$$+g\mathbf{x}O\mathbf{E}(\mathbf{X},t)S^\dagger + g\mathbf{x}O^\dagger\mathbf{E}(\mathbf{X},t)S + \frac{g}{2}\mathbf{x}OO^\dagger\mathbf{E}(\mathbf{X},t) + \frac{g}{2}\mathbf{x}O^\dagger O\mathbf{E}(\mathbf{X},t)\Bigg\}$$

pNRQCD

Assumption: $\Lambda_{QCD} \lesssim m_Q \mathrm{v}^2 \ \equiv$ weak coupling regime

- General structure (x=r)

$$\mathcal{L}_{\rm pNRQCD} = \int d^3\mathbf{r} \left(\mathrm{Tr} \left\{ \mathrm{S}^\dagger \left(i\partial_0 - h_s(\mathbf{r}, \mathbf{p}, \mathbf{P_R}, \mathbf{S}_1, \mathbf{S}_2, \mu) \right) \mathrm{S} + \right. \right.$$

$$\left. + \mathrm{O}^\dagger \left(iD_0 - h_o(\mathbf{r}, \mathbf{p}, \mathbf{P_R}, \mathbf{S}_1, \mathbf{S}_2, \mu) \right) \mathrm{O} \right\}$$

$$+ V_A(r,\mu) \mathrm{Tr} \left\{ \mathrm{O}^\dagger \mathbf{r} \cdot g\mathbf{E}\, \mathrm{S} + \mathrm{S}^\dagger \mathbf{r} \cdot g\mathbf{E}\, \mathrm{O} \right\} +$$

$$\left. + \frac{V_B(r,\mu)}{2} \mathrm{Tr} \left\{ \mathrm{O}^\dagger \mathbf{r} \cdot g\mathbf{E}\, \mathrm{O} + \mathrm{O}^\dagger \mathrm{O} \mathbf{r} \cdot g\mathbf{E} \right\} + \mathcal{O}\left(1/m_Q\right) \right)$$

h_s, $h_o =$ quantum mechanical Hamiltonians with scale dependent potentials calculable in pertubation theory in $\alpha_{\rm s}(m_Q \mathrm{v})$

pNRQCD

$$
\begin{aligned}
h_s \;=\;& c_k \frac{\mathbf{p}^2}{m} - c_4 \frac{\mathbf{p}^4}{4m^3} - C_f \frac{\alpha_{V_s}}{r} - \frac{C_f C_A D_s^{(1)}}{2mr^2} \\
& - \frac{C_f D_{1,s}^{(2)}}{2m^2} \left\{ \frac{1}{r}, \mathbf{p}^2 \right\} + \frac{C_f D_{2,s}^{(2)}}{2m^2} \frac{1}{r^3} \mathbf{L}^2 + \frac{\pi C_f D_{d,s}^{(2)}}{m^2} \delta^{(3)}(\mathbf{r}) \\
& + \frac{4\pi C_f D_{S^2,s}^{(2)}}{3m^2} \mathbf{S}^2 \delta^{(3)}(\mathbf{r}) + \frac{3 C_f D_{LS,s}^{(2)}}{2m^2} \frac{1}{r^3} \mathbf{L} \cdot \mathbf{S} + \frac{C_f D_{S_{12},s}^{(2)}}{4m^2} \frac{1}{r^3} S_{12}(\hat{\mathbf{r}})
\end{aligned}
$$

$m = m_Q$

- Lorentz invariance implies $c_k = c_4 = 1$, and relations between potentials at different orders of $1/m_Q$ (Gromes 84; Barchielli, Brambilla, Prospieri 90; Berwein, Brambilla, Hwang, Vairo 18)

pNRQCD

$$\alpha_{V_s} = \alpha_s(r)\left\{1 + (a_1 + 2\gamma_E\beta_0)\,\frac{\alpha_s(r)}{4\pi}\right.$$

$$\left. + \left[\gamma_E\,(4a_1\beta_0 + 2\beta_1) + \left(\frac{\pi^2}{3} + 4\gamma_E^2\right)\beta_0^2 + a_2\right]\frac{\alpha_s^2(r)}{16\,\pi^2} + \frac{C_A^3}{12}\,\frac{\alpha_s^3}{\pi}\,\ln\mu r\right\}$$

$$D_s^{(1)} = \alpha_s^2(r)\left\{1 + \frac{2}{3}(4C_f + 2C_A)\frac{\alpha_s}{\pi}\,\ln\mu r\right\}$$

$$D_{1,s}^{(2)} = \alpha_s(r)\left\{1 + \frac{4}{3}C_A\frac{\alpha_s}{\pi}\,\ln\mu r\right\},$$

$$D_{2,s}^{(2)} = \alpha_s(r)$$

$$D_{d,s}^{(2)} = \alpha_s(r)\left\{1 + \frac{\alpha_s}{\pi}\left(\frac{2C_f}{3} + \frac{17C_A}{3}\right)\ln mr + \frac{16}{3}\,\frac{\alpha_s}{\pi}\left(\frac{C_A}{2} - C_f\right)\ln\mu r\right\}$$

$$D_{S^2,s}^{(2)} = \alpha_s(r)\left\{1 - \frac{7C_A}{4}\,\frac{\alpha_s}{\pi}\,\ln mr\right\}$$

$$D_{LS,s}^{(2)} = \alpha_s(r)\left\{1 - \frac{2C_A}{3}\,\frac{\alpha_s}{\pi}\,\ln mr\right\}$$

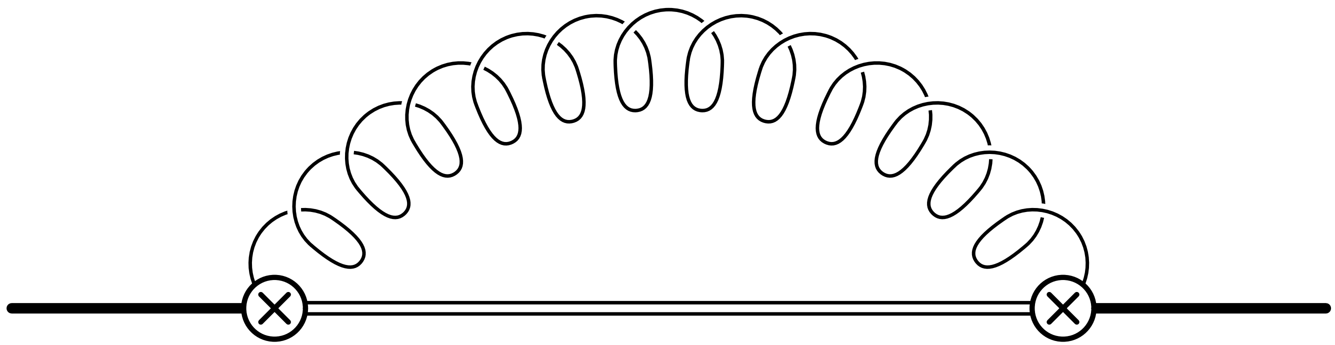
$$D_{S_{12},s}^{(2)} = \alpha_s(r)\left\{1 - C_A\frac{\alpha_s}{\pi}\,\ln mr\right\}$$

pNRQCD

- Expanding the US scales, and using DR $\overline{\rm MS}$ for both IR and UV divergences only a NRQCD calculation is necessary
- The potentials can be calculated in both an $1/m$ and $\alpha_{\rm s}(1/r)$ ($1/r \sim m{\rm v}$) expansion (the pinch singularities cancel in the matching)
- The NRQCD operators in the one NR quark one NR antiquark sector are inherited as $\delta(\mathbf{r})$-type potentials
- As matching coefficients, the potentials obey renormalization group equations (!), (see Pineda, 11, for a review)

pNRQCD

- The μ-dependence at the order shown before (NNNLO, $m\alpha_s^5$) is compensated by ultrasoft contributions



solid (double) line $=$ singlet (octet) propagator

- Beyond that order there are also further μ-dependences inherited from the hard scale (m), that are conpensate by potential contributions from quantum mechanics perturbation theory

pNRQCD

- When is the weak coupling regime reliable?
  - Up to NNLO $(m\alpha_s^4)$ we only need mv $\sim m\alpha_s \gg \Lambda_{QCD}$ (no US contributions)
  - Beyond NNLO (US loops) we need $m\text{v}^2 \sim m\alpha_s^2 \gg \Lambda_{QCD}$
- When $m\text{v}^2 \sim \Lambda_{QCD}$, the leading non-perturbative effects in the spectrum go like $r^2\Lambda_{QCD}^3$
- For Coulombic states $r \sim n^2/m\alpha_s$, $n =$ principal quantum number $\Longrightarrow$ non-perturbative effects become very important for excited states $\sim n^4\Lambda_{QCD}^3/(m\alpha_s)^2$ (Leutwyler, 80)
- In practise it appears to be relible for $\Upsilon(1S)$ and to a lesser extend for J/ψ

E.g. Extraction of α_s

E.g. Extraction of $\alpha_{\rm s}$

- Old formula ($X =$ light hadrons)

$$R_\gamma \equiv \frac{\Gamma(\Upsilon(1S) \to \gamma\, X)}{\Gamma(\Upsilon(1S) \to X)} = \frac{36}{5}\frac{e_b^2\alpha}{\alpha_{\rm s}}\frac{N}{D},$$

$$N = 1 + C_{gg\gamma}\frac{\alpha_{\rm s}}{\pi} + C_{\gamma\mathcal{P}_1(^3S_1)}\mathcal{R}_{\mathcal{P}_1(^3S_1)}$$

$$D = 1 + C_{ggg}\frac{\alpha_{\rm s}}{\pi} + C_{\mathcal{P}_1(^3S_1)}\mathcal{R}_{\mathcal{P}_1(^3S_1)}$$

E.g. Extraction of α_s

- Old formula + Color octet contributions

$$R_\gamma \equiv \frac{\Gamma(\Upsilon(1S) \to \gamma\, X)}{\Gamma(\Upsilon(1S) \to X)} = \frac{36}{5} \frac{e_b^2 \alpha}{\alpha_s} \frac{N}{D},$$

$$\begin{aligned} N &= 1 + C_{gg\gamma}\frac{\alpha_s}{\pi} + C_{\mathcal{P}_1({}^3S_1)}\mathcal{R}_{\mathcal{P}_1({}^3S_1)} \\ &\quad + \frac{\pi}{\alpha_s} C_{\gamma\, O_8({}^1S_0)} \mathcal{R}_{O_8({}^1S_0)} + \frac{\pi}{\alpha_s} C_{\gamma\, O_8({}^3P_0)} \mathcal{R}_{O_8({}^3P_0)}, \\ D &= 1 + C_{ggg}\frac{\alpha_s}{\pi} + C_{\mathcal{P}_1({}^3S_1)}\mathcal{R}_{\mathcal{P}_1({}^3S_1)} + \frac{\pi}{\alpha_s} C_{O_8({}^3S_1)} \mathcal{R}_{O_8({}^3S_1)} \\ &\quad + \frac{\pi}{\alpha_s} C_{O_8({}^1S_0)} \mathcal{R}_{O_8({}^1S_0)} + \frac{\pi}{\alpha_s} C_{O_8({}^3P_0)} \mathcal{R}_{O_8({}^3P_0)}, \end{aligned}$$

E.g. Extraction of α_s

- C_O , $C_{\gamma O}$ $\sim$ imaginary part of the matching coefficient of O

$$C_{O_1(^3S_1)} = 1 + C_{ggg}\frac{\alpha_s}{\pi} \quad , \quad C_{\gamma O_1(^3S_1)} = 1 + C_{gg\gamma}\frac{\alpha_s}{\pi}$$

$$O = O_1(^3S_1)\,,\ \mathcal{P}_1(^3S_1)\,,\ O_8(^3P_0)\,,\ O_8(^1S_0)\,,\ O_8(^3S_1)$$

$$\mathcal{P}_1(^3S_1) = \frac{1}{2}\left[\psi^\dagger \boldsymbol{\sigma}\chi \cdot \chi^\dagger \boldsymbol{\sigma}(-\tfrac{i}{2}\overleftrightarrow{\mathbf{D}})^2\psi \ + \ \text{h.c.}\right]$$

$$\mathcal{R}_O = \frac{\langle \Upsilon(1S)|O|\Upsilon(1S)\rangle}{m_b^{\Delta d}\,\langle \Upsilon(1S)|O_1(^3S_1)|\Upsilon(1S)\rangle}$$

E.g. Extraction of $\alpha_{\rm s}$

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  - Continuum: $O_8({}^1S_0)$, $O_8({}^3P_0)$ (X. Garcia i Tormo, JS, PRD69 (2004)114006)

E.g. Extraction of α_s

E.g. Extraction of $\alpha_{\rm s}$

- A precise value of $\alpha_{\rm s}$ can be obtained from $\Gamma(\Upsilon(1S) \to \gamma\, X)/\Gamma(\Upsilon(1S) \to X)$:

$$\alpha_{\rm s}(M_{\Upsilon(1S)}) = 0.184^{+0.012}_{-0.018} \quad \longrightarrow \quad \alpha_{\rm s}(M_Z) = 0.119^{+0.005}_{-0.007},$$

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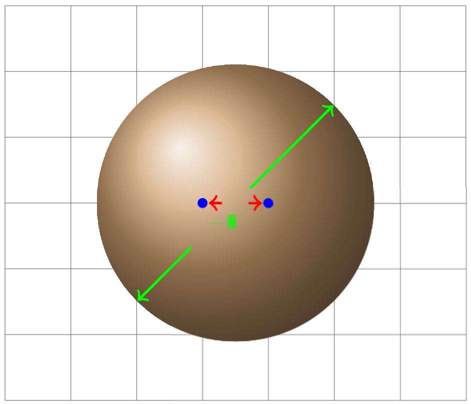
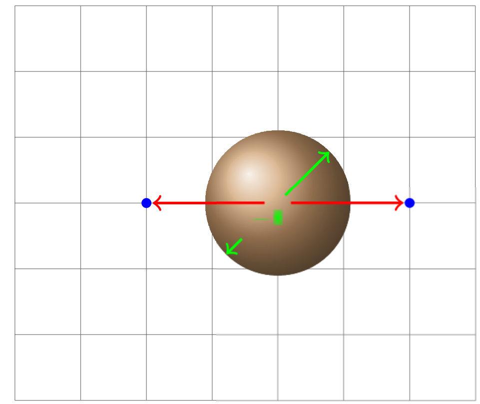
- Compatible with the PDG 2023 average ($\alpha_s(M_Z) = 0.1179 \pm 0.0009$), with errors not competitive anymore (see JS in 1512.05194 for a discussion)

How does the hadron look like ?





$m_Q v \sim 1/r \gg m_Q v^2 \gtrsim \Lambda_{QCD}$ $m_Q v \sim 1/r \gtrsim \Lambda_{QCD} \gg m_Q v^2$

weak coupling pNRQCD strong coupling pNRQCD

$|||$

Figures: Najjar, Bali, 2009 Born-Oppenheimer EFT

Born-Oppenheimer EFT

$\Lambda_{QCD} \lesssim mv \sim 1/r$: strong coupling regime

$$L_{\rm pNRQCD} = \int d^3\mathbf{R} \int d^3\mathbf{r}\; S^\dagger \big(i\partial_0 - h_s(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2, \mathbf{S}_1, \mathbf{S}_2)\big) S,$$

$$h_s(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2, \mathbf{S}_1, \mathbf{S}_2) = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + V_s(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2, \mathbf{S}_1, \mathbf{S}_2),$$

$$V_s = V_s^{(0)} + \frac{V_s^{(1,0)}}{m_1} + \frac{V_s^{(0,1)}}{m_2} + \frac{V_s^{(2,0)}}{m_1^2} + \frac{V_s^{(0,2)}}{m_2^2} + \frac{V_s^{(1,1)}}{m_1 m_2} + \ldots$$

All V_ss can be, and most of them have been, calculated on the lattice

$$\mathbf{p}_j = -i\nabla_j \quad , \quad \mathbf{L}_j = \mathbf{r} \times \mathbf{p}_j \quad , \quad j = 1, 2 \quad , \quad \mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$$

$$\mathbf{S}_{12}(\hat{\mathbf{r}}) = 3\hat{\mathbf{r}} \cdot \boldsymbol{\sigma}_1\, \hat{\mathbf{r}} \cdot \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \quad , \quad r = |\mathbf{r}| \quad , \quad \hat{\mathbf{r}} = \mathbf{r}/r$$

Born-Oppenheimer EFT

- $V_s^{(0)}$ and $V_s^{(1,0)}$, $V_s^{(0,1)}$ are central (Spin Symmetry holds)
- $V_s^{(2,0)}$, $V_s^{(0,2)}$, $V_s^{(1,1)}$ contain spin and velocity dependent terms

$$V^{(2,0)} = V_{SD}^{(2,0)} + V_{SI}^{(2,0)} \quad , \quad V^{(1,1)} = V_{SD}^{(1,1)} + V_{SI}^{(1,1)}$$

  - SI terms respect Spin Symmetry

$$V_{SI}^{(2,0)} = \frac{1}{2}\left\{\mathbf{p}_1^2, V_{\mathbf{p}^2}^{(2,0)}(r)\right\} + \frac{V_{\mathbf{L}^2}^{(2,0)}(r)}{r^2}\mathbf{L}_1^2 + V_r^{(2,0)}(r)$$

$$V_{SI}^{(1,1)} = -\frac{1}{2}\left\{\mathbf{p}_1\cdot\mathbf{p}_2, V_{\mathbf{p}^2}^{(1,1)}(r)\right\} - \frac{V_{\mathbf{L}^2}^{(1,1)}(r)}{2r^2}(\mathbf{L}_1\cdot\mathbf{L}_2+\mathbf{L}_2\cdot\mathbf{L}_1)+V_r^{(1,1)}(r)$$

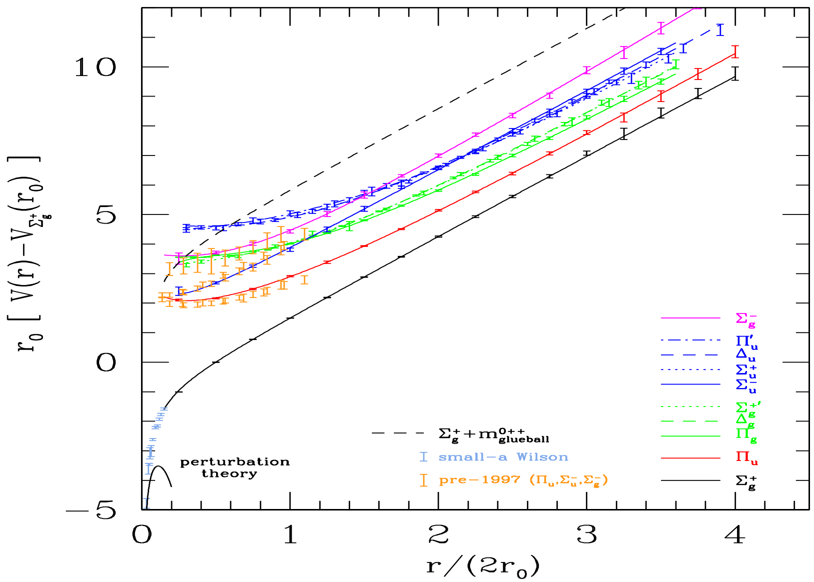
  - SD terms break Spin Symmetry

$$V_{SD}^{(2,0)} = V_{LS}^{(2,0)}(r)\mathbf{L}_1\cdot\mathbf{S}_1$$

$$V_{SD}^{(1,1)} = V_{L_1S_2}^{(1,1)}(r)\mathbf{L}_1\cdot\mathbf{S}_2 - V_{L_2S_1}^{(1,1)}(r)\mathbf{L}_2\cdot\mathbf{S}_1 + V_{S^2}^{(1,1)}(r)\mathbf{S}_1\cdot\mathbf{S}_2 + V_{\mathbf{S}_{12}}^{(1,1)}(r)\mathbf{S}_{12}(\hat{\mathbf{r}})$$

Born-Oppenheimer EFT at LO

- Matching to NRQCD in the static limit $\Rightarrow V_s^{(0)}$ is the ground state energy of two static color sources separated at a distance r
- Can be extracted from lattice calculations of the Wilson loop



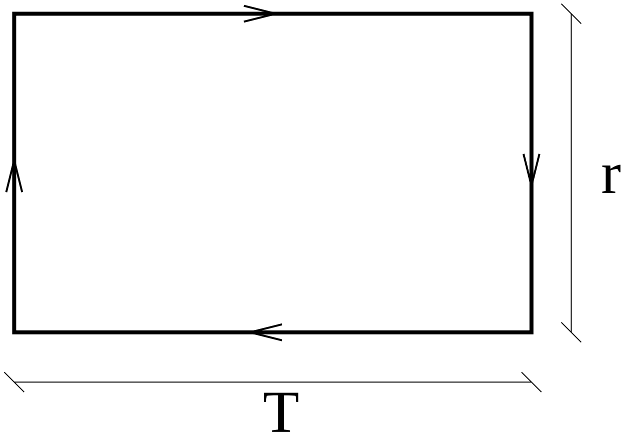
Meyer, Swanson, 15; Juge, Kuti, Morningstar, 02

- Well fitted by the Cornell potential

$$V_s^{(0)} = V_{\Sigma_g^+}(r) \approx -\frac{k_g}{r} + \kappa r + E_g^{Q\bar{Q}} \quad , \; k_g = 0.489 \quad , \kappa = 0.187\,\mathrm{GeV}^2$$

Born-Oppenheimer EFT at LO

- Wilson loop



$$W_\Box = \mathrm{P}\exp\left\{-ig\oint_{r\times T} dz^\mu A_\mu(z)\right\}$$

- The ground state potential is obtained from

$$V_{\Sigma_g^+}(r) = \lim_{T\to\infty} i\ln\langle W_\Box\rangle/T$$

- The excited potentials are the hybrid potentials ($V_{\Pi_u}(r)$, $V_{\Sigma_u^-}(r)$, ...)
  - They are obtained from an analogous formula by inserting suitable gluonic operators in the vertical Wilson lines ($\mathbf{B}-\hat{\mathbf{r}}(\hat{\mathbf{r}}\mathbf{B})$, $\hat{\mathbf{r}}\mathbf{B}$, ...)

Born-Oppenheimer EFT at LO: $\bar{c}c$ (MeV)

nL	$M_{Q\bar{Q}}$	$<p>$	$\frac{1}{<r>}$	$\frac{V^{(1)}}{m_Q}$	$\frac{V^{(2)}_{vd}}{m_Q^2}$	$\frac{V^{(2)}_{vi}}{m_Q^2}$	$\frac{p^4}{8m_Q^3}$	$\Delta M_{Q\bar{Q}}$	E_{exp}
$1s$	3068	738	518	54	71	35	12	96	3068
$2s$	3678	836	259	109	129	30	19	173	3674
$3s$	4130	935	186	109	162	30	30	199	4039
$4s$	4517	1019	149	109	192	30	42	227	4421
$5s$	4865	1097	127	109	223	30	57	256	?
$1p$	3494	753	317	109	105	30	13	155	3525
$2p$	3968	871	209	109	140	30	23	182	3927
$3p$	4369	966	162	109	173	30	34	209	4286
$4p$	4726	1048	135	109	203	30	47	237	4700
$5p$	5055	1136	119	109	239	30	66	272	?

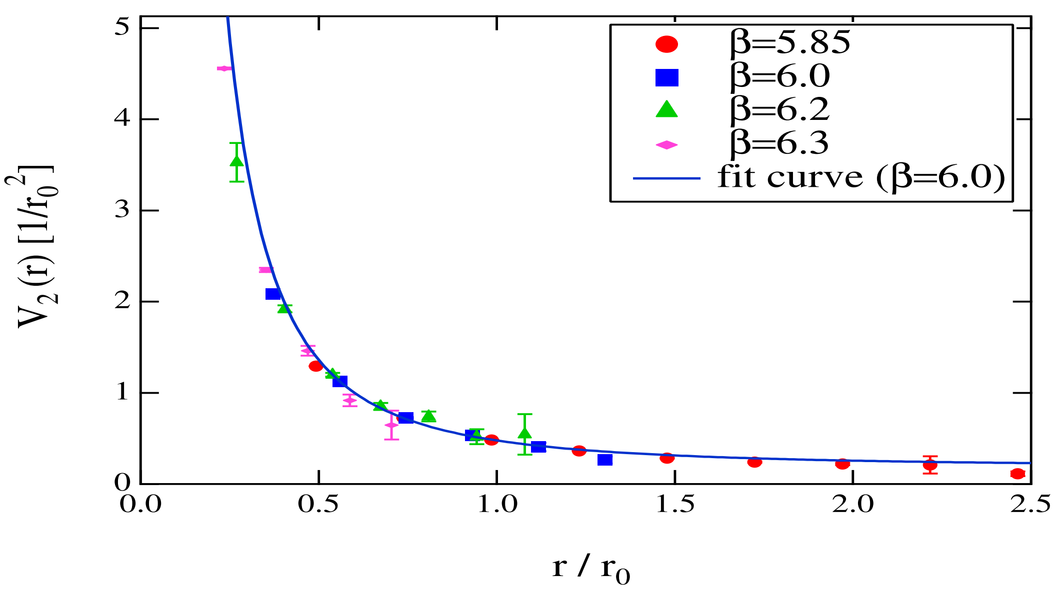
Born-Oppenheimer EFT at LO: $\bar{b}b$ (MeV)

nL	$M_{Q\bar{Q}}$	$<p>$	$\frac{1}{<r>}$	$\frac{V^{(1)}}{m_Q}$	$\frac{V^{(2)}_{vd}}{m_Q^2}$	$\frac{V^{(2)}_{vi}}{m_Q^2}$	$\frac{p^4}{8m_Q^3}$	$\Delta M_{Q\bar{Q}}$	E_{exp}
$1s$	9442	1546	1028	29	37	17	6	50	9445
$2s$	10009	1408	432	14	22	2	4	26	10017
$3s$	10356	1494	295	33	38	3	5	50	10355
$4s$	10638	1594	232	33	43	3	7	54	10579
$5s$	10885	1692	195	33	48	3	9	59	10876
$1p$	9908	1268	531	17	19	3	3	26	9900
$2p$	10265	1386	332	33	32	3	4	46	10260
$3p$	10553	1504	252	33	38	3	5	51	10520
$4p$	10806	1612	207	33	44	3	7	55	?
$5p$	11035	1727	180	33	50	3	10	61	?

Born-Oppenheimer EFT beyond LO

Ex. at $\mathcal{O}(1/m_Q^2)$: $V_{L_2 S_1}^{(1,1)}$ spin-orbit potential (Eichten, Feinberg , 79)

$$V_{L_2 S_1}^{(1,1)}(r) = -\frac{c_F}{r^2} i\mathbf{r} \cdot \lim_{T\to\infty} \int_0^T dt\, t \,\langle\!\langle g\mathbf{B}(t,\mathbf{r}/2) \times g\mathbf{E}(0,-\mathbf{r}/2)\rangle\!\rangle$$



$V_2 = V_{L_2 S_1}^{(1,1)}/c_F$, Koma, Koma, 09

Born-Oppenheimer EFT beyond LO

An example at $\mathcal{O}(1/m_Q^2)$: the $V_{L_2S_1}^{(1,1)}$ spin-orbit potential

- Short distance constraint: it must coincide with the perturbative evaluation,

$$V_{L_2S_1}^{(1,1)}(r) \sim c_F \frac{C_f \alpha_s}{r^3} \quad , r \to 0$$

(Gupta, Radford, 81)

- Long distance constraint: it must coincide with the QCD effective string theory result

$$V_{L_2S_1}^{(1,1)}(r) = -\frac{c_F g^2 \Lambda^2 \Lambda^{'}}{\kappa\, r^2} \quad , r \to \infty$$

(Perez-Nadal, JS, 08)

Hadrons with two heavy quarks

- $QQ+$ light quarks and gluons
  - Double Heavy Baryons: QQq
  - Tetraquarks: $QQ\bar{q}\bar{q}$
  - Pentaquarks: $QQqq\bar{q}$
  - Hybrids: $QQqg$
  - $\ldots$
- $Q\bar{Q}+$ light quarks and gluons
  - Heavy Quarkonium: $Q\bar{Q}$
  - Hybrids: $Q\bar{Q}g$
  - Tetraquarks: $Q\bar{Q}q\bar{q}$
  - Pentaquarks: $Q\bar{Q}qqq$
  - $\ldots$

$QQ/Q\bar{Q}$ + light quarks and gluons

$$\mathcal{L}_{\rm HEH} = \sum_{\kappa^p} \Psi^{\dagger}_{\kappa^p} \left[i\partial_t - h_{\kappa^p} \right] \Psi_{\kappa^p}$$

(JS, Tarrús Castellà, 20)

$$h_{\kappa^p} = \frac{\boldsymbol{p}^2}{m_Q} + \frac{\boldsymbol{P}^2}{4m_Q} + V^{(0)}_{\kappa^p}(\boldsymbol{r}) + \frac{1}{m_Q} V^{(1)}_{\kappa^p}(\boldsymbol{r}, \boldsymbol{p}) + \mathcal{O}\left(\frac{1}{m_Q^2}\right)$$

- LDF$\equiv$ light quarks $+$ gluons, characterized by their quantum numbers $(\kappa, \, p \, \ldots)$
  - $\kappa \equiv$ total angular momentum, $p \equiv$ parity (P/CP)
  - Quantum numbers not explicitely displayed: baryon number (B), isospin (I), strangeness (S), principal quantum number
- $V^{(0)}_{\kappa^p}$, $V^{(1)}_{\kappa^p}$, $\ldots$ must be calculated non-perturbatively
- A truncation of $\mathcal{L}_{\rm HEH}$ needed for practical calculations

- $V^{(0)}_{\kappa^p}$ is a $(2\kappa+1)\times(2\kappa+1)\times\mathbb{I}_{2\,Q_1}\times\mathbb{I}_{2\,Q_2}$ matrix, which can be decomposed into irreducible representations of $D_{\infty\,h}$

$$V^{(0)}_{\kappa^p}(\boldsymbol{r}) = \sum_\Lambda V^{(0)}_{\kappa^p\Lambda}(r)\mathcal{P}_{\kappa\Lambda}$$

$\mathcal{P}_{\kappa\Lambda}$ projects onto LDF angular momenta $\pm\Lambda$ in the direction joining the two heavy quarks, $\Lambda = \kappa,\ \kappa-1,\ \ldots,\ \kappa-[\kappa]$

$$\mathcal{P}_{\frac{1}{2}\frac{1}{2}} = \mathbb{I}^{\rm lq}_2$$
$$\mathcal{P}_{\frac{3}{2}\frac{1}{2}} = \frac{9}{8}\mathbb{I}^{\rm lq}_4 - \frac{1}{2}\left(\hat{\boldsymbol{r}}\cdot\boldsymbol{S}_{3/2}\right)^2$$
$$\mathcal{P}_{\frac{3}{2}\frac{3}{2}} = -\frac{1}{8}\mathbb{I}^{\rm lq}_4 + \frac{1}{2}\left(\hat{\boldsymbol{r}}\cdot\boldsymbol{S}_{3/2}\right)^2$$
$$\mathcal{P}_{10} = \mathbb{I}^{\rm lq}_3 - \left(\hat{\boldsymbol{r}}\cdot\boldsymbol{S}_1\right)^2$$
$$\mathcal{P}_{11} = \left(\hat{\boldsymbol{r}}\cdot\boldsymbol{S}_1\right)^2$$
$$\ldots$$

$D_{\infty h}$

- The symmetry group of a diatomic molecule (two equal atoms separated at a distance r)
- The generators are
  - Rotations around the z-axis, labeled by $|L| = 0, 1, 2, \cdots \equiv \Sigma, \Pi, \Delta, \ldots$
  - Reflections about the xz plain, labeled by $\pm$ (only important for Σ states)
  - Parity, labeled by g (positive) and u (negative). In the case of a $Q\bar{Q}$ pair is replaced by CP.
  - When $r \to 0$ reduces to $O(3)$ (plus C in the case of $Q\bar{Q}$), which implies short distance degeneracies

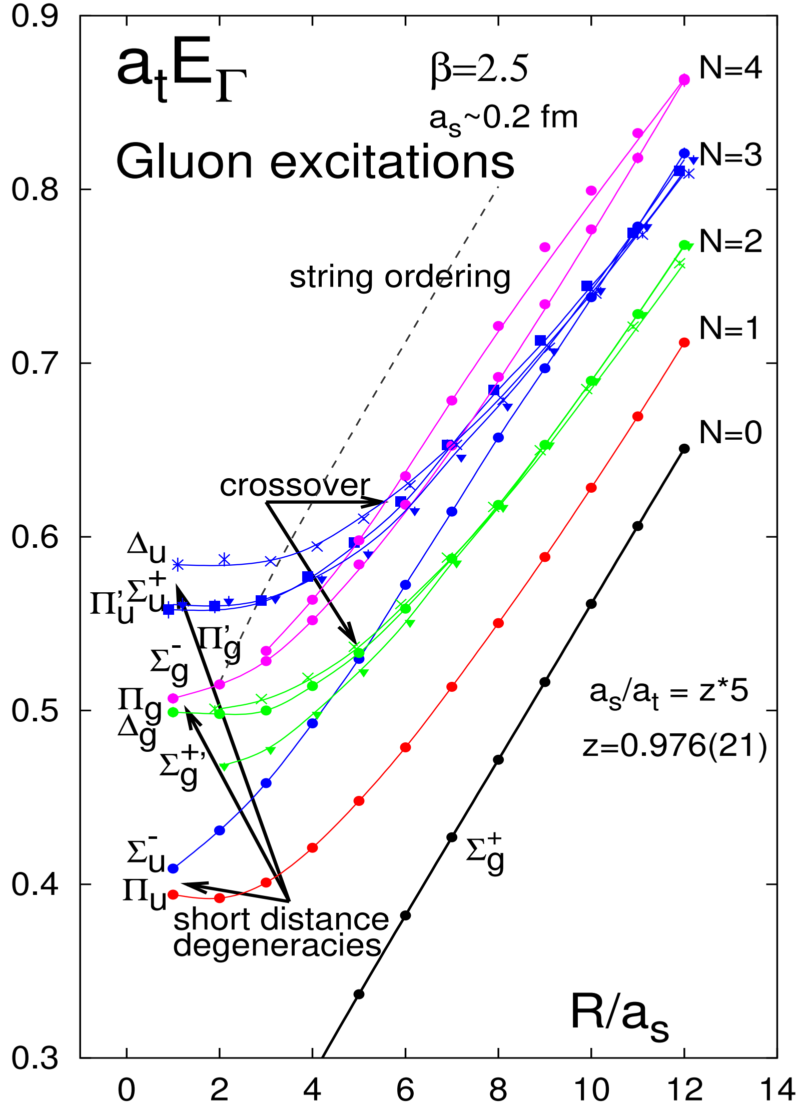
$$\kappa^p \quad \underset{1^+}{(\Sigma_u^-, \Pi_u)} \quad , \quad \underset{2^-}{(\Sigma_g^-, \Pi_g, \Delta_g)} \quad , \quad \underset{1^-}{(\Sigma_g^{+\prime}, \Pi_g^\prime)} \quad , \quad \underset{2^+}{(\Sigma_u^+, \Pi_u^\prime, \Delta_u)} \quad , \quad \ldots$$

  - When $r \to \infty$ the degeneracies of the QCD string must be reproduced

$$(\Pi_u) \quad , \quad (\Sigma_g^{+\prime}, \Pi_g, \Delta_g) \quad , \quad (\Sigma_u^+, \Sigma_u^-, \Pi_u^\prime, \Pi_u^{\prime\prime}, \Delta_u, \Phi_u) \quad , \quad \ldots$$

$D_{\infty h}$, e.g. Hybrids

$$(\Sigma_u^+,\, \Pi_u',\, \Delta_u)$$
$$(\Sigma_g^{+\prime},\, \Pi_g')$$
$$(\Sigma_g^-,\, \Pi_g,\, \Delta_g)$$
$$(\Sigma_u^-,\, \Pi_u)$$



$$(\Sigma_u^+, \Sigma_u^-, \Pi_u', \Pi_u'', \Delta_u, \Phi_u)$$
$$(\Sigma_g^{+\prime}, \Pi_g, \Delta_g)$$
$$(\Pi_u)$$

Juge, Kuti, Morningstar, 02

- $V^{(1)}_{\kappa^p} = V^{(1)}_{\kappa^p\mathrm{SI}} + V^{(1)}_{\kappa^p\mathrm{SD}}$
- $V^{(1)}_{\kappa^p\mathrm{SI}}$ does not depend on the spin or orbital angular momentum of the heavy quarks $\Longrightarrow$ admits the same decomposition as $V^{(0)}_{\kappa^p}$
- $V^{(1)}_{\kappa^p\mathrm{SD}}$ depends on the spin and orbital angular momentum of the heavy quarks

$$V^{(1)}_{\kappa^p\mathrm{SD}}(\boldsymbol{r}) = \sum_{\Lambda\Lambda'} \mathcal{P}_{\kappa\Lambda} \left[V^{sa}_{\kappa^p\Lambda\Lambda'}(\boldsymbol{r}) \boldsymbol{S}_{QQ} \cdot (\mathcal{P}_{10} \cdot \boldsymbol{S}_\kappa) + V^{sb}_{\kappa^p\Lambda\Lambda'}(\boldsymbol{r}) \boldsymbol{S}_{QQ} \cdot (\mathcal{P}_{11} \cdot \boldsymbol{S}_\kappa) \right.$$
$$\left. + V^{l}_{\kappa^p\Lambda\Lambda'}(\boldsymbol{r}) \left(\boldsymbol{L}_{QQ} \cdot \boldsymbol{S}_\kappa \right) \right] \mathcal{P}_{\kappa\Lambda'}$$

$$2\boldsymbol{S}_{QQ} = \boldsymbol{\sigma}_{QQ} = \boldsymbol{\sigma}_{Q_1} \mathbb{I}_{2\,Q_2} + \mathbb{I}_{2\,Q_1} \boldsymbol{\sigma}_{Q_2} \quad , \quad \mathcal{P}^{ij}_{10} = \hat{\boldsymbol{r}}^i \hat{\boldsymbol{r}}^j \quad , \quad \mathcal{P}^{ij}_{11} = \delta^{ij} - \hat{\boldsymbol{r}}^i \hat{\boldsymbol{r}}^j$$

Matching to NRQCD

- Build an NRQCD operator with the quantum numbers of Ψ_{κ^p}

$$\mathcal{O}^{Q\bar{Q}}_{\kappa^p}(t,\,\boldsymbol{r},\,\boldsymbol{R}) = \chi_c^\top(t,\,\boldsymbol{x}_2)\phi(t,\,\boldsymbol{x}_2,\,\boldsymbol{R})\mathcal{Q}_{Q\bar{Q}\kappa^p}(t,\,\boldsymbol{R})\phi(t,\,\boldsymbol{R},\,\boldsymbol{x}_1)\psi(t,\,\boldsymbol{x}_1)$$
$$\mathcal{O}^{QQ}_{\kappa^p}(t,\,\boldsymbol{r},\,\boldsymbol{R}) = \psi^\top(t,\,\boldsymbol{x}_2)\phi^\top(t,\,\boldsymbol{R},\,\boldsymbol{x}_2)\mathcal{Q}_{QQ\kappa^p}(t,\,\boldsymbol{R})\phi(t,\,\boldsymbol{R},\,\boldsymbol{x}_1)\psi(t,\,\boldsymbol{x}_1)$$

- Examples:
  - Hybrid

$$\mathcal{Q}^{\alpha}_{1^{+-}}(t,\boldsymbol{x}) = \left(\boldsymbol{e}^\dagger_\alpha\cdot\boldsymbol{B}(t,\boldsymbol{x})\right)$$

  - $Q\bar{Q}q\bar{q}$ tetraquark

$$\mathcal{Q}_{0^{++}}(t,\boldsymbol{x}) = \left[\bar{q}(t,\boldsymbol{x})T^a q(t,\boldsymbol{x})\right]T^a$$

  - Doubly heavy baryons

$$\mathcal{Q}^{\alpha}_{(1/2)^{+}}(t,\boldsymbol{x}) = \underline{T}^l\left[P_+ q^l(t,\boldsymbol{x})\right]^\alpha$$

  - $QQ\bar{q}\bar{q}$ tetraquark

$$\mathcal{Q}_{0^{-}}(t,\boldsymbol{x}) = \left[\bar{q}(t,\boldsymbol{x})\underline{T}^l\gamma^2 q^*(t,\boldsymbol{x})\right]\underline{T}^l$$

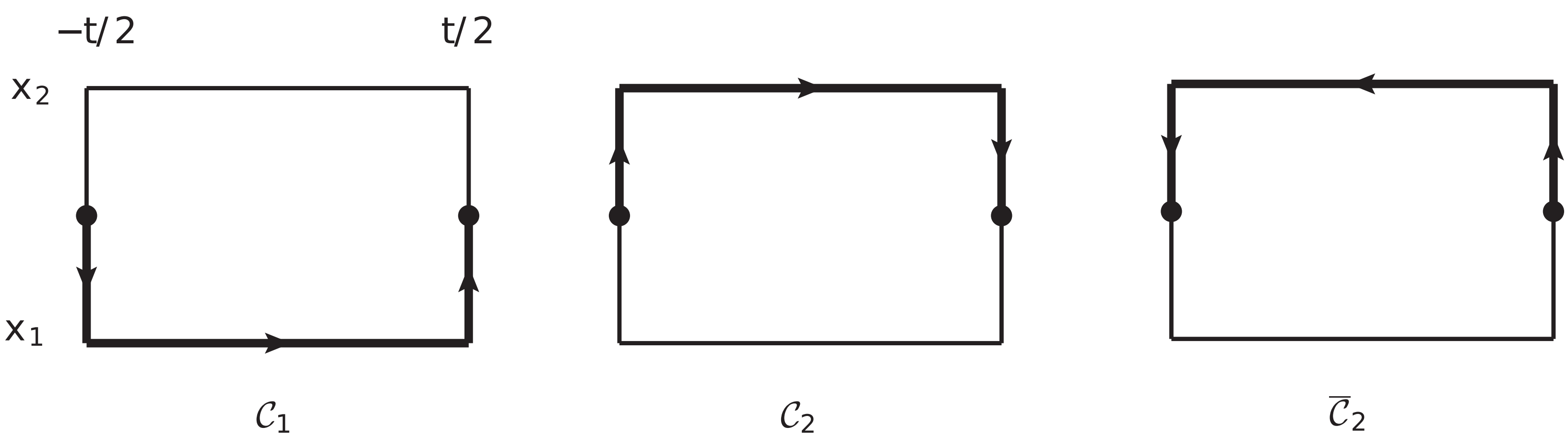
Matching to NRQCD

- Impose $\quad \mathcal{O}^h_{\kappa^p}(t,\,\boldsymbol{r},\,\boldsymbol{R}) = \sqrt{Z_{h\kappa^p}}\,\Psi_{h\kappa^p}(t,\,\boldsymbol{r},\,\boldsymbol{R})\,, \quad h = QQ\,, Q\bar{Q}.$

$$\langle 0|T\{\mathcal{O}^h_{\kappa^p}(t/2)\mathcal{O}^{h\dagger}_{\kappa^p}(-t/2)\}|0\rangle = \sqrt{Z_{h\kappa^p}}\langle 0|T\{\Psi_{h\kappa^p}(t/2)\Psi^\dagger_{h\kappa^p}(-t/2)\}|0\rangle\sqrt{Z^\dagger_{h\kappa^p}}$$

  - Then at $\mathcal{O}(1)$

$$V^{(0)}_{h\kappa^p\Lambda}(\boldsymbol{r}) = \lim_{t\to\infty}\frac{i}{t}\log\left(\mathrm{Tr}\left[\mathcal{P}_{\kappa\Lambda}\langle 1\rangle^{h\kappa^p}_{\Box}\right]\right)$$



- At $\mathcal{O}\left(\frac{1}{m_Q}\right)$, for instance,

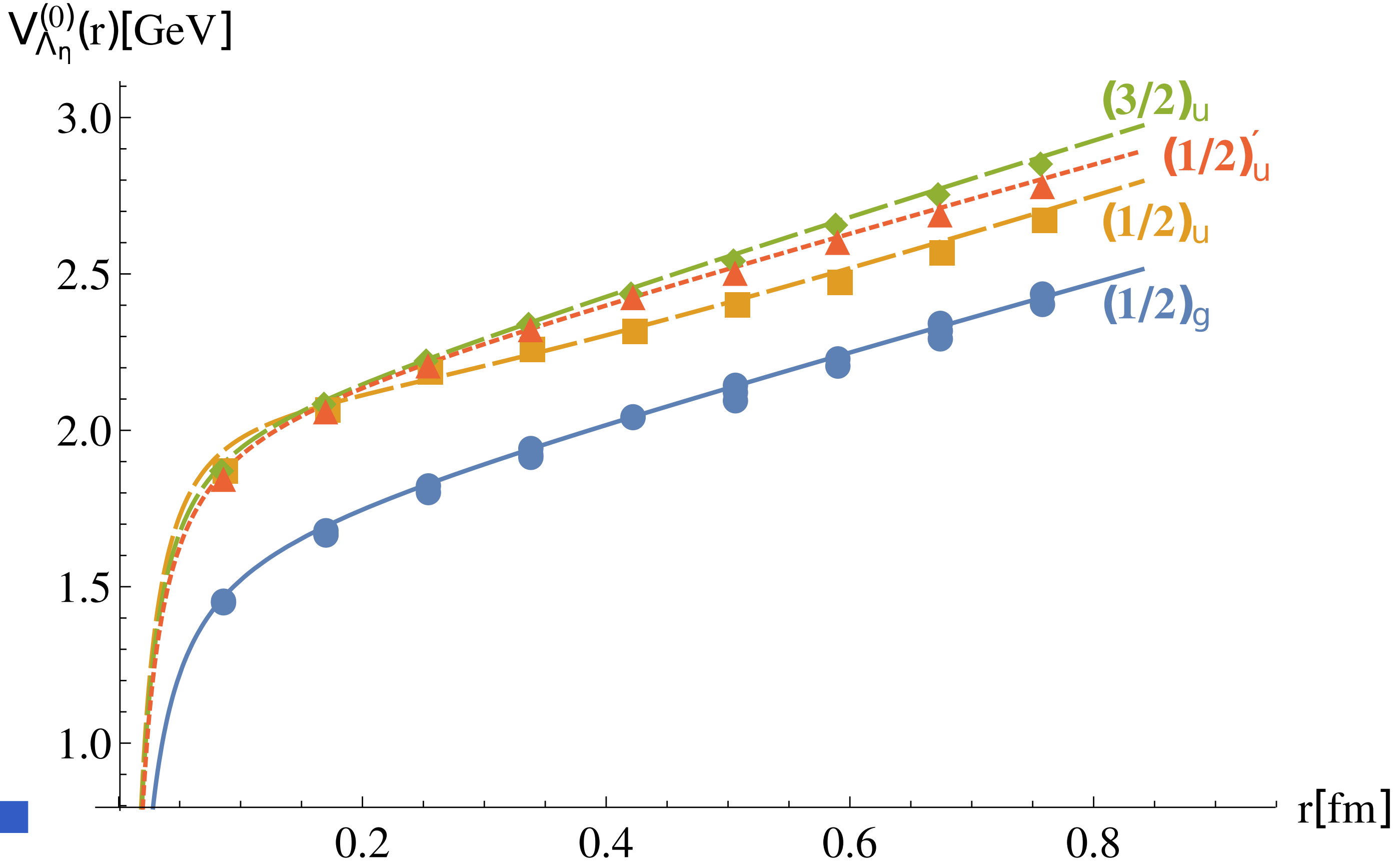
$$V^{sb}_{\kappa^p\Lambda\Lambda'} = -\,c_F \lim_{t\to\infty} \sqrt{\frac{\mathrm{Tr}\left[\mathcal{P}_{\kappa\Lambda}\right]\mathrm{Tr}\left[\mathcal{P}_{\kappa\Lambda'}\right]}{\mathrm{Tr}\left[\mathcal{P}_{\kappa\Lambda}\langle 1\rangle_{\Box}^{h\kappa^p}\right]\mathrm{Tr}\left[\mathcal{P}_{\kappa\Lambda'}\langle 1\rangle_{\Box}^{h\kappa^p}\right]}}$$

$$\times \frac{\ln\left(\frac{\mathrm{Tr}\left[\mathcal{P}_{\kappa\Lambda}\langle 1\rangle_{\Box}^{h\kappa^p}\right]\mathrm{Tr}[\mathcal{P}_{\kappa\Lambda'}]}{\mathrm{Tr}\left[\mathcal{P}_{\kappa\Lambda'}\langle 1\rangle_{\Box}^{h\kappa^p}\right]\mathrm{Tr}[\mathcal{P}_{\kappa\Lambda}]}\right)}{2t\sinh\left(\ln\sqrt{\frac{\mathrm{Tr}\left[\mathcal{P}_{\kappa\Lambda}\langle 1\rangle_{\Box}^{h\kappa^p}\right]\mathrm{Tr}[\mathcal{P}_{\kappa\Lambda'}]}{\mathrm{Tr}\left[\mathcal{P}_{\kappa\Lambda'}\langle 1\rangle_{\Box}^{h\kappa^p}\right]\mathrm{Tr}[\mathcal{P}_{\kappa\Lambda}]}}\right)}$$

$$\times \int_{-t/2}^{t/2} dt' \frac{\mathrm{Tr}\left[\left(\boldsymbol{S}_\kappa\cdot\mathcal{P}_{11}\right)\cdot\left(\mathcal{P}_{\kappa\Lambda}\langle g\boldsymbol{B}(t',\boldsymbol{x}_1)\rangle_{\Box}^{h\kappa^p}\mathcal{P}_{\kappa\Lambda'}\right)\right]}{\mathrm{Tr}\left[\left(\boldsymbol{S}_\kappa\cdot\mathcal{P}_{11}^{\mathrm{c.r.}}\right)\cdot\left(\mathcal{P}_{\kappa\Lambda}\boldsymbol{S}_\kappa\mathcal{P}_{\kappa\Lambda'}\right)\right]}$$

E.g. QQq ($m_Q v$, $\Lambda_{QCD} \gg m_Q v^2$)

- We apply the general results to the case $B = 1/3$, $I = 1/2$, $S = 0$ in the BO approximation (Tarrús Castellà, JS, 20)
- There is available lattice data (Najjar, Bali, 09): $N_f = 2$, $a \simeq 0.084$ fm, $L \simeq 1.3$ fm, $m_\pi \simeq 783$ MeV



$O(3)$	$D_{\infty h}$
$(1/2)^+$	$(1/2)_g$
$(3/2)^-$	$(1/2)_u$, $(3/2)_u$
$(1/2)^-$	$(1/2)'_u$

- At $\mathcal{O}(1)$

$$\mathcal{L}^{\rm LO}_{QQq} \simeq \Psi^\dagger_{(1/2)^+}\left(i\partial_t - \frac{\boldsymbol{p}^2}{m_Q} + V^{(0)}_{(1/2)^+}(r)\right)\Psi^\dagger_{(1/2)^+}$$
$$+ \Psi^\dagger_{(1/2)^-}\left(i\partial_t - \frac{\boldsymbol{p}^2}{m_Q} + V^{(0)}_{(1/2)^-}(r)\right)\Psi^\dagger_{(1/2)^-}$$
$$+ \Psi^\dagger_{(3/2)^-}\left(i\partial_t - \frac{\boldsymbol{p}^2}{m_Q} + V^{(0)}_{(3/2)^-(1/2)}(r)\mathcal{P}_{\frac{3}{2}\frac{1}{2}} + V^{(0)}_{(3/2)^-(3/2)}(r)\mathcal{P}_{\frac{3}{2}\frac{3}{2}}\right)\Psi^\dagger_{(3/2)^-}$$

- $V^{(0)}_{\kappa^p\Lambda}(r)$ are obtained from fits to lattice data taking into account
  - Short distance constraints

$$V^{(0)}_{\kappa^p\Lambda}(r) \underset{r\to 0}{\simeq} -\frac{2}{3}\frac{\alpha_s(\nu_{\rm lat})}{r} + \overline{\Lambda}_{\kappa^p}$$

$$M_{\bar{Q}q} = m_Q + \overline{\Lambda}_{\kappa^p}$$

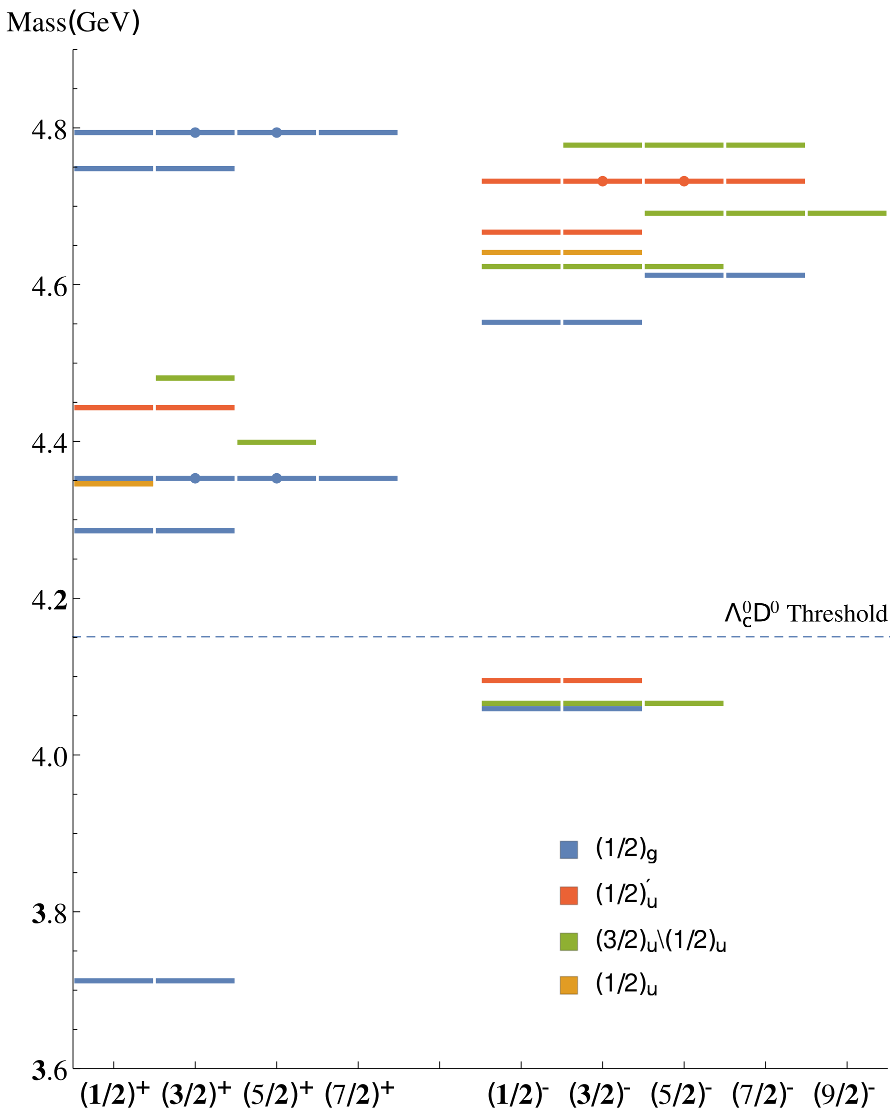
Heavy quark-diquark symmetry

  - Long distance constraints

$$V^{(0)}_{\kappa^p\Lambda}(r) \underset{r\to \infty}{\simeq} \sigma r$$

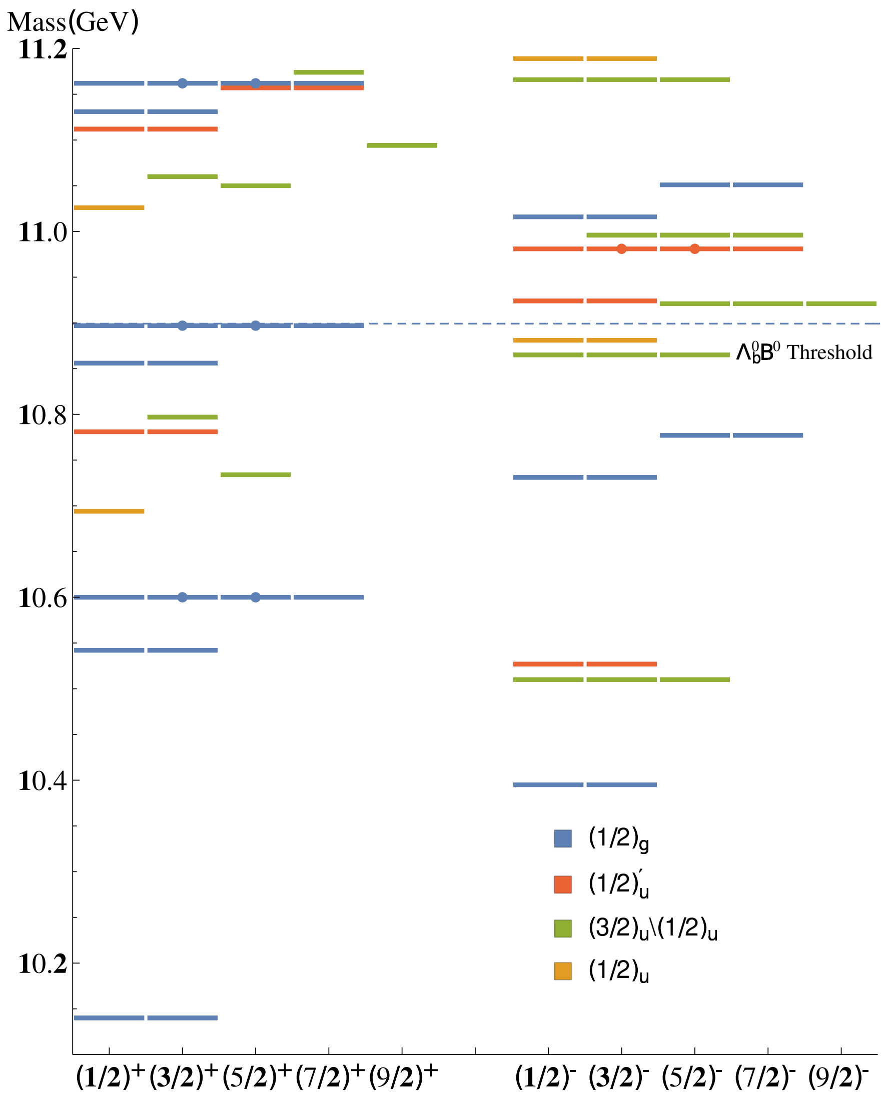
QCD string

Ξ_{cc}



Mass(GeV)
4.8
4.6
4.4
4.2
4.0
3.8
3.6
$\Lambda_c^0 D^0$ Threshold
(1/2)$_g$
(1/2)$'_u$
(3/2)$_u$\(1/2)$_u$
(1/2)$_u$
(1/2)$^+$ (3/2)$^+$ (5/2)$^+$ (7/2)$^+$
(1/2)$^-$ (3/2)$^-$ (5/2)$^-$ (7/2)$^-$ (9/2)$^-$

Ξ_{bb}



Mass(GeV)
11.2
11.0
10.8
10.6
10.4
10.2
$\Lambda_b^0 B^0$ Threshold
(1/2)g
(1/2)'u
(3/2)u\(1/2)u
(1/2)u
(1/2)+ (3/2)+ (5/2)+ (7/2)+ (9/2)+
(1/2)- (3/2)- (5/2)- (7/2)- (9/2)-

- At $\mathcal{O}(1/m_Q)$
  - There is no lattice data available
  - We can derive general properties of the splittings for $\kappa = 1/2$ (only three unknown potentials)

$$V^{(1)}_{(1/2)\pm\mathrm{SD}}(\boldsymbol{r}) = V^{s1}_{(1/2)\pm}(r)\boldsymbol{S}_{QQ}\cdot\boldsymbol{S}_{1/2} + V^{s2}_{(1/2)\pm}(r)\boldsymbol{S}_{QQ}\cdot\left(\boldsymbol{\mathcal{T}}_2\cdot\boldsymbol{S}_{1/2}\right) + V^{l}_{(1/2)\pm}(r)\left(\boldsymbol{L}_{QQ}\cdot\boldsymbol{S}_{1/2}\right)$$

$$\left(\boldsymbol{\mathcal{T}}_2\right)^{ij} = \hat{\boldsymbol{r}}^i\hat{\boldsymbol{r}}^j - \frac{1}{3}\delta^{ij}$$

- Let us write the doubly heavy baryon mass as,

$$M_{njl\ell} = M_{nl}^{(0)} + M_{njl\ell}^{(1)} + \ldots$$

  - $n =$ principal quantum number
  - $j =$ total angular momentum
  - $l =$ orbital angular momentum
  - $\ell =$ orbital angular momentum $+$ LDF angular momentum
  - Pauli principle $\Longrightarrow$ $s_{QQ} = 1$ for l even, $s_{QQ} = 0$ for l odd

- For $l = 0$,

$$M^{(1)}_{nj0\frac{1}{2}} = \frac{1}{2}\left(j(j+1) - \frac{11}{4}\right)\frac{\langle V^{s1}_{(1/2)\pm}\rangle_{n0}}{m_Q} \quad \Longrightarrow \quad 2M_{n\frac{3}{2}0\frac{1}{2}} + M_{n\frac{1}{2}0\frac{1}{2}} = 3M^{(0)}_{n0}$$

- For $l = 1$,

$$M^{(1)}_{nj1j} = \frac{1}{2}\left(j(j+1) - \frac{11}{4}\right)\frac{\langle V^{l}_{(1/2)\pm}\rangle_{n1}}{m_Q} \quad \Longrightarrow \quad 2M_{n\frac{3}{2}1\frac{3}{2}} + M_{n\frac{1}{2}1\frac{1}{2}} = 3M^{(0)}_{n1}$$

- If we take $M_{n\frac{1}{2}1\frac{1}{2}} = M_{\Xi^{++}_{cc}} = 3621.2 \pm 0.7$ MeV,

$$\Longrightarrow \quad M_{n\frac{3}{2}1\frac{1}{2}} = M_{\Xi^{*}{}^{++}_{cc}} = 3757(68) \ \text{MeV}$$

More on pNRQCD

- Factorization formulas for NRQCD matrix elements can be worked out (N. Brambilla, D. Eiras, A. Pineda, JS and A. Vairo, Phys. Rev. Lett. **88**, 012003 (2002); Phys. Rev. D **67**, 034018 (2003))

$$\langle \Upsilon(n)|O_8(^1S_0)|\Upsilon(n)\rangle = C_A \frac{|R_n(0)|^2}{2\pi}\left(-\frac{(C_A/2-C_f)c_F^2\mathcal{B}_1}{3m_Q^2}\right)$$

  - $R_n(0)$, wave function at the origin
  - $\mathcal{B}_1 \sim \Lambda_{QCD}^2$, independent of n
  - c_F, short distance matching coefficient

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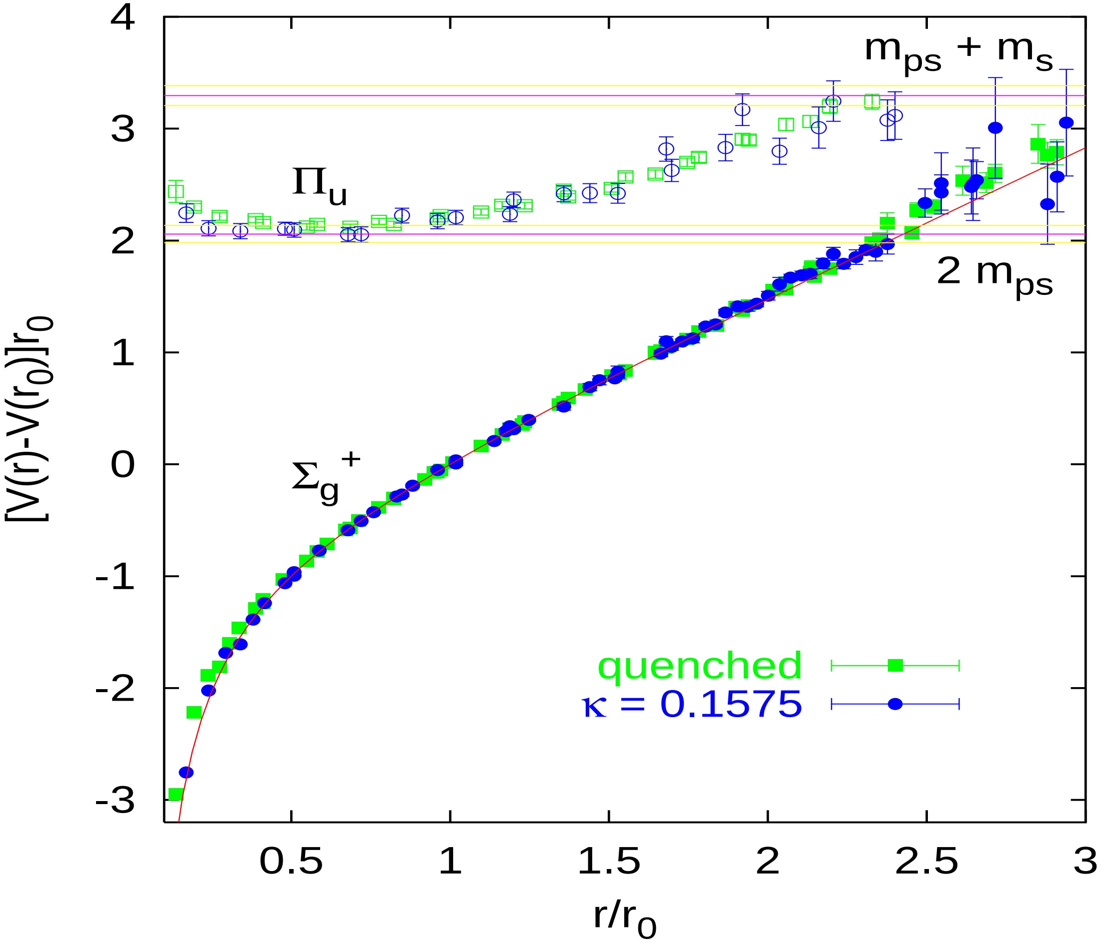
  - $R_n(0)$, wave function at the origin
  - $\mathcal{B}_1 \sim \Lambda^2_{QCD}$, independent of n
  - c_F, short distance matching coefficient

- New predictions can be put forward

$$\frac{\Gamma(\chi_{b0}(1P)\to \mathrm{LH})}{\Gamma(\chi_{b1}(1P)\to \mathrm{LH})} = \frac{\Gamma(\chi_{b0}(2P)\to \mathrm{LH})}{\Gamma(\chi_{b1}(2P)\to \mathrm{LH})} = 8.0 \pm 1.3,$$

More on pNRQCD



G.S. Bali at al. (TXL Collaboration), Phys. Rev. **D62**,(2000):054503