Methods of EFT and LFT, Bad Honnef Physics School July 2023
Problems for Lecture 1 (Yannick Meurice)

General references: Yannick Meurice, Ryo Sakai, and Judah Unmuth-Yockey, Rev. Mod. Phys. 94, 025005 (2022) https://arxiv.org/abs/2010.06539; Yannick Meurice, Quantum Field theory: a quantum computation approach, IoP (2021) https://iopscience.iop.org/book/mono/978-0-7503-2187-7 (often available online in university libraries).

1. The partition function for the (0+1) dimensional $O(2)$ model reads

$$Z_{O(2)} = \prod_{x=1}^{N} \int_{-\pi}^{\pi} \frac{d\varphi_x}{2\pi} e^{\beta \sum_x \cos(\varphi_{x+1}-\varphi_x)}.$$

Use the Fourier expansion to expand the Boltzmann weights

$$e^{\beta\cos(\varphi_{x+1}-\varphi_x)} = \sum_{n_x=-\infty}^{\infty} e^{in_x(\varphi_{x+1}-\varphi_x)} I_{n_x}(\beta),$$

and integrate over the angles φ to calculate the partition function with periodic (PBC) and open (OBC) boundary conditions. For PBC in the large β limit, discuss approximate classical solutions (in the original φ formulation) with winding number n. Resum the classical solution contributions with their quadratic fluctuations and show that after using Poisson resummation formula,

$$\sum_{\ell=-\infty}^{\infty} e^{-\frac{B}{2}\ell^2} = \sqrt{\frac{2\pi}{B}} \sum_{n=-\infty}^{\infty} e^{-\frac{(2\pi)^2}{2B}n^2},$$

you recover the tensor result in the large β limit.

2. The lattice compact Abelian Higgs model is a non-perturbative regularized formulation of scalar quantum electrodynamics (scalar electrons-positrons + photons with compact fields). The partition function reads

$$Z_{CAHM} = \prod_x \int_{-\pi}^{\pi} \frac{d\varphi_x}{2\pi} \prod_{x,\mu} \int_{-\pi}^{\pi} \frac{dA_{x,\mu}}{2\pi} e^{-S_{gauge}-S_{matter}},$$

$$S_{gauge} = \beta_{plaquette} \sum_{x,\mu<\nu} (1-\cos(A_{x,\mu}+A_{x+\hat{\mu},\nu}-A_{x+\hat{\nu},\mu}-A_{x,\nu})),$$

$$S_{matter} = \beta_{link} \sum_{x,\mu} (1-\cos(\varphi_{x+\hat{\mu}}-\varphi_x+A_{x,\mu})).$$

The local invariance is $\varphi'_x = \varphi_x + \alpha_x$ and $A'_{x,\mu} = A_{x,\mu} - (\alpha_{x+\hat{\mu}} - \alpha_x)$. φ is the Nambu-Goldstone mode of the original model. The Brout-Englert-Higgs mode is decoupled (heavy). Derive the tensor reformulation of the compact Abelian Higgs model in (1+1) and (2+1) dimensions. Discuss the discrete analog of the inhomogeneous Maxwell's equations. Take the time continuum limit in (1+1) dimensions and derive the Hamiltonian

$$H = \frac{U}{2}\sum_{i=1}^{N_s} (L_i^z)^2 + \frac{Y}{2}\sum_i (L_{i+1}^z - L_i^z)^2 - X\sum_{i=1}^{N_s} U_i^x$$

with $U^x \equiv \frac{1}{2}(U^+ + U^-)$ and $L^z|m\rangle = m|m\rangle$ and $U^{\pm}|m\rangle = |m\pm 1\rangle$; m is a discrete electric field quantum number ($-\infty < m < +\infty$). In practice, we apply truncations: $U^{\pm}|\pm m_{max}\rangle = 0$.