EFT Exercises

Problem 2

The Lagrangian for a weakly interacting Bose gas consisting of atoms with scattering length a is

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\left(\psi^\dagger i\frac{\partial}{\partial t}\psi + \text{h.c.}\right) - \frac{1}{2m}\nabla\psi^\dagger\cdot\nabla\psi - \mathcal{V},\\ \mathcal{V} &= -\mu\psi^\dagger\psi + \frac{2\pi a}{m}\big(\psi^\dagger\psi\big)^2.\end{aligned}$$

The Bose gas with number density $n = \langle\psi^\dagger\psi\rangle$ can be obtained by adjusting the chemical potential μ.

(a) Show that if $a > 0$, the potential energy density $\mathcal{V}$ for a classical field ψ is minimized by a nonzero value that corresponds to the number density

$$n = \psi^*\psi = \frac{m}{4\pi a}\mu.$$

(b) Expand the field ψ around its vacuum value by expressing it in the form

$$\psi(\vec{r},t) = \sqrt{n} + \xi(\vec{r},t) + i\,\eta(\vec{r},t),$$

where ξ and η are real scalar fields. Show that the quadratic terms in the Lagrangian can be reduced to the form

$$\mathcal{L}_2 = \big(\eta\,\dot{\xi} - \xi\,\dot{\eta}\big) - \frac{1}{2m}\left(\nabla\xi\cdot\nabla\xi + \nabla\eta\cdot\nabla\eta\right) - 2\mu\xi^2.$$

(c) Find the dispersion relation $\omega(k)$ for the quasiparticles by expressing $\mathcal{L}_2$ in terms of a 2-component field $\binom{\xi}{\eta}$, finding the matrix equation of motion, and looking for solutions of the form

$$\begin{pmatrix}\xi(\vec{r},t)\\ \eta(\vec{r},t)\end{pmatrix} = \begin{pmatrix}\xi_0\\ \eta_0\end{pmatrix}\exp\big(i\vec{k}\cdot\vec{r} - i\omega t\big).$$

(d) Show that the dispersion relation is

$$\omega(k) = \frac{k\sqrt{k^2 + k_B^2}}{2m},$$

where the Bogoliubov momentum is $k_B = \sqrt{16\pi an}$.

(e) The leading term sin the effective Lagrangian for the Goldstone mode have the form

$$\mathcal{L}_{\text{eff}} = \tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2}v^2 \nabla\phi \cdot \nabla\phi.$$

Find the dispersion relation for the Goldstone mode.

(f) Determine the Goldstone boson velocity v by matching with the dispersion relation for the weakly interacting Bose gas.