Exercises: Perturbative Gradient Flow

R. Harlander

RWTH Aachen University

Bad Honnef, 2023

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1 Flow lines

The generating functional which implies the flow equation reads

$$Z[J_B,J_L]\sim\int\mathcal{D}L\mathcal{D}B\,\exp\Big[-\int\mathrm{d}^4x\int_0^\infty\mathrm{d}t\,\Big(L^a_\mu(t,x)(\partial_t-\Box_x)B^a_\mu(t,x)\\ -J_{B,\mu}(t,x)B_\mu(t,x)-L_\mu(t,x)J_{L,\mu}(t,x)\Big)\Big]\,, \tag{1}$$

where we only kept the quadratic terms in the fields in the exponent. Using the fact that the integral measure is invariant under the shifts

$$\begin{aligned}L^a_\mu(t,x)&\to L^a_\mu(t,x)+\int\mathrm{d}^4y\int_0^\infty\mathrm{d}s\,J_{B,\mu}(s,y)P(s-t,y-x)\,,\\ B^a_\mu(t,x)&\to B^a_\mu(t,x)+\int\mathrm{d}^4y\int_0^\infty\mathrm{d}s\,P(t-s,x-y)J_{L,\mu}(s,y)\,,\end{aligned} \tag{2}$$

with

$$(\partial_t-\Box_x)P(t-s,x-y)=\delta(t-s)\delta(x-y)\,, \tag{3}$$

show that the mixed propagator is

$$\langle 0|TB^a_\mu(t,x)L^b_\nu(s,y)|0\rangle=\delta^{ab}\delta_{\mu\nu}\,P(t-s,x-y)\,. \tag{4}$$

2 Asymptotic expansion of flow-time integrals

Evaluate the integral

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{e^{-t[k^2+(k-q)^2]}}{k^2(k-q)^2} \tag{5}$$

in the limit $tq^2 \ll 1$ up to the first non-vanishing order in $q^2 t$.

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3 Flowed anomalous dimension

Consider the small-flow-time expansion

$$\tilde{\mathcal{O}}(t) = \zeta(t)\mathcal{O}^R + \cdots , \tag{6}$$

where $\mathcal{O}^R$ is a basis of renormalized operators of mass dimension n, $\tilde{\mathcal{O}}(t)$ are the corresponding flowed operators, and $\zeta(t)$ is the matching matrix. Terms or order t are neglected. The flowed operators obey a flow equation [1]:

$$t\frac{\partial}{\partial t}\tilde{\mathcal{O}}(t) = \tilde{\gamma}(t)\tilde{\mathcal{O}}(t) + \cdots \tag{7}$$

where again terms of order t have been neglected. Express $\tilde{\gamma}(t)$ in terms of $\zeta(t)$.

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4 Method of projectors

Consider the operator

$$\mathcal{O}_D = \bar{\psi}\not{D}\psi\,. \tag{8}$$

where D_μ is the covariant derivative of QCD.

1. Write down the Feynman rules for this operator.
2. Construct a projector onto this operator.

3. Apply the projector to the flowed operator

$$\tilde{\mathcal{O}}_D = \bar{\chi}(t) \not{\mathcal{D}} \chi(t) \,, \tag{9}$$

where $\mathcal{D}_\mu$ is the flowed covariant derivative. Draw two diagrams that contribute at one-loop level. Calculate one of them (which is non-zero).

You can use Ref. [2] for the Feynman rules, for example.

References

[1] R. V. Harlander, F. Lange, and T. Neumann, *Hadronic vacuum polarization using gradient flow*, *JHEP* **08** (2020) 109, `arXiv:2007.01057 [hep-lat]`.

[2] J. Artz, R. V. Harlander, F. Lange, T. Neumann, and M. Prausa, *Results and techniques for higher order calculations within the gradient-flow formalism*, *JHEP* **06** (2019) 121, `arXiv:1905.00882 [hep-lat]`. [Erratum: JHEP 10, 032 (2019)].