Bad Honnef summer school exerices:
Nuclear physics from (lattice) QCD

1) Nucleon Wick contractions

Consider a theory of free quarks without gluonic interactions. The propagator for an off-shell quark with momentum $p^\mu = (0,0,0,0)$ takes the particularly simple form

$$S^{ab}_{\alpha\beta}(p=0) = \frac{1}{m_q}\delta_{\alpha\beta}\delta^{ab}, \tag{1}$$

where m_q is the quark mass and α, β are Dirac spinor indices and a, b are color indices.

Compute the correlation function

$$\left\langle p_\uparrow \bar{p}_\uparrow \right\rangle, \tag{2}$$

using the proton interpolating operator defined in class composed of a product of such free quark fields with $p^\mu = 0$. This can be done either using the Dirac spinor representation of the proton field or the spin-color weight representation

$$p_\uparrow = w^{abc}_{\alpha\beta\gamma} u^a_\alpha d^b_\beta u^c_\gamma, \tag{3}$$

where

$$w^{abc}_{\uparrow\downarrow\uparrow} = -w^{abc}_{\downarrow\uparrow\uparrow} = \varepsilon^{abc}. \tag{4}$$

2) Deuteron Wick contractions

Using the same simple theory of free-field quarks, compute the deuteron correlation function

$$\left\langle d_1 d_1^\dagger \right\rangle, \tag{5}$$

where

$$d_1 = p_\uparrow n_\uparrow, \tag{6}$$

with the neutron interpolating operator

$$n_\uparrow = w^{abc}_{\alpha\beta\gamma} d^a_\alpha u^b_\beta d^c_\gamma, \tag{7}$$

defined using the same weights as the proton operator.

3) Finite volume energies

Compute the finite-volume energy spectrum for two nucleons in the deuteron channel in a cubic box of side length L for all states that lie below the 3π threshold for $L \lesssim 7$ fm. How many L-independent states have the right quantum numbers?

4) Quantization condition

Derive Lüscher's quantization condition for two nucleons in nonrelativistic pionless EFT quoted in the lectures. The Lagrange density is

$$\mathcal{L}(x) = \psi^\dagger(x)\left[i\partial_0 - \frac{\nabla^2}{2M}\right]\psi(x) - \frac{1}{2}\left(\frac{4\pi a}{M}\right)\psi^\dagger(x)\psi(x)\psi^\dagger(x)\psi(x). \tag{8}$$

Note that the nonrelativistic FV propagator is given by

$$S\left(E, \mathbf{p} = \frac{2\pi}{L}\mathbf{n}\right) = \frac{-i}{E - \left(\frac{2\pi}{L}\right)^2\mathbf{n}^2 + i\epsilon}, \tag{9}$$

which only has a pole in one half of the complex E plane.

Details can be found in S. R. Beane, P. F. Bedaque, A. Parreno and M. J. Savage, "Two nucleons on a lattice," Phys. Lett. B **585**, 106-114 (2004) [arXiv:hep-lat/0312004 [hep-lat]].

5) Contour deformations

Consider the toy model of a plaquette in 2D $U(1)$ gauge theory with open boundary conditions,

$$\int_{-\pi}^{\pi}\frac{d\phi}{2\pi}e^{i\phi}e^{\beta\cos(\phi)} = I_1(\beta). \tag{10}$$

Find the minimum-variance integration contour within the family of constant shifts

$$\phi \to \phi + if, \tag{11}$$

and compute the signal-to-noise ratios for the original and minimum-variance contours.

Details can be found in W. Detmold, G. Kanwar, H. Lamm, M. L. Wagman and N. C. Warrington, "Path integral contour deformations for observables in $SU(N)$ gauge theory," Phys. Rev. D **103**, no.9, 094517 (2021) [arXiv:2101.12668 [hep-lat]].