Basics of Finite-Temperature Field Theory: Exercises

These are a few problems that might be useful to work through. As I will be gone after my lecture on Thursday, I won't be able to cover the parts related to Thursday's lecture during the school, but if you already feel like trying to do them on Wednesday, we can talk about them during the tutoring session. I've also written down some extra exercises related to thermal effective field theories, which I'm unlikely to have time to cover.

1. Imaginary-time Formalism (Wednesday's lecture)

The following two problems are "core", fundamental calculations in thermal field theory

- By using standard rules of Grassmann integration and performing the logarithmic fermionic sum-integral, confirm that the formula stated in the main text for the pressure of a free fermionic field is correct.
- Using the QCD Feynman rules, check the contractions and evaluate the NLO pressure of finite-temperature QCD.

The next two problems are mostly there in case you feel unsatisfied with the heuristics of the lecture notes and want to fill in the gaps (or have nothing better to do)

- The derivation of the imaginary-time formalism was quite heuristic, and required the reader to believe certain definitions and associations. If you want to further convince yourself of its worth, directly obtain a path-integral of a quantum mechanical particle with a Hamiltonian $H = \frac{p^2}{2m} + V$ from $Z = \mathrm{Tr} e^{-\beta H}$. You should obtain $Z = \int_{x(\beta)=x(0)} \mathscr{D}x e^{-\int_0^\beta \mathrm{d}t \left[\frac{m}{2} x_t'' + V(x_t)\right]}$, and see eg. the periodic boundaries arise in a natural way. The QFT result then follows by the same arguments as when constructing the path-integral formalism for QFT from QM. *Hint:* This is discussed extensively in practically all textbooks, eg. Laine & Vuorinen.
- Confirm the contour representation of the Bosonic Matsubara sum by evaluating the residues, and derive an equivalent fermionic expression. If you are mathematically inclined, make sure you understand the conditions f (and do a "there exists $\delta > 0$" proof) must fulfill in the construction. Compute the standard bosonic Matsubara sum $\sum_{p_0} \left[p_0^2 + p^2\right]^{-1}$ and the corresponding sum-integral by hand, eg with a contour integral or a Fourier transform.

2. Real-time Formalism (Thursday's lecture)

- Check the r/a change-of-basis-rules explicitly.
- In the $\lambda\phi^4$-theory and the real-time formalism, write down the r/a-basis diagrams for the two-loop corrections to the retarted self-energy $\Pi^R(P)$.

3. Thermal EFTs (Bonus)

- Evaluate the pressure of the scalar DR theory to next-to-leading order.
- Evaluate the matching coefficient m_E^2 (and λ_E , if it is not obvious to you) in the scalar DR theory to leading order.
- (A fair bit of work, but essential if you want to do HTL, and done in LV 8.4) Starting from the gluon self-energy

$$\Pi^{ab}_{\mu\nu}(K) = \frac{1}{2}\delta^{ab} g^2 N_c \sum\!\!\!\!\!\!\int_P \frac{1}{P^2 (K-P)^2} \left[\left(-4K^2 + 2(D-2)P^2\right)\delta_{\mu\nu} + (D+2)K_\mu K_\nu - 4(D-2)P_\mu P_\nu\right]$$

$$-\delta^{ab} g^2 N_f \sum\!\!\!\!\!\!\int_{\{P\}} \frac{1}{P^2 (K-P)^2} \left[\left(2P^2 - K^2\right)\delta_{\mu\nu} + 2K_\mu K_\nu - 4P_\mu P_\nu\right]$$

derive the HTL self-energy

$$H^{ab}_{\mu\nu}(K) = \frac{m_E^2}{2}\left[-\frac{k_0^2}{k^2} + \frac{ik_0}{2k}\left(1 + \frac{k_0^2}{k^2}\right)\ln\frac{ik_0 + k}{ik_0 - k}\right]\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\delta_{\mu i}\delta_{\nu j}$$

$$+ m_E^2\left[1 + \frac{k_0^2}{k^2}\right]\left[1 - \frac{ik_0}{2k}\ln\frac{ik_0 + k}{ik_0 - k}\right]\left[\delta_{\mu\nu} - \frac{K_\mu K_\nu}{K^2} - \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\delta_{\mu i}\delta_{\nu j}\right].$$